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L^p ESTIMATES FOR MULTILINEAR OPERATORS OF STRONGLY SINGULAR INTEGRAL OPERATORS*†

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Abstract. In this paper, the authors get the L^p estimates for the commutators generated by strongly singular integral operators and BMO functions and the corresponding multilinear operators by the scale changing method introduced by Carleson and Sjölin.

§1. Introduction

Let T be a linear operator, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. The commutator generated by T and b is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x),$$

where f is a suitable function. Coifman, Rochberg and Weiss [CRW] proved a celebrated result, when T is a standard Carderón-Zygmund singular integral operator, $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$, where $1 < p < \infty$. Later, Chanillo [C1] got the L^p boundedness of the commutator generated by a BMO function and a fractional integral. In 1993, Alvarez, Bagby, Kurtz and Pérez [ABKP] studied the L^p bounedness of the commutator generated by a BMO function and a general linear operator, and got the following relation between the L^p boundeness of the commutator and the weighted L^p bounedness of the corresponding linear operator.

THEOREM A. ([ABKP]) *Let $1 < p, q < \infty$. Suppose that a linear operator T satisfies the weighted norm estimate*

$$\|Tf\|_{p,w} \le C\|f\|_{p,w}$$

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for all $w \in A_q$, where the constant C depends only on n, p and the A_q constant of w, but not on the weight w. Then for any $b \in \mathrm{BMO}(\mathbb{R}^n)$, the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$.

It is easy to prove that Theorem A is suitable to many operators in harmonic analysis, such as standard Calderón-Zygmund integral operators, oscillatory integral operators with polynomial phases and Calderón-Zygmund kernels, the Bochner-Riesz operator with the critical index and etc. But, it should be pointed out that Theorem A can not be used to some important operators. For instance, Hu and the second author of this paper in [HL] considered the Bochner-Riesz operator below the critical index. In this paper, we consider the commutators of strongly singular integral operators which have important background in multiple Fourier series. Let us first state some definitions.

Given a suitable function f, its Fourier transform is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x\xi}\, dx.$$

Let $\theta(\xi)$ be a smooth radial cut-off function, $\theta(\xi) = 1$ if $|\xi| \geq 1$ and $\theta(\xi) = 0$ if $|\xi| \leq 1/2$. The strongly singular integral operator is defined by

$$\mathcal{F}(T^{s,\alpha} f)(\xi) = \theta(\xi) \frac{e^{i|\xi|^s}}{|\xi|^\alpha} \hat{f}(\xi),$$

where $0 < s < 1$, $0 < \alpha \leq ns/2$. Let $\lambda = \frac{ns/2-\alpha}{1-s}$, the convolution form of $T^{s,\alpha}$ can be roughly written as

$$T^{s,\alpha} f(x) = p.v. \int \frac{e^{i|x-y|^{-s'}}}{|x-y|^{n+\lambda}} \chi(|x-y|) f(y)\, dy. \tag{1}$$

Here $s' = s/(1-s)$ and χ denotes the characteristic function of the unit interval $(0, 1) \subset \mathbb{R}$. Since s and α do not appear in the convolution form apparently, we would like to use $T^{s',\lambda}$ instead of $T^{s,\alpha}$. When $\alpha = ns/2$, it turns out that $\lambda = 0$. This operator will be denoted by T, and the convolution form of T will be

$$Tf(x) = p.v. \int \frac{e^{i|x-y|^{-s'}}}{|x-y|^n} \chi(|x-y|)\, f(y)\, dy. \tag{2}$$

It is known that $T^{s',\lambda}$ is bounded on $L^p(\mathbb{R}^n)$ for $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{\alpha}{n}\left[\frac{n/2+\lambda}{\alpha+\lambda}\right] = \frac{1}{2}-\frac{\lambda}{ns'}$ (see [Hi], [F]), and the range of p is the best. We now consider two cases $\lambda = 0$ and $\lambda > 0$. If $\lambda = 0$, $T^{s',\lambda}$ turns out to be T in (2). For this kind of operators, the author of [C2] has already got their boundeness on weighted L^p space for $1 < p < \infty$ as follows:

THEOREM B. ([C2]) *Let $\omega \in A_p$, $1 < p < \infty$, and T be a strongly singular integral operator defined by* (2)*. Then*

$$\|Tf\|_{p,\omega} \leq C_p \|f\|_{p,\omega}. \tag{3}$$

It follows from Theorem A and Theorem B that when $\lambda = 0$, the commutator generated by $T^{s',\lambda} = T$ and a BMO function b is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In addition, the weighted norm inequalities for this commutator was also obtained in [GHST]. The more difficulty case is $\lambda > 0$. In this case, we can not expect that the inequality $\|T^{s',\lambda} f\|_p \leq C\|f\|_p$ holds for all $1 < p < \infty$. In other words, we can not use Theorem A to get the L^p boundedness of the commutator $[b, T^{s',\lambda}]$. And the method used in [GHST] can not be used to this case either, since that the method in which essentially is an estimate of the sharp function of $[b, T]f(x)$ and only suite to the operators that are L^p bounded for all $r_0 < p < \infty$, here $1 \leq r_0 < \infty$. Then an interesting question arises naturally, that is whether $[b, T^{s',\lambda}]$ is bounded on some $L^p(\mathbb{R}^n)$ under the assumption $\lambda > 0$? Precisely, whether $[b, T^{s',\lambda}]$ is bounded on $L^p(\mathbb{R}^n)$ for $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{2}-\frac{\lambda}{ns'}$? In this paper we give an affirmative answer by using the scale changing method which is introduced by Carleson and Sjölin in [CS]. We get

THEOREM 1.1. *Let $0 < s' < \infty$, $0 < \lambda < ns'/2$, $T^{s',\lambda}$ be a strongly singular integral operator defined by* (1)*, $\left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{2}-\frac{\lambda}{ns'}$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then the commutator $[b, T^{s',\lambda}]$ is a bounded operator on $L^p(\mathbb{R}^n)$.*

More generally, we consider the multilinear operator of strongly singular integral operator defined by

$$T_A^{s',\lambda} f(x) = p.v. \int \frac{e^{i|x-y|^{-s'}}}{|x-y|^{n+\lambda}} \chi(|x-y|) \frac{R_{m+1}(A; x, y)}{|x-y|^m} f(y)\, dy.$$

Here $R_{m+1}(A; x, y) = A(x) - \sum_{|\gamma| \leq m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma$ is the $(m+1)$-th order Taylor series remainder of A.

THEOREM 1.2. *Let* $0 < s' < \infty$, $0 < \lambda < ns'/2$, $T^{s',\lambda}$ *be a strongly singular integral operator defined in* (1), $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2} - \frac{\lambda}{ns'}$, *and* $D^\gamma A \in \mathrm{BMO}(\mathbb{R}^n)$ *for any* n*-tuple index* γ *with* $|\gamma| = m > 0$. *Then the multilinear operator* $T_A^{s',\lambda}$ *can be extended to a bounded operator on* $L^p(\mathbb{R}^n)$.

Remark 1.1. It is clear that when $m = 0$, the multilinear operator turns to be a commutator, then Theorem 1.2 can be regarded as an extension of Theorem 1.1.

§2. Some elementary results and lemmas

In this section let us give some lemmas which will be used in the proofs of our theorems.

