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ADDENDUM TO THE PAPER "A NOTE ON WEIGHTED BERGMAN SPACES AND THE CESÀRO OPERATOR"

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Abstract. Let $H(\mathbf{D}_n)$ be the space of holomorphic functions on the unit polydisk $\mathbf{D}_n$, and let $\mathcal{L}_\alpha^{p,q}(\mathbf{D}_n)$, where $p, q > 0$, $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j > -1$, $j = 1, \dots, n$, be the class of all measurable functions f defined on $\mathbf{D}_n$ such that

$$\int_{[0,1)^n} M_p^q(f, r) \prod_{j=1}^n (1 - r_j)^{\alpha_j} \, dr_j < \infty,$$

where $M_p(f, r)$ denote the p-integral means of the function f. Denote the weighted Bergman space on $\mathbf{D}_n$ by $\mathcal{A}_\alpha^{p,q}(\mathbf{D}_n) = \mathcal{L}_\alpha^{p,q}(\mathbf{D}_n) \cap H(\mathbf{D}_n)$. We provide a characterization for a function f being in $\mathcal{A}_\alpha^{p,q}(\mathbf{D}_n)$. Using the characterization we prove the following result: Let $p > 1$, then the Cesàro operator is bounded on the space $\mathcal{A}_\alpha^{p,p}(\mathbf{D}_n)$.

§1. Introduction

Let $\mathbf{D}_1 = \mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk in the complex plane and let $\mathbf{D}_n$ be the unit polydisk in the complex vector space $\mathbf{C}^n$. Denote the space of all holomorphic functions on $\mathbf{D}_n$ by $H(\mathbf{D}_n)$. For $z, w \in \mathbf{C}^n$, we write $z \cdot w = (z_1 w_1, \dots, z_n w_n)$; $e^{i\theta}$ is an abbreviation for $(e^{i\theta_1}, \dots, e^{i\theta_n})$; $d\theta = d\theta_1 \cdots d\theta_n$ and r, θ, α are vectors in $\mathbf{C}^n$. We say $0 \le r = (r_1, \dots, r_n) < 1$ whenever $0 \le r_j < 1$ for $j = 1, \dots, n$.

For $f \in H(\mathbf{D}_n)$ and $p \in (0, \infty)$,

$$M_p(f, r) = \left(\frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta \right)^{1/p}, \quad \text{for } 0 \le r < 1$$

denote the integral means of f.

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Let $\mathcal{L}^{p,q}_\alpha = \mathcal{L}^{p,q}_\alpha(\mathbf{D}_n)$, where $p, q > 0$ and $\alpha_j > -1$, $j = 1, \dots, n$, be the class of all measurable functions f defined on $\mathbf{D}_n$ such that

$$\|f\|^q_{\mathcal{L}^{p,q}_\alpha} = \int_{[0,1)^n} M^q_p(f, r) \prod_{j=1}^n (1 - r_j)^{\alpha_j} dr_j < \infty.$$

The weighted Bergman space (with classical weight) $\mathcal{A}^{p,q}_\alpha$ is the intersection of $\mathcal{L}^{p,q}_\alpha$ and $H(\mathbf{D}_n)$. When $p = q$ we denote $\mathcal{A}^{p,q}_\alpha$ by $\mathcal{A}^p_\alpha$ and $\mathcal{L}^{p,q}_\alpha$ by $\mathcal{L}^p_\alpha$. Weighted Bergman spaces of holomorphic or harmonic functions with weights other than classical weights have been studied, for example, in [2], [3], [4], [6], [7], [8], see also the references therein.