Let Q denote a cube in $\mathbb{R}^n$ with sides parallel to the axes, $|Q|$ denote the Lebesgue measure of Q. Let $m_Q(g) = \frac{1}{|Q|}\int_Q g(x)\,dx$ and $S_q(g)(x) = \sup_{x\in Q}\left\{\frac{1}{|Q|}\int_Q |g(x) - m_Q(g)|^q\,dx\right\}^{1/q}$. It is known that if $g \in \mathrm{BMO}(\mathbb{R}^n)$, then $\|S_q(g)\|_\infty \approx \|g\|_{\mathrm{BMO}}$.

LEMMA 2.1. ([Hu]) *Let* Q_1 *and* Q_2 *be two cubes whose intersections are not empty. If* $d(Q_1) \geq d(Q_2)$ ($d(Q)$ *is the diameter of* Q), $p \geq 1$, *then*

$$|m_{Q_1}(g) - m_{Q_2}(g)| \leq C\left(1 + \log\frac{d(Q_1)}{d(Q_2)}\right)S_p(g)(x_1),$$

where $x_1 \in Q_2$ *and* C *is independent of* Q_1 *and* Q_2.

It follows that if $g \in \mathrm{BMO}(\mathbb{R}^n)$ and Q_1, Q_2 are two cubes whose intersections are not empty, then

$$|m_{Q_1}(g) - m_{Q_2}(g)| \leq C\left(1 + \left|\log\frac{d(Q_1)}{d(Q_2)}\right|\right)\|g\|_{\mathrm{BMO}}. \tag{4}$$

LEMMA 2.2. ([CG]) *Let* $A(x)$ *be a function on* $\mathbb{R}^n$ *with* m*-th order derivatives in* $L^q(\mathbb{R}^n)$ *where* $q > n$. *Then*

$$|R_m(A;x,y)| \leq C_{m,n}|x-y|^m \sum_{|\gamma|=m}\left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)}|D^\gamma A(z)|^q\,dz\right)^{1/q},$$

where $Q(x,y)$ *is the cube centered at* x *with edges parallel to the axes and having diameter* $5\sqrt{n}|x-y|$.

Let Ψ be a smooth function with compact support in both x and ξ, and Φ be real valued and smooth. We assume that on the support of Ψ, the Hessian determinant of Φ is nonvanishing, i.e.

$$\det\left(\frac{\partial^2\Phi(x,\xi)}{\partial x_i \partial \xi_j}\right) \neq 0. \tag{5}$$

We consider oscillatory integral

$$(T_\lambda f)(\xi) = \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,\xi)}\Psi(x,\xi)f(x)\,dx.$$

LEMMA 2.3. ([St]) *Under the above assumptions on Φ and Ψ, we have that*

$$\|T_\lambda f\|_{L^2(\mathbb{R}^n)} \leq C\lambda^{-n/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

Obviously, we also have

$$\|T_\lambda f\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_{L^\infty(\mathbb{R}^n)} \quad \text{and} \quad \|T_\lambda f\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{L^1(\mathbb{R}^n)}.$$

By interpolations, we get

$$\|T_\lambda f\|_{L^p(\mathbb{R}^n)} \leq C\lambda^{-n/p}\|f\|_{L^p(\mathbb{R}^n)}, \quad 2 \leq p < \infty, \tag{6}$$

and

$$\|T_\lambda f\|_{L^p(\mathbb{R}^n)} \leq C\lambda^{-n/p'}\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leq p < 2,\ 1/p + 1/p' = 1. \tag{7}$$

Let $I = [0,1]^n$ be the unit cube in $\mathbb{R}^n$. Let tI denote a cube with the same center as I and side length t, for any $t > 0$. Denote $F(I) = 5I \setminus 2I$, we will use Lemma 2.3 in different places with $\Phi(x,y) = |x-y|^{-s'/2}$, but with $\Psi(x,y)$ being different functions supported on $F(I) \times I$. Thus we need to show that $\Phi(x,y)$ satisfies (5) on $F(I) \times I$. Write $r = \sum_{i=1}^n (x_i - y_i)^2$, then $\Phi(x,y) = r^{-s'/2}$. It is easy to get

$$\frac{\partial^2\Phi(x,y)}{\partial x_i \partial y_j} = -s'(s'+2)r^{-\frac{s'}{2}-2}(x_i-y_i)(x_j-y_j), \quad i \neq j,$$

and

$$\frac{\partial^2\Phi(x,y)}{\partial x_i \partial y_i} = -s'(s'+2)r^{-\frac{s'}{2}-2}(x_i-y_i)^2 + s'r^{-\frac{s'}{2}-1}.$$

Denote $C_r = (-s')^n r^{-(\frac{s'}{2}+2)n}$, thus

$$
\begin{aligned}
&\det\left(\frac{\partial^2 \Phi(x,y)}{\partial x_i \partial y_j}\right) \\
&= C_r \begin{vmatrix}
(s'+2)(x_1-y_1)^2-r & (s'+2)(x_1-y_1)(x_2-y_2) & \cdots & (s'+2)(x_1-y_1)(x_n-y_n) \\
(s'+2)(x_2-y_2)(x_1-y_1) & (s'+2)(x_2-y_2)^2-r & \cdots & (s'+2)(x_2-y_2)(x_n-y_n) \\
\cdots & \cdots & \cdots & \cdots \\
(s'+2)(x_n-y_n)(x_1-y_1) & (s'+2)(x_n-y_n)(x_2-y_2) & \cdots & (s'+2)(x_n-y_n)^2-r
\end{vmatrix}
\end{aligned}
$$

$$
\begin{vmatrix}
(s'+2)(x_1-y_1)^2-r & (s'+2)(x_1-y_1)(x_2-y_2) & \cdots & (s'+2)(x_1-y_1)(x_n-y_n) \\
(s'+2)(x_2-y_2)(x_1-y_1) & (s'+2)(x_2-y_2)^2-r & \cdots & (s'+2)(x_2-y_2)(x_n-y_n) \\
\cdots & \cdots & \cdots & \cdots \\
(s'+2)(x_n-y_n)(x_1-y_1) & (s'+2)(x_n-y_n)(x_2-y_2) & \cdots & (s'+2)(x_n-y_n)^2-r
\end{vmatrix}
$$

$$
\begin{aligned}
&= \begin{vmatrix}
1 & (s'+2)(x_1-y_1) & (s'+2)(x_2-y_2) & \cdots & (s'+2)(x_n-y_n) \\
0 & (s'+2)(x_1-y_1)^2-r & (s'+2)(x_1-y_1)(x_2-y_2) & \cdots & (s'+2)(x_1-y_1)(x_n-y_n) \\
0 & (s'+2)(x_2-y_2)(x_1-y_1) & (s'+2)(x_2-y_2)^2-r & \cdots & (s'+2)(x_2-y_2)(x_n-y_n) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & (s'+2)(x_n-y_n)(x_1-y_1) & (s'+2)(x_n-y_n)(x_2-y_2) & \cdots & (s'+2)(x_n-y_n)^2-r
\end{vmatrix} \\
&= \begin{vmatrix}
1 & (s'+2)(x_1-y_1) & (s'+2)(x_2-y_2) & \cdots & (s'+2)(x_n-y_n) \\
-(x_1-y_1) & -r & 0 & \cdots & 0 \\
-(x_2-y_2) & 0 & -r & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-(x_n-y_n) & 0 & 0 & \cdots & -r
\end{vmatrix} \\
&= \begin{vmatrix}
-(s'+1) & (s'+2)(x_1-y_1) & (s'+2)(x_2-y_2) & \cdots & (s'+2)(x_n-y_n) \\
0 & -r & 0 & \cdots & 0 \\
0 & 0 & -r & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & -r
\end{vmatrix} \\
&= -(s'+1)(-r)^n.
\end{aligned}
$$

When $x \in F(I)$ and $y \in I$,

$$\det\left(\frac{\partial^2 \Phi(x,y)}{\partial x_i \partial y_j}\right) = -(s'+1)s'^n r^{-(\frac{s'}{2}+1)n} \neq 0.$$

This confirms the above assertion.

Let $K_{s',\lambda}(x) = \frac{e^{i|x|^{-s'}}}{|x|^{n+\lambda}}\chi(|x|)$, we define

$$S_N^{s',\lambda} f(x) = N^n \int_I K_{s',\lambda}(N(x-y)) f(y)\, dy, \tag{8}$$

$$S_{N,b}^{s',\lambda}f(x) = N^n \int_I K_{s',\lambda}(N(x-y))[b(x)-b(y)]f(y)\,dy, \tag{9}$$

and

$$S_{N,A}^{s',\lambda}f(x) = N^n \int_I K_{s',\lambda}(N(x-y))\frac{R_{m+1}(A;x,y)}{|x-y|^m}f(y)\,dy. \tag{10}$$

LEMMA 2.4. *For any $1 < p < \infty$, we have*

$$\|S_{N,b}^{s',\lambda}f\|_{L^p(F(I))} \leq CN^{-\lambda}\|b\|_{\mathrm{BMO}}\|f\|_{L^p(I)}. \tag{11}$$

Proof. Set $0 < r < n/p$ and $\sigma > 0$ such that

$$\frac{1}{p+\sigma} = \frac{1}{p} - \frac{r}{n}.$$

Observe that if $x \in F(I)$, then

$$\begin{aligned}|S_N^{s',\lambda}f(x)| &\leq CN^{-\lambda}\int_I |f(y)|\,dy \leq C_r N^{-\lambda}\int_I \frac{|f(y)|\,dy}{|x-y|^{n-r}} \\ &\leq C_r N^{-\lambda} I_r(|f\chi_I|)(x),\end{aligned}$$

where $\chi_I(x)$ is the character function of I, and I_r is the fractional integral operator of order r. By the Hardy-Littlewood-Sobolev theorem, we get

$$\|S_N^{s',\lambda}f\|_{L^{p+\sigma}(F(I))} \leq CN^{-\lambda}\|f\|_{L^p(I)}. \tag{12}$$

In the same way, we can choose σ very small, such that

$$\|S_N^{s',\lambda}f\|_{L^p(F(I))} \leq CN^{-\lambda}\|f\|_{L^{p-\sigma}(I)}. \tag{13}$$

Let $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) = 1$ if $|x| \leq 10\sqrt{n}$ and $\operatorname{supp}\phi \subset \{x : |x| \leq 20\sqrt{n}\}$. Set

$$\tilde{b}(y) = [b(y) - m_{4I}(b)]\phi(y),$$

where $m_{4I}(b)$ denotes the mean value of b on $4I$. If $x \in F(I)$, then

$$S_{N,b}^{s',\lambda}f(x) = \tilde{b}(x)S_N^{s',\lambda}f(x) - S_N^{s',\lambda}(\tilde{b}f)(x) = I + II.$$

For I, we choose $1 < r < \infty$ such that $1/r + 1/(p+\sigma) = 1/p$. By using Hölder's inequality and inequality (12), we get

$$\begin{aligned}\|I\|_{L^p(F(I))} &\leq \left\{\int_{4I} |b(x) - m_{4I}(b)|^r\,dx\right\}^{1/r} \|S_N^{s',\lambda}f\|_{L^{p+\sigma}(F(I))} \\ &\leq CN^{-\lambda}\|b\|_{\mathrm{BMO}}\|f\|_{L^p(I)}.\end{aligned}$$

For II, we choose $1 < r < \infty$ such that $1/r + 1/p = 1/(p-\sigma)$. From inequality (13) and Hölder's inequality, it follows that

$$\begin{aligned}\|II\|_{L^p(F(I))} &\le CN^{-\lambda}\bigg\{\int_I |b(y)-m_{4I}(b)|^{p-\sigma}|f(y)|^{p-\sigma}\,dy\bigg\}^{1/(p-\sigma)}\\ &\le CN^{-\lambda}\bigg\{\int_I |b(y)-m_{4I}(b)|^{r}\,dy\bigg\}^{1/r}\|f\|_{L^p(I)}\\ &\le CN^{-\lambda}\|b\|_{\mathrm{BMO}}\|f\|_{L^p(I)}.\end{aligned}$$

Combining the two estimates above, we finish the proof of this lemma. ∎

Similarly, we get

Lemma 2.5. *For any* $1 < p < \infty$, *we have*

$$\|S^{s',\lambda}_{N,A}f\|_{L^p(F(I))} \le C(n,p,s',\lambda,m)N^{-\lambda}\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}\|f\|_{L^p(I)}. \tag{14}$$

Proof. As in the proof of Lemma 2.4, select $\sigma > 0$ and $0 < r < n$ such that

$$\frac{1}{p} = \frac{1}{p-\sigma} - \frac{r}{n}.$$

Let

$$\bar{A}(z) = \bigg[A(z) - \sum_{|\gamma|=m}\frac{z^\gamma}{\gamma!}m_I(D^\gamma A)\bigg]\phi(z).$$

Since $R_{m+1}(A;x,y) = R_{m+1}(\bar{A};x,y)$, for any $x \in F(I)$ and $y \in I$, we have

$$\begin{aligned}|S^{s',\lambda}_{A}f(x)| \le CN^{-\lambda}&\int_I |R_m(\bar{A};x,y)||f(y)|\,dy\\ &+ C_rN^{-\lambda}\sum_{|\gamma|=m}\int_I\frac{|D^\gamma\bar{A}(y)f(y)|}{|x-y|^{n-r}}\,dy.\end{aligned}$$

From Lemma 2.2 and Lemma 2.1, it follows that

$$\int_I |R_m(\bar{A};x,y)||f(y)|\,dy \le C\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}M(f\chi_I)(x),$$

where $M(f\chi_I)$ is the Hardy-Littlewood maximal function of $f\chi_I$. By the Hardy-Littlewood-Sobolev theorem, we get

$$\begin{aligned}
&\sum_{|\gamma|=m}\left\|\int_I \frac{|D^\gamma \bar{A}(y)f(y)|}{|\cdot - y|^{n-r}}\,dy\right\|_{L^p(F(I))}\\
&\leq C\sum_{|\gamma|=m}\left(\int_I |D^\gamma A(y)-m_I(D^\gamma A)|^{p-\sigma}|f(y)|^{p-\sigma}\,dy\right)^{1/(p-\sigma)}\\
&\leq C\sum_{|\gamma|=m}\left(\int_I |D^\gamma A(y)-m_I(D^\gamma A)|^{n/r}\,dy\right)^{r/n}\|f\|_{L^p(I)}\\
&\leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}\|f\|_{L^p(I)}.
\end{aligned}$$

By the L^p boundedness of Hardy-Littlewood maximal function and the above estimates, we get

$$\|S_A^{s',\lambda}f\|_{L^p(F(I))} \leq CN^{-\lambda}\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}\|f\|_{L^p(I)}.$$

□

Let K be a distribution with compact support in $\mathbb{R}^n$, locally integrable outside the origin and satisfying the conditions

$$A(\beta) \qquad |\hat{K}(\xi)| \leq B(1+|\xi|)^{-n\beta/2}, \quad \xi\in\mathbb{R}^n,$$

and

$$B(\theta) \qquad \int_{|x|>2|y|^{1-\theta}} |K(x-y)-K(x)|\,dx \leq B, \quad |y|<d.$$

Here $\hat{K}$ is the Fourier transform of K, B and d denote positive constants and $0\leq\beta\leq\theta<1$.