In [5] a family of Cesàro operators $\mathcal{C}^{\vec{\gamma}}$, called *the generalized Cesàro* operators, was introduced on the polydisk $\mathbf{D}_n$, by

$$\mathcal{C}^{\vec{\gamma}}(f)(z) = \sum_{|\delta|=0}^{\infty} \left(\frac{\sum_{\beta \le \delta} a_{\delta-\beta} \prod_{j=1}^n A^{\gamma_j}_{\beta_j}}{\prod_{j=1}^n A^{\gamma_j+1}_{\delta_j}} \right) z^\delta,$$

where $\vec{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathbf{C}^n$, $\mathrm{Re}(\gamma_j) > -1$, $j = 1, \dots, n$, whenever $f(z) = \sum_{|\delta|=0}^\infty a_\delta z^\delta$ is an analytic function on $\mathbf{D}_n$ (β and δ are multi-indices from $(\mathbf{Z}_+)^n$). A simple calculation with power series then gives

$$\mathcal{C}^{\vec{\gamma}}(f)(z) = \int_0^1 \cdots \int_0^1 f(\tau_1 z_1, \dots, \tau_n z_n) \prod_{j=1}^n \frac{(\gamma_j + 1)(1 - \tau_j)^{\gamma_j}}{(1 - \tau_j z_j)^{\gamma_j+1}} d\tau, \tag{1}$$

where $d\tau = d\tau_1 \cdots d\tau_n$.

From (1), the following formula also holds

$$\mathcal{C}^{\vec{\gamma}}(f)(z) = \left[\prod_{j=1}^n \frac{\gamma_j + 1}{z_j^{\gamma_j+1}} \right] \int_0^{z_1} \cdots \int_0^{z_n} f(\omega_1, \dots, \omega_n) \prod_{j=1}^n \frac{(z_j - \omega_j)^{\gamma_j}}{(1 - \omega_j)^{\gamma_j+1}} d\omega_j. \tag{2}$$

It was shown in [5] that the generalized Cesàro operator is bounded on the Hardy space when $p \in (0, 1]$:

Theorem A. *Let $0 < p \le 1$, $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$ such that $\mathrm{Re}(\gamma_j) > -1$, $j = 1, \dots, n$, and $0 \le r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,2\pi]^n} |\mathcal{C}^{\vec{\gamma}}(f)(r \cdot e^{i\theta})|^p d\theta \le C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta,$$

for all $f \in H(\mathbf{D}_n)$.

It is easy to see by Theorem A that the generalized Cesàro operator is bounded on the weighted Bergman space $\mathcal{A}^{p,q}_{\alpha}(\mathbf{D}_n)$, when $p \in (0,1]$ and $q > 0$.

In [1], G. Benke and the first author independently introduced and considered the case $\vec{\gamma} = \vec{0}$. They also considered the boundedness of the operator $\mathcal{C}^{\vec{0}}$ on the weighted Bergman space in the case $1 < p < \infty$. The main ingredient of their method is based on the following result (Theorem 1.8 in [1]):

Theorem B. *Let $p \in [1,\infty)$, $\alpha_j > -1$, $j = 1,\dots,n$ and m be a fixed positive integer and let $\mathbf{k} = (k_1,\dots,k_n) \in (\mathbf{Z}_+)^n$. Let f be a holomorphic function defined on the polydisk $\mathbf{D}_n$ in $\mathbf{C}^n$. Then for $\vec{\alpha} = (\alpha_1,\dots,\alpha_n)$, $f \in \mathcal{A}^p_\alpha$ if and only if*

$$\left[\prod_{j=1}^{n}(1-|z_j|^2)^{k_j}\right]\frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z) \in \mathcal{L}^p_\alpha, \quad \forall\, \mathbf{k} \text{ with } |\mathbf{k}| = m.$$

Moreover,

$$\|f\|_{\mathcal{A}^p_\alpha} \asymp \left(\sum_{|\mathbf{k}|=0}^{m-1}\left|\frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(0)\right| + \sum_{|\mathbf{k}|=m}\left\|\left[\prod_{j=1}^{n}(1-|z_j|^2)^{k_j}\right]\frac{\partial^{m} f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\right\|_{\mathcal{L}^p_\alpha}\right).$$

However, there is an apparent typo in the statement of the theorem. In the paper [1], the authors did not mention the condition: for all $\mathbf{k} \in \mathbf{Z}^n$ with $|\mathbf{k}| = m$, as above (Theorem 1.8 in [1]). This gap caused a misunderstanding in the proof of Theorem 2.4 in [1]. However, it is a good idea to use this kind of method to investigate the boundedness of Cesàro operator on the Bergman spaces. In this note we would like to provide a complete proof of Theorem 2.4 in [1], which is based on the idea in that paper when $1 < p < \infty$. In order to do that we put aside Theorem B and use another characterization for $f \in H(\mathbf{D}_n)$ to be in $\mathcal{A}^p_\alpha(\mathbf{D}_n)$ (see Theorem 2 below). Our main result is the following theorem.

Theorem 1. *Let $1 < p < \infty$, $\alpha = (\alpha_1,\dots,\alpha_n)$ such that $\alpha_j > -1$, $j = 1,\dots,n$. Then the Cesàro operator is bounded on $\mathcal{A}^p_\alpha(\mathbf{D}_n)$.*

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§2. Auxiliary results

In order to prove Theorem 1 we need some auxiliary results which can be of independent interest. For $f \in H(\mathbf{D}_n)$, set

$$\partial_n f(z) = \frac{\partial^n f(z)}{\partial z_1 \cdots \partial z_n}.$$

LEMMA 1. *Let $f \in H(\mathbf{D}_n)$ such that $f(z) = 0$ when $\prod_{j=1}^n z_j = 0$. Then for $p, q \in [1, \infty)$ and $\alpha_j > -1$, $j = 1, \dots, n$, there is a positive constant C independent of f such that*

$$\int_{[0,1)^n} M_p^q(f, r) \prod_{j=1}^n (1-r_j)^{\alpha_j} dr \le C \int_{[0,1)^n} M_p^q(\partial_n f, r) \prod_{j=1}^n (1-r_j)^{q+\alpha_j} dr. \tag{3}$$

Proof. Let

$$I = \int_0^1 M_p^q(f, r)(1 - r_1)^{\alpha_1} dr_1.$$

First suppose that $f \in H(\overline{\mathbf{D}_n})$. Using integration by parts, and $f(0, z_2, \dots, z_n) \equiv 0$ in $\mathbf{D}_{n-1}$, we obtain

$$I = \int_0^1 M_p^q(f, r)(1 - r_1)^{\alpha_1} dr_1 = \frac{1}{\alpha_1 + 1} \int_0^1 \frac{\partial}{\partial r_1} M_p^q(f, r)(1 - r_1)^{\alpha_1 + 1} dr_1.$$

At points $z = r \cdot e^{i\theta}$ where f is not zero (almost everywhere) we have

$$\begin{aligned}
\frac{\partial}{\partial r_1} |f(r \cdot e^{i\theta})|^p &= p|f(r \cdot e^{i\theta})|^{p-1} \frac{\partial}{\partial r_1} |f(r \cdot e^{i\theta})| \\
&= p|f(r \cdot e^{i\theta})|^{p-1} \lim_{h \to 0} \frac{|f((r_1 + h, r_2, \dots, r_n) \cdot e^{i\theta})| - |f(r \cdot e^{i\theta})|}{h} \\
&\le p|f(r \cdot e^{i\theta})|^{p-1} \lim_{h \to 0} \frac{|f((r_1 + h, r_2, \dots, r_n) \cdot e^{i\theta}) - f(r \cdot e^{i\theta})|}{|h|} \\
&= p|f(r \cdot e^{i\theta})|^{p-1} \left| \frac{\partial f(r \cdot e^{i\theta})}{\partial r_1} \right| = p|f(r \cdot e^{i\theta})|^{p-1} \left| \frac{\partial f(r \cdot e^{i\theta})}{\partial z_1} \right|.
\end{aligned}$$