Lemma 2.6. ([Sj]) *If $0<\beta<\theta<1$, then $T_K f(x) = K * f(x)$ can be extended to bounded linear operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all K satisfying the above assumption if and only if $p\leq q$ and*

$$\beta/2 > 1/p - 1/q + b\max(1/2-1/p,\, 1/q-1/2,\, 0),$$

where $b = (n\beta(1-\theta)+2\theta)/(n(1-\theta)+2)$.

Let $1-\theta = 1/(\lambda + s' + 1)$ and $\beta = 2\alpha/n$. In [Sj], the author pointed out that $K_{s',\lambda}$ satisfied condition $A(\beta)$ and $B(\theta)$. Recently, we [LL] get that for any $|\gamma| = m$, $K_{s',\lambda,\gamma}(x) = K_{s',\lambda}(x)\frac{x^\gamma}{|x|^m}$ satisfies $A(\beta)$ and $B(\theta)$. Thus we have for any $p \le q$ satisfying

$$\alpha/n > 1/p - 1/q + s\max(1/2 - 1/p,\, 1/q - 1/2,\, 0), \quad p \le q, \tag{15}$$

$T^{s',\lambda}$ and $T_\gamma^{s',\lambda} f(x) = K_{s',\lambda,\gamma} * f(x)$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

§3. Proof of Theorem 1.1

If we want to show that $[b, T^{s',\lambda}]$ is bounded on $L^p(\mathbb{R}^n)$ for $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2} - \frac{\lambda}{ns'}$, it suffices to prove that for any positive number $N \ge 1$,

$$\begin{aligned}\int_{[0,N]^n} \left| \int_{[0,N]^n} K_{s',\lambda}(x-y)(b(x)-b(y))f(y)\,dy \right|^p dx \\ \le C\|b\|_{\mathrm{BMO}}^p \int_{[0,N]^n} |f(y)|^p\,dy.\end{aligned}$$

By changing the scale, we need to show that

$$\begin{aligned}\int_I \left| N^n \int_I K_{s',\lambda}(N(x-y))(b(Nx)-b(Ny))F(y)\,dy \right|^p dx \\ \le C\|b\|_{\mathrm{BMO}}^p \int_I |F(y)|^p\,dy,\end{aligned}$$

where $F(y) = f(Ny)$. Note that if $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$, then $b(tx) \in \mathrm{BMO}(\mathbb{R}^n)$ and $\|b(\cdot)\|_{\mathrm{BMO}} = \|b(t\,\cdot)\|_{\mathrm{BMO}}$ for any $t > 0$, it is equal to show

$$\begin{aligned}\int_I \left| N^n \int_I K_{s',\lambda}(N(x-y))(b(x)-b(y))f(y)\,dy \right|^p dx \\ \le C\|b\|_{\mathrm{BMO}}^p \int_I |f(y)|^p\,dy.\end{aligned}$$

We reduce our proof into proving the following lemma.

LEMMA 3.1. *Under the same conditions of Theorem* 1.1, *we have*

$$\|S_{N,b}^{s',\lambda} f\|_{L^p(I)} \le C(n,p,s',\lambda)\|b\|_{\mathrm{BMO}}\|f\|_{L^p(I)}. \tag{16}$$

Proof. Let Ω_μ, $\mu = 0, 1, \ldots$, denote the set of all dyadic cubes in $(-2,\, 2)^n$ with side length $2^{-\mu}$, and let Ω_μ^* denote the set of all cubes which are the union of 2^n cubes in Ω_μ. Let $f \in L^p(I)$ and set f equal to zero outside I. If $x \in I$ and x does not belong to the boundary of any dyadic cubes, let $w_\mu^*(x)$ be the unique element of Ω_μ^* which satisfies $x \in \frac{1}{2} w_\mu^*(x)$, and set $w_{-1}^*(x) = (-2,\, 2)^n$.

For a measurable set $D \subset I$, we define $E(x, D)$ by

$$E(x, D) = N^{-\lambda} \int_D \frac{e^{iN^{-s'}|x-y|^{-s'}}}{|x-y|^{n+\lambda}} \chi(N|x-y|)(b(x) - b(y)) f(y)\, dy,$$

where $x \in I$ and we also set $E_\mu(x) = E(x, w_{\mu-1}^*(x) \setminus w_\mu^*(x) \cap I)$, $\mu \geq 0$. Defining μ_N by $2^{-\mu_N - 1} < N^{-1} \leq 2^{-\mu_N}$, we have

$$S_{N,b}^{s',\lambda} f(x) = \sum_{\mu=0}^{\mu_N} E_\mu(x) + E(x, w_{\mu_N}^*(x) \cap I). \tag{17}$$

From the construction of $w_\mu^*(x)$ it follows that $|E_\mu(x)| \leq \sum_{\substack{w \in \Omega_\mu \\ \omega \cap I \neq \emptyset}} |E(x, w)|$ $\chi_{F(w)}(x)$, where $F(w) = 5\omega \setminus 2\omega$ and $\chi_{F(w)}$ is the characteristic function of $F(w)$. Since $\sum_{w \in \Omega_\mu} \chi_{F(w)}(x) \leq 5^n - 2^n$, Hölder's inequality yields $|E_\mu(x)|^p \leq C \sum_{\substack{w \in \Omega_\mu \\ \omega \cap I \neq \emptyset}} |E(x, w)|^p \chi_{F(w)}(x)$ and hence for any $\mu \leq \mu_N$,

$$\int_I |E_\mu(x)|^p\, dx \leq C \sum_{\substack{w \in \Omega_\mu \\ \omega \cap I \neq \emptyset}} \int_{F(w)} |E(x, w)|^p\, dx. \tag{18}$$

Performing a change of scale and using Lemma 2.4, we obtain

$$\int_{F(w)} |E(x, w)|^p\, dx \leq C(N 2^{-\mu})^{-\lambda p} \|b\|_{\mathrm{BMO}}^p \int_w |f(x)|^p\, dx.$$

A combination of this inequality with (18) yields

$$\|E_\mu\|_{L^p(I)} \leq C N^{-\lambda} 2^{\lambda \mu} \|b\|_{\mathrm{BMO}} \|f\|_{L^p(I)}. \tag{19}$$

And assume that we have

$$\left\{ \int_I |E(x, w_{\mu_N}^*(x) \cap I)|^p\, dx \right\}^{1/p} \leq C \|b\|_{\mathrm{BMO}} \|f\|_{L^p(I)}, \tag{20}$$

a combination of (17), (19) with (20) yields (16). Now we suffer to prove inequality (20). From the construction of $\omega^*(x)$ it follows that $E(x,\omega^*_{\mu_N}(x)\cap I)=\sum_{\substack{\omega\in\Omega^*_{\mu_N}\\ \omega\cap I\neq\emptyset}}E(x,\omega)\chi_{\frac{1}{2}\omega}(x)$, where $\chi_{\frac{1}{2}\omega}(x)$ is the characteristic function of $\frac{1}{2}\omega$. Since $\sum_{\substack{\omega\in\Omega^*_{\mu_N}\\ \omega\cap I\neq\emptyset}}\chi_{\frac{1}{2}\omega}(x)=1$, Hölder's inequality yields

$$|E(x,\omega^*_{\mu_N}(x)\cap I)|^p\le\sum_{\substack{\omega\in\Omega^*_{\mu_N}\\ \omega\cap I\neq\emptyset}}|E(x,\omega)|^p\chi_{\frac{1}{2}\omega}(x).$$