By the Dominated Convergence Theorem we have

$$\begin{aligned}
\frac{\partial}{\partial r_1} M_p^p(f, r) &= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \frac{\partial}{\partial r_1} |f(r \cdot e^{i\theta})|^p d\theta \\
&\le \frac{p}{(2\pi)^n} \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^{p-1} \left| \frac{\partial f(r \cdot e^{i\theta})}{\partial z_1} \right| d\theta.
\end{aligned}$$

Applying Hölder's inequality with exponents $p/(p-1)$ and p (when $p>1$), we have

$$\frac{\partial}{\partial r_1} M_p^p(f,r) \le p M_p^{p-1}(f,r) M_p(\partial f/\partial z_1, r). \tag{4}$$

Case $p=1$ is clear. Now let us turn to the case $1<p<\infty$. Note that

$$\frac{\partial}{\partial r_1} M_p^q(f,r) = \frac{q}{p}(M_p^p(f,r))^{q/p-1}\frac{\partial}{\partial r_1} M_p^p(f,r).$$

Using this and (4), we obtain

$$\frac{\partial}{\partial r_1} M_p^q(f,r) \le q M_p^{q-1}(f,r) M_p(\partial f/\partial z_1, r).$$

It follows that

$$\begin{aligned} I &\le \frac{q}{\alpha_1+1}\int_0^1 M_p^{q-1}(f,r)M_p(\partial f/\partial z_1,r)(1-r_1)^{\alpha_1+1}dr_1 \\ &\le \frac{q}{\alpha_1+1} I^{\frac{q-1}{q}}\left(\int_0^1 M_p^q(\partial f/\partial z_1,r)(1-r_1)^{\alpha_1+q}dr_1\right)^{1/q}, \end{aligned}$$

where we used Hölder's inequality with exponents $q/(q-1)$ and q when $q>1$. When $q=1$ the last inequality is obvious. Hence

$$\int_0^1 M_p^q(f,r)(1-r_1)^{\alpha_1}dr_1 \le \left(\frac{q}{\alpha_1+1}\right)^q \int_0^1 M_p^q(\partial f/\partial z_1,r)(1-r_1)^{\alpha_1+q}dr_1.$$

Multiplying this inequality by $(1-r_2)^{\alpha_2}dr_2$, then integrating over $[0,1)$ and applying Fubini's theorem it follows that

$$\begin{aligned} &\int_0^1\int_0^1 M_p^q(f,r)(1-r_1)^{\alpha_1}dr_1(1-r_2)^{\alpha_2}dr_2 \\ &\quad \le \left(\frac{q}{\alpha_1+1}\right)^q \int_0^1\int_0^1 M_p^q(\partial f/\partial z_1,r)(1-r_2)^{\alpha_2}dr_2(1-r_1)^{\alpha_1+q}dr_1. \end{aligned}$$

Applying the above procedure on the integral

$$\int_0^1 M_p^q(\partial f/\partial z_1,r)(1-r_2)^{\alpha_2}dr_2$$

and using the fact that

$$M_p^q(\partial f/\partial z_1,r)\big|_{r_2=0} = 0$$

since

$$\frac{\partial f}{\partial z_1}(z_1,0,z_2,\dots,z_n) = \lim_{h\to 0}\frac{f(z_1+h,0,z_2,\dots,z_n)-f(z_1,0,z_2,\dots,z_n)}{h}=0,$$

we obtain

$$\int_0^1 M_p^q(\partial f/\partial z_1,r)(1-r_2)^{\alpha_2}dr_2 \le \Big(\frac{q}{\alpha_2+1}\Big)^q\int_0^1 M_p^q(\partial^2 f/\partial z_1\partial z_2,r)(1-r_2)^{\alpha_2+q}dr_2$$

and consequently

$$\int_0^1\int_0^1 M_p^q(f,r)(1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2}dr_1dr_2 \le \prod_{j=1}^2\Big(\frac{q}{\alpha_j+1}\Big)^q \times\int_0^1\int_0^1 M_p^q(\partial^2 f/\partial z_1\partial z_2,r)(1-r_1)^{\alpha_1+q}(1-r_2)^{\alpha_2+q}dr_1dr_2.$$

Repeating the same procedure for $r_3,\dots,r_n$, we obtain the result in this case with the constant

$$C=\prod_{j=1}^n\Big(\frac{q}{\alpha_j+1}\Big)^q.$$

If $f\in H(\mathbf{D}_n)$, we use the functions $f(\rho z)$ where $\rho\in[0,1)$, and the Monotone Convergence Theorem to obtain the result.