Hence

$$\int_I|E(x,\omega^*_N(x)\cap I)|^p\,dx\le\sum_{\substack{\omega\in\Omega^*_{\mu_N}\\ \omega\cap I\neq\emptyset}}\int_{\frac{1}{2}\omega}|E(x,\omega)|^p\,dx:=\sum_{\substack{\omega\in\Omega^*_{\mu_N}\\ \omega\cap I\neq\emptyset}}B_\omega.$$

For any fixed $\omega\in\Omega^*_{\mu_N}$, and $\omega\cap I\neq\emptyset$, denote $x_\omega=(x_1,x_2,\dots,x_n)$ such that for any $y=(y_1,y_2,\dots,y_n)\in\omega$, $x_i\le y_i$, $i=1,2,\dots,n$. Let $b_\omega(x)=b(x+x_\omega)$, $f_\omega(x)=f(x+x_\omega)$, and $J=[0,2]^n$. Noticing that $2^{-\mu_N-1}<N^{-1}\le 2^{-\mu_N}$, we may assume that $N^{-1}=2^{-\mu_N}$. Since the side length of w is $2^{-\mu_N+1}$,

$$\begin{aligned}
B_\omega&=\int_{\frac{1}{2}\omega}\Big|\int_\omega N^nK_{s,\lambda}(N(x-y))[b(x)-b(y)]f(y)\,dy\Big|^p\,dx\\
&=\int_{\frac{1}{2}\omega-x_\omega}\Big|\int_{[0,2N^{-1}]^n}N^nK_{s',\lambda}(N(x-y))[b_\omega(x)-b_\omega(y)]f_\omega(y)\,dy\Big|^p\,dx\\
&=N^{-n}\int_Q\Big|\int_J K_{s,\lambda}(x-y)[b_\omega(N^{-1}x)-b_\omega(N^{-1}y)]f_\omega(N^{-1}y)\,dy\Big|^p\,dx\\
&\le N^{-n}\int_J\Big|\int_J K_{s',\lambda}(x-y)[b_\omega(N^{-1}x)-b_\omega(N^{-1}y)]f_\omega(N^{-1}y)\,dy\Big|^p\,dx.
\end{aligned}$$

Here, $Q=N(\frac{1}{2}\omega-x_\omega)$ is a cube in J. Recall that $\phi(x)\in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x)=1$ if $|x|\le 10\sqrt{n}$ and $\operatorname{supp}\phi\subset\{x:|x|\le 20\sqrt{n}\}$. Set

$$\tilde b(y)=[b_\omega(N^{-1}y)-m_J(b_\omega(N^{-1}\cdot))]\phi(y),$$

where $m_J(b_\omega)$ denotes the mean value of b_ω on J. Letting $f_\omega(N^{-1}y)\chi_J(y)=F(y)$, we write

$$N^nB_\omega\le\int_J|\tilde b(x)T^{s',\lambda}F(x)|^p\,dx+\int_J|T^{s',\lambda}(\tilde bF)(x)|^p\,dx:=I+II.$$

It follows from Lemma 2.6, that for any $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{2}-\frac{\lambda}{ns'}=\frac{\alpha}{ns}$, there exist $1<q_1<p<q_2<\infty$, such that $T^{s',\lambda}$ is a bounded linear operator from $L^{q_1}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, and from $L^p(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$.

For I, let $1<r<\infty$ such that $\frac{1}{r}+\frac{1}{q_2}=\frac{1}{p}$, by using Hölder's inequality and noting that $T^{s',\lambda}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$, we have

$$\begin{aligned}
I &\leq \left\{\int_J |b(N^{-1}x+x_\omega)-m_J(b(N^{-1}\ \cdot\ +x_\omega))|^r\,dx\right\}^{p/r}\|T^{s',\lambda}F(y)\|_{L^{q_2}}^p\\
&\leq C\|b\|_{\mathrm{BMO}}^p\|F\|_{L^p(J)}^p\\
&\leq CN^n\|b\|_{\mathrm{BMO}}^p\|f\|_{L^p(\omega)}^p.
\end{aligned}$$

For the second inequality, we used Hölder's inequality again. During the estimate, C is independent of N.

For II, we choose $1<r<\infty$ such that $\frac{1}{r}+\frac{1}{p}=\frac{1}{q_1}$. The L^{q_1} to L^p boundedness of $T^{s',\lambda}$ and Hölder's inequality yield

$$\begin{aligned}
II &\leq C\|\tilde{b}F\|_{L^{q_1}(J)}^p\\
&\leq C\left\{\int_J |b(N^{-1}x+x_\omega)-m_J(b(N^{-1}\ \cdot\ +x_\omega))|^r dx\right\}^{p/r}\|F\|_{L^p(J)}^p\\
&\leq C\|b\|_{\mathrm{BMO}}^p\|F\|_{L^p(J)}^p\\
&\leq CN^n\|b\|_{\mathrm{BMO}}^p\|f\|_{L^p(\omega)}^p.
\end{aligned}$$

Combining the estimates of I with II, we get

$$B_\omega \leq C\|b\|_{\mathrm{BMO}}^p\|f\|_{L^p(\omega)}^p,$$

where C is independent of N. For a fixed $\omega\in\Omega_{\mu_N}^*$ the number of $\omega'\in\Omega_{\mu_N}^*$ such that $\omega'\cap\omega\neq\phi$ is at most 3^n-1. Thus we have

$$\int_I |E(x,\omega_{\mu_N}^*(x))|^p\,dx \leq C\|b\|_{\mathrm{BMO}}^p\|f\|_{L^p(I)}^p.$$

We finish the proof of Lemma 3.1. □

Turn back to the very beginning. It is clear that

$$\|[b,T^{s',\lambda}]f\|_{L^p(\mathbb{R}^n)} \leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^p(\mathbb{R}^n)},$$

for $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{2}-\frac{\lambda}{ns'}$. We finish the proof of Theorem 1.1.

§4. Proof of Theorem 1.2

Noticing that $D_x^\gamma A(Nx) = N^m D^\gamma A(\mu)\big|_{\mu=Nx}$ for $|\gamma| = m$, in the same way as the proof of Theorem 1.1, our aim becomes to prove under the conditions of Theorem 1.2,

$$\|S_{N,A}^{s',\lambda} f\|_{L^p(I)} \leq C(n, p, s', \lambda, m) \sum_{|\gamma|=m} \|D^\gamma A\|_{\mathrm{BMO}} \|f\|_{L^p(I)}. \tag{21}$$

As in the proof of Theorem 1.1, for a measurable set $S \subset I$, we define $E(x, S)$ by

$$E(x, S) = N^n \int_S \frac{K(N(x-y))}{|x-y|^m} R_{m+1}(A; x, y) f(y)\, dy.$$

Set $E_\mu(x) = E(x, \omega_{\mu-1}^*(x) \setminus \omega_\mu^*(x) \cap I)$. μ_N denotes a number such that $2^{-\mu_N - 1} < N^{-1} \leq 2^{-\mu_N}$. We have

$$S_{N,A}^{s',\lambda} f(x) = \sum_{\mu=0}^{\mu_N} E_\mu(x) + E(x, \omega_{\mu_N}^*(x) \cap I). \tag{22}$$

Similarly we have

$$\int_I |E_\mu(x)|^p\, dx \leq C \sum_{\substack{w \in \Omega_\mu \\ \omega \cap I \neq \emptyset}} \int_{F(w)} |E(x, w)|^p\, dx. \tag{23}$$

For a fixed $\omega \in \Omega_\mu$, denote $x_\omega = (x_1, x_2, \ldots, x_n)$ such that for any $y = (y_1, y_2, \ldots, y_n)$ $x_i \leq y_i$, $i = 1, 2, \ldots, n$. Noting that the side length of $\omega = 2^{-\mu}$, we have

$$\begin{aligned}
&\int_{F(\omega)} |E(x, \omega)|^p\, dx \\
&\quad = \int_{F(\omega)} \Big| \int_\omega N^n \frac{K_{s',\lambda}(N(x-y))}{|x-y|^m} R_{m+1}(A; x, y) f(y)\, dy \Big|^p dx \\
&\quad = \int_{F(\omega) - x_\omega} \Big| \int_{[0, 2^{-\mu}]^n} N^n \frac{K_{s',\lambda}(N(x-y))}{|x-y|^m} \\
&\qquad\qquad \times R_{m+1}(A; x + x_\omega, y + x_\omega) f(y + x_\omega)\, dy \Big|^p dx \\
&\quad = \int_{F(I)} 2^{-\mu n} \Big| \int_I 2^{(-\mu)(n-m)} N^n \frac{K_{s',\lambda}(2^{-\mu} N(x-y))}{|x-y|^m} \\
&\qquad\qquad \times R_{m+1}(A; 2^{-\mu} x + x_\omega, 2^{-\mu} y + x_\omega) f(2^{-\mu} y + x_\omega)\, dy \Big|^p dx.
\end{aligned}$$

By Lemma 2.5 and noting that $\|D^\gamma A(2^{-\mu}\cdot)\|_{\mathrm{BMO}} = 2^{-\mu m}\|D^\gamma A\|_{\mathrm{BMO}}$ for any $|\gamma| = m$, we obtain

$$\int_{F(w)} |E(x,w)|^p\,dx \le C(N2^{-\mu})^{-\lambda p}\sum_{|\gamma|=m}\|D^\gamma A\|^p_{\mathrm{BMO}}\int_w |f(x)|^p\,dx.$$

A combination of this inequality with (23) yields

$$\|E_\mu\|_{L^p(I)} \le CN^{-\lambda}2^{\lambda\mu}\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}\|f\|_{L^p(I)}. \tag{24}$$

Corresponding to inequality (20), we suffer to show

$$\left\{\int_I |E(x,\omega^*_{\mu_N}(x)\cap I)|^p\,dx\right\}^{1/p} \le C\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}\|f\|_{L^p(I)}. \tag{25}$$

For a fixed $\bar\omega \in \Omega^*_{\mu_N}$ with $\bar\omega\cap I \ne \emptyset$, denote $x_{\bar\omega} = (x_1, x_2, \ldots, x_n)$ such that for any $y = (y_1, y_2, \ldots, y_n) \in \bar\omega$, $x_i \le y_i$, $i = 1, 2, \ldots, n$. Recall that $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x) = 1$ if $|x| \le 10\sqrt{n}$ and $\operatorname{supp}\phi \subset \{x\,;\,|x| \le 20\sqrt{n}\}$. Set

$$\tilde A(z) = \left[A(z) - \sum_{|\gamma|=m}\frac{1}{\gamma!}m_I(D^\gamma A(N^{-1}\,\cdot\,+x_{\bar\omega}))z^\gamma\right]\phi(z).$$

It is known that $R_{m+1}(A;x,y) = R_{m+1}(\tilde A;x,y)$ for $x \in F(\bar\omega)$ and $y \in \bar\omega$. We write

$$\begin{aligned}E(x,\bar\omega) &= \int_{\bar\omega} N^n\frac{K_{s',\lambda}(N(x-y))}{|x-y|^m}R_m(\tilde A;x,y)f(y)\,dy\\ &\quad+\sum_{|\gamma|=m}\int_{\bar\omega} N^nK_{s',\lambda,\gamma}(N(x-y))D^\gamma\tilde A(y)f(y)\,dy\\ &:= E^{(1)}(x,\bar\omega)+E^{(2)}(x,\bar\omega).\end{aligned}$$

Since $\sum_{\substack{\bar\omega\in\Omega^*_{\mu_N}\\ \bar\omega\cap I\ne\emptyset}}\chi_{\frac12\bar\omega}(x) = 1$, Hölder's inequality yields

$$\begin{aligned}&|E(x,\Omega^*_{\mu_N}(x)\cap I)|^p\\ &\quad\le \sum_{\substack{\bar\omega\in\Omega^*_{\mu_N}\\ \bar\omega\cap I\ne\emptyset}}|E^{(1)}(x,\bar\omega)|^p\chi_{\frac12\bar\omega}(x)+\sum_{\substack{\bar\omega\in\Omega^*_{\mu_N}\\ \bar\omega\cap I\ne\emptyset}}|E^{(2)}(x,\bar\omega)|^p\chi_{\frac12\bar\omega}(x).\end{aligned}$$

Hence

$$\begin{aligned}\int_I |E(x,\Omega^*_{\mu_N}(x)\cap I)|^p\,dx \\ \le \sum_{\substack{\bar\omega\in\Omega^*_{\mu_N}\\ \bar\omega\cap I\neq\emptyset}} \int_{\frac12\bar\omega} |E^{(1)}(x,\bar\omega)|^p\,dx + \sum_{\substack{\bar\omega\in\Omega^*_{\mu_N}\\ \bar\omega\cap I\neq\emptyset}} \int_{\frac12\bar\omega} |E^{(2)}(x,\bar\omega)|^p\,dx.\end{aligned}$$

For a fixed $\bar\omega\in\Omega^*_{\mu_N}$, noticing that $2^{-\mu_N-1}<N^{-1}\le 2^{-\mu_N}$, we may assume that $N^{-1}=2^{-\mu_N}$. Since the side length of $\bar\omega$ is $2^{-\mu_N+1}$,

$$\begin{aligned}&\int_{\frac12\bar\omega}|E^{(2)}(x,\bar\omega)|^p\,dx\\ &= \sum_{|\gamma|=m}\int_{\frac12\bar\omega-x_{\bar\omega}}\Big|\int_{[0,2N^{-1}]^n} N^nK_{s',\lambda,\gamma}(N(x-y))\\ &\qquad\times D^\gamma\tilde A(y+x_{\bar\omega})f(y+x_{\bar\omega})\,dy\Big|^p\,dx\\ &= N^{-n}\sum_{|\gamma|=m}\int_Q\Big|\int_{[0,2]^n}K_{s',\lambda,\gamma}(x-y)\Big[D^\gamma A(N^{-1}y+x_{\bar\omega})\\ &\qquad - m_I(D^\gamma A(N^{-1}\,\cdot\,+x_{\bar\omega}))\Big]f(N^{-1}y+x_{\bar\omega})\,dy\Big|^p\,dx.\end{aligned}$$