Now we formulate and prove a useful characterization for $f\in H(\mathbf{D}_n)$ to be in $\mathcal{A}_\alpha^{p,q}(\mathbf{D}_n)$, which was discovered by the second author several years ago and have already been presented at several talks. Here is a good occasion to present the result since we apply it in the proof of Theorem 1.

Theorem 2. (Binomial criterion) *Let $p,q\in[1,\infty)$, $\alpha=(\alpha_1,\dots,\alpha_n)$, with $\alpha_j>-1$ for $j=1,\dots,n$, and $f\in H(\mathbf{D}_n)$. Then $f\in\mathcal{A}_\alpha^{p,q}(\mathbf{D}_n)$ if and only if the functions*

$$T_Sf=\prod_{j\in S}(1-|z_j|)\frac{\partial^{|S|}f}{\prod_{j\in S}\partial z_j}(\chi_S(1)z_1,\chi_S(2)z_2,\dots,\chi_S(n)z_n), \tag{5}$$

belong to the space $\mathcal{L}_\alpha^{p,q}(\mathbf{D}_n)$*, for every* $S \subseteq \{1,2,\dots,n\}$*, where* $\chi_S(\cdot)$ *is the characteristic function of* S*,* $|S|$ *is the cardinal number of* S*, and* $\prod_{j\in S}\partial z_j = \partial z_{j_1}\cdots\partial z_{j_{|S|}}$*, where* $j_k \in S$*,* $k = 1,\dots,|S|$*.*

Moreover, $\|\cdot\|_{\mathcal{A}_\alpha^{p,q}}$ *and* $\|\cdot\|_*$ *are equivalent norms on* $\mathcal{A}_\alpha^{p,q}(\mathbf{D}_n)$*, where*

$$\|f\|_* = |f(0,\dots,0)| + \sum_{S\subseteq\{1,\dots,n\},\, S\neq\emptyset} \|T_S f\|_{\mathcal{L}_\alpha^{p,q}}.$$

Remark 1. To be more suggestive and to give an explanation why it is called the Binomial criterion, we explain here what condition (5) exactly means when $n = 2$ and $n = 3$. When $n = 2$, it means that the following four functions

$$f(0,0), \quad g_1(z_1,z_2) = (1-|z_1|)\frac{\partial f(z_1,0)}{\partial z_1}, \quad g_2(z_1,z_2) = (1-|z_2|)\frac{\partial f(0,z_2)}{\partial z_2},$$

and

$$g_3(z_1,z_2) = (1-|z_1|)(1-|z_2|)\frac{\partial^2 f(z_1,z_2)}{\partial z_1\partial z_2}$$

belong to the space $\mathcal{L}_\alpha^{p,q}(\mathbf{D}_2)$.

Moreover, the norms $\|f\|_{\mathcal{A}_\alpha^{p,q}}$ and

$$\|f\|_* = |f(0,0)| + \sum_{i=1}^{3} \|g_i\|_{\mathcal{L}_\alpha^{p,q}}.$$

are equivalent.

When $n = 3$, it means that the following eight functions

$$f(0,0,0),\ (1-|z_1|)\frac{\partial f(z_1,0,0)}{\partial z_1},\ (1-|z_2|)\frac{\partial f(0,z_2,0)}{\partial z_2},\ (1-|z_3|)\frac{\partial f(0,0,z_3)}{\partial z_2},$$

$$(1-|z_1|)(1-|z_2|)\frac{\partial^2 f(z_1,z_2,0)}{\partial z_1\partial z_2},\ (1-|z_1|)(1-|z_3|)\frac{\partial^2 f(z_1,0,z_3)}{\partial z_1\partial z_3},$$

$$(1-|z_2|)(1-|z_3|)\frac{\partial^2 f(0,z_2,z_3)}{\partial z_2\partial z_3},\ (1-|z_1|)(1-|z_2|)(1-|z_3|)\frac{\partial^3 f(z_1,z_2,z_3)}{\partial z_1\partial z_2\partial z_3},$$

are in $\mathcal{L}_\alpha^{p,q}(\mathbf{D}_3)$.

Proof of Theorem 2. Sufficiency. First, we assume that $f(0,\dots,0) = 0$ and $f \in H(\overline{\mathbf{D}_n})$. In the case we have that

$$f(z) = \sum_{S\subset\{1,\dots,n\},\, S\neq\emptyset} f(\chi_S(1)z_1,\dots,\chi_S(n)z_n) + g(z),$$

where the function g is of the form $z_1 z_2 \cdots z_n h(z)$, $h \in H(\mathbf{D}_n)$.

By Lemma 1 we have

$$\begin{aligned}\|g\|^q_{\mathcal{A}^{p,q}_\alpha} &\le C \int_{[0,1)^n} M_p^q(\partial_n g, r) \prod_{j=1}^n (1-r_j)^{q+\alpha_j}\,dr\\ &= C \int_{[0,1)^n} M_p^q(\partial_n f, r) \prod_{j=1}^n (1-r_j)^{q+\alpha_j}\,dr,\end{aligned}$$

since $\partial_n g = \partial_n f$.

We show that for each $S \subset \{1, \dots, n\}$, $S \ne \emptyset$

$$\|f(\chi_S(1)z_1, \dots, \chi_S(n)z_n)\|_{\mathcal{A}^{p,q}_\alpha}$$

can be estimated by $\|T_S f\|_{\mathcal{L}^{p,q}_\alpha}$, i.e., by the integral

$$\left\| \prod_{j\in S}(1-|z_j|)\frac{\partial^{|S|} f}{\prod_{j\in S}\partial z_j}(\chi_S(1)z_1, \chi_S(2)z_2, \dots, \chi_S(n)z_n)\right\|_{\mathcal{L}^{p,q}_\alpha}.$$

Define $f_S(z) = f(\chi_S(1)z_1, \dots, \chi_S(n)z_n)$ for $z \in \mathbf{D}_n$. Let $S = \{j_1, \dots, j_{|S|}\}$, $1 \le j_1 < \cdots < j_{|S|} \le n$, and $\alpha_S = (\alpha_{j_1}, \dots, \alpha_{j_{|S|}})$. Then there exists an $\tilde{f}_S \in H(\mathbf{D}_{|S|})$ such that $f_S(z) = \tilde{f}(z_{j_1}, \dots, z_{j_{|S|}})$ for any $z \in \mathbf{D}_n$. A simple calculation gives

$$\begin{aligned}\|f_S\|^q_{\mathcal{A}^{p,q}_\alpha} &= \prod_{j\notin S}\frac{1}{\alpha_j+1}\|\tilde{f}_S\|^q_{\mathcal{A}^{p,q}_{\alpha_S}}\\ &= \prod_{j\notin S}\frac{1}{\alpha_j+1}\int_{[0,1)^{|S|}} M_p^q(\tilde{f}_S, r_S)\prod_{k\in S}(1-r_k)^{\alpha_k}\,dr_k,\end{aligned}$$

where $r_S = (r_{j_1}, \dots, r_{j_{|S|}})$. As in the proof of Lemma 1, we have

$$\begin{aligned}\|\tilde{f}_S\|^q_{\mathcal{A}^{p,q}_{\alpha_S}} &\le \prod_{j\in S}\left(\frac{q}{\alpha