Here $Q=N\big[\frac12\bar\omega-x_{\bar\omega}\big]$ is a cube in $[0,2]^n$. Noting that $\big|\frac1p-\frac12\big|<\frac12-\frac{\lambda}{ns'}=\frac{\alpha}{ns}$, it follows from Lemma 2.6 that there exists $q<p$ such that $T^{s',\lambda}_\gamma$ is bounded from $L^q(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Choosing $1<r<\infty$ such that $\frac1r+\frac1p=\frac1q$, we have

$$\begin{aligned}&N^n\int_{\frac12\bar\omega}|E^{(2)}(x,\bar\omega)|^p\,dx\\ &\le C\sum_{|\gamma|=m}\Big\{\int_{[0,2]^n}|D^\gamma A(N^{-1}y+x_{\bar\omega})-m_I(D^\gamma A(N^{-1}\,\cdot\,+x_{\bar\omega}))|^q\\ &\qquad\times|f(N^{-1}y+x_{\bar\omega})|^q\,dy\Big\}^{p/q}\\ &\le C\sum_{|\gamma|=m}\Big\{\int_{[0,2]^n}|D^\gamma A(N^{-1}y+x_{\bar\omega})-m_I(D^\gamma A(N^{-1}\,\cdot\,+x_{\bar\omega}))|^r\,dy\Big\}^{p/r}\\ &\qquad\times\|f(N^{-1}\,\cdot\,+x_{\bar\omega})\|^p_{L^p([0,2]^n)}\end{aligned}$$

$$\leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{\mathrm{BMO}}^p \|f(N^{-1}\ \cdot\ + x_{\bar\omega})\|_{L^p([0,2]^n)}$$
$$\leq CN^n \sum_{|\gamma|=m} \|D^\gamma A\|_{\mathrm{BMO}}^p \|f\|_{L^p(\bar\omega)}.$$

Here C is a constant independent of N. Noting that the number of $\omega \in \Omega^*_{\mu_N}$ with $\omega \cap \bar\omega \neq \emptyset$ is at most $3^n - 1$, we have

$$\sum_{\substack{\bar\omega \in \Omega^*_{\mu_N} \\ \bar\omega \cap I \neq \emptyset}} \int_{\frac12 \bar\omega} |E^{(2)}(x,\bar\omega)|^p\, dx \leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{\mathrm{BMO}}^p \|f\|_{L^p(I)}.$$

For $E^{(1)}(x,\bar\omega)$, we define $E^{(1)}_\mu(x) = E^{(1)}(x, \omega^*_{\mu-1}(x) \setminus \omega^*_\mu(x) \cap \bar\omega)$ and write

$$E^{(1)}(x,\bar\omega) = \sum_{\mu=\mu_N}^{\infty} E^{(1)}_\mu(x). \tag{26}$$

And it is clear that $|E^{(1)}_\mu(x)| \leq \sum_{\substack{\omega\in\Omega_\mu \\ \omega\cap\bar\omega\neq\emptyset}} |E^{(1)}(x,\omega)|\chi_{F(\omega)}(x)$, in the same way as in the case $\mu < \mu_N$, we get

$$\sum_{\substack{\bar\omega \in \Omega^*_{\mu_N} \\ \bar\omega \cap I \neq \emptyset}} \int_{\frac12 \bar\omega} |E^{(1)}(x,\bar\omega)|^p\, dx$$
$$\leq C \sum_{\substack{\bar\omega \in \Omega^*_{\mu_N} \\ \bar\omega \cap I \neq \emptyset}} \left\{ \sum_{\mu \geq \mu_N} \left\{ \sum_{\substack{\omega\in\Omega_\mu \\ \omega\cap\bar\omega\neq\emptyset}} \int_{F(\omega)} |E^{(1)}(x,\omega)|^p\, dx \right\}^{1/p} \right\}^p.$$

Fix $\mu \geq \mu_N$, $\bar\omega \in \Omega^*_{\mu_N}$, and $\omega \in \Omega_\mu$. Denote $x_\omega = (x_1, x_2, \ldots, x_n)$ such that for any $y = (y_1, y_2, \ldots, y_n) \in \omega$, $x_i \leq y_i$, $i = 1, 2, \ldots, n$. Noting that the side length of $\omega = 2^{-\mu}$, we have

$$\int_{F(\omega)} |E^{(1)}(x,\omega)|^p\, dx$$
$$= \int_{F(\omega)} \left| \int_\omega N^n \frac{K_{s',\lambda}(N(x-y))}{|x-y|^m} R_m(\tilde A; x, y) f(y)\, dy \right|^p dx$$
$$= \int_{F(\omega)-x_\omega} \bigg| \int_{[0,2^{-\mu}]^n} N^n \frac{K_{s',\lambda}(N(x-y))}{|x-y|^m}$$
$$\times R_m(\tilde A; x + x_\omega, y + x_\omega) f(y + x_\omega)\, dy \bigg|^p dx$$

$$= \int_{F(I)} 2^{-\mu n} \Big| \int_I 2^{(-\mu)(n-m)} N^n \frac{K_{s',\lambda}(2^{-\mu}N(x-y))}{|x-y|^m} \times R_m(\tilde{A}; 2^{-\mu}x + x_\omega, 2^{-\mu}y + x_\omega) f(2^{-\mu}y + x_\omega)\, dy \Big|^p \, dx.$$

Select $x_0 \in 8I \setminus 6I$, and denote $K_{s',\lambda}^{\mu,N,m}(x) = 2^{(-\mu)(n-m)} N^n \frac{K_{s',\lambda}(2^{-\mu}N(x))}{|x|^m}$. Then

$$\begin{aligned} &\int_{F(\omega)} |E^{(1)}(x,\omega)|^p \, dx \\ &= \int_{F(I)} 2^{-\mu n} \Big| \int_I K_{s',\lambda}^{\mu,N,m}(x-y) \Big[R_m(\tilde{A}; 2^{-\mu}x + x_\omega, 2^{-\mu}y + x_\omega) \\ &\qquad - R_m(\tilde{A}; 2^{-\mu}x + x_\omega, 2^{-\mu}x_0 + x_\omega) \Big] f(2^{-\mu}y + x_\omega)\, dy \Big|^p \, dx \\ &\quad + \int_{F(I)} 2^{-\mu n} |R_m(\tilde{A}; 2^{-\mu}x + x_\omega, 2^{-\mu}x_0 + x_\omega)|^p \\ &\qquad \times \Big| \int_I K_{s',\lambda}^{\mu,N,m}(x-y) f(2^{-\mu}y + x_\omega)\, dy \Big|^p \, dx. \end{aligned}$$

Recall that for any g with m-th order derivatives [CG],

$$R_m(g; x, y) - R_m(g; x, x_0) = \sum_{|\alpha|<m} \frac{(x-x_0)^\alpha}{\alpha!} R_{m-|\alpha|}(D^\alpha g; x_0, y).$$

We have

$$\begin{aligned} &\int_{F(\omega)} |E^{(1)}(x,\omega)|^p \, dx \\ &\le C \sum_{|\alpha|<m} \int_{F(I)} 2^{-\mu n} \Big| 2^{-\mu|\alpha|} \int_I K_{s',\lambda}^{\mu,N,m}(x-y)(x-x_0)^\alpha \\ &\qquad \times R_{m-|\alpha|}(D^\alpha \tilde{A}; 2^{-\mu}x_0 + x_\omega, 2^{-\mu}y + x_\omega) f(2^{-\mu}y + x_\omega)\, dy \Big|^p \, dx \\ &\quad + \int_{F(I)} 2^{-\mu n} |R_m(\tilde{A}; 2^{-\mu}x + x_\omega, 2^{-\mu}x_0 + x_\omega)|^p \\ &\qquad \times \Big| \int_I K_{s',\lambda}^{\mu,N,m}(x-y) f(2^{-\mu}y + x_\omega)\, dy \Big|^p \, dx. \end{aligned}$$

Denote $Q^{-\mu}_{x_0,y} = Q(2^{-\mu}x_0 + x_\omega, 2^{-\mu}y + x_\omega)$. From Lemma 2.2, it follows that for any $|\alpha| < m$,

$$\begin{aligned}
&|R_{m-|\alpha|}(D^\alpha \tilde{A}; 2^{-\mu}x_0 + x_\omega, 2^{-\mu}y + x_\omega)| \\
&\quad \le 2^{-\mu(m-|\alpha|)}|x_0 - y|^{m-|\alpha|} \\
&\qquad \times \sum_{|\gamma|=m} \left(\frac{1}{|Q^{-\mu}_{x_0,y}|} \int_{Q^{-\mu}_{x_0,y}} |D^\gamma A(z) - m_I(D^\gamma A(N^{-1}\ \cdot\ + x_{\bar\omega}))|^q \right)^{1/q} \\
&\quad \le C 2^{-\mu(m-|\alpha|)}|x_0 - y|^{m-|\alpha|} \\
&\qquad \times \sum_{|\gamma|=m} \Big\{ \|D^\gamma A\|_{\mathrm{BMO}} + |m_{Q^{-\mu}_{x_0,y}}(D^\gamma A) - m_I(D^\gamma A(N^{-1}\ \cdot\ + x_{\bar\omega}))| \Big\}.
\end{aligned}$$

Noting that $m_I(D^\gamma A(N^{-1}\ \cdot\ + x_{\bar\omega})) = m_{\bar\omega}(D^\gamma A)$, $Q^{-\mu}_{x_0,y}$ and $\bar\omega$ are not disjoint, $d(\bar\omega) \ge d(Q^{-\mu}_{x_0,y})$ and the side length of $Q^{-\mu}_{x_0,y}$ is approximately $2^{-\mu}$, we get by inequality (4) that

$$\begin{aligned}
&|R_{m-|\alpha|}(D^\alpha \tilde{A}; 2^{-\mu}x_0 + x_\omega, 2^{-\mu}y + x_\omega)| \\
&\quad \le C 2^{-\mu(m-|\alpha|)}(\mu - \mu_N) \sum_{|\gamma|=m} \|D^\gamma A\|_{\mathrm{BMO}}.
\end{aligned}$$

In the same way, we obtain

$$|R_m(\tilde{A}; 2^{-\mu}x + x_\omega, 2^{-\mu}x_0 + x_\omega)| \le C 2^{-\mu(m)}(\mu - \mu_N) \sum_{|\gamma|=m} \|D^\gamma A\|_{\mathrm{BMO}}.$$

For $p \ge 2$, we use inequality (6) with $\Psi(x,y)$ approximating to $\frac{(x-x_0)^\alpha}{|x-y|^{n+\lambda+m}}$ on $F(I) \times I$ for the case that $|\alpha| < m$, and $\Psi(x,y)$ approximating to $\frac{1}{|x-y|^{n+m+\lambda}}$ on $F(I) \times I$ for the case that $|\alpha| = 0$.

$$\begin{aligned}
&\int_{F(\omega)} |E^{(1)}(x,\omega)|^p\, dx \\
&\quad \le C \sum_{|\alpha|<m} 2^{-\mu n} 2^{-\mu(-m+|\alpha|)p} 2^{\mu\lambda p} N^{-\lambda p} (2^{-\mu}N)^{ns'} (\mu - \mu_N)^p \\
&\qquad\quad \times \|R_{m-|\alpha|}(D^\alpha \tilde{A}; 2^{-\mu}x_0 + x_\omega, 2^{-\mu}\ \cdot\ + x_\omega) f(2^{-\mu}\ \cdot\ + x_\omega)\|^p_{L^p(I)} \\
&\qquad + C 2^{-\mu n} 2^{\mu\lambda p} N^{-\lambda p} (2^{-\mu}N)^{ns'} (\mu - \mu_N)^p \\
&\qquad\quad \times \sum_{|\gamma|=m} \|D^\gamma A\|^p_{\mathrm{BMO}} \|f(2^{-\mu} \cdot + x_\omega)\|^p_{L^p(I)}
\end{aligned}$$

$$\begin{aligned}&\le C2^{-\mu n}2^{\mu\lambda p}N^{-\lambda p}(2^{-\mu}N)^{ns'}(\mu-\mu_N)^p\\&\qquad\times\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}^p\|f(2^{-\mu}\ \cdot\ +x_\omega)\|_{L^p(I)}^p\\&=C(2^{-\mu}N)^{ns'-\lambda p}(\mu-\mu_N)^p\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}^p\|f\|_{L^p(\omega)}^p.\end{aligned}$$

In the same way, for $1<p<2$ by using inequality (7), we get

$$\begin{aligned}&\int_{F(\omega)}|E^{(1)}(x,\omega)|^p\,dx\\&\qquad\le C(2^{-\mu}N)^{ns'(p-1)-\lambda p}(\mu-\mu_N)^p\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}^p\|f\|_{L^p(\omega)}^p.\end{aligned}$$

Recalling that $2^{-\mu-1}<N^{-1}\le 2^{-\mu}$, and $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{2}-\frac{\lambda}{ns'}$, there exists $\sigma>0$ such that

$$\begin{aligned}&\int_{F(\omega)}|E^{(1)}(x,\omega)|^p\,dx\\&\qquad\le C2^{-(\mu-\mu_N)\sigma p}(\mu-\mu_N)^p\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}^p\|f\|_{L^p(\omega)}^p.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{\substack{\omega\in\Omega_\mu\\ \omega\cap\bar\omega\ne\emptyset}}\int_{F(\omega)}|E^{(1)}(x,\omega)|^p\,dx\\&\qquad\le C2^{-(\mu-\mu_N)\sigma p}(\mu-\mu_N)^p\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}^p\|f\|_{L^p(\bar\omega)}^p.\end{aligned}$$

And

$$\sum_{\mu\ge\mu_N}\Bigg\{\sum_{\substack{\omega\in\Omega_\mu\\ \omega\cap\bar\omega\ne\emptyset}}\int_{F(\omega)}|E^{(1)}(x,\omega)|^p\,dx\Bigg\}^{1/p}\le C\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}\|f\|_{L^p(\bar\omega)}.$$

As the same as in the estimate of $E^{(2)}(x,\omega^*_{\mu_N}(x)\cap I)$, we can obtain

$$\sum_{\substack{\bar\omega\in\Omega^*_{\mu_N}\\ \bar\omega\cap I\ne\emptyset}}\int_{\frac{1}{2}\bar\omega}|E^{(1)}(x,\bar\omega)|^p\,dx\le C\sum_{|\gamma|=m}\|D^\gamma A\|_{\mathrm{BMO}}^p\|f\|_{L^p(I)}^p.$$

Turn back to the very beginning, we obtain

$$\|T_A^{s',\lambda} f\|_{L^p(\mathbb{R}^n)} \leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{\mathrm{BMO}} \|f\|_{L^p(\mathbb{R}^n)},$$

where $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{2} - \frac{\lambda}{ns'}$. We finish the proof of Theorem 1.2.

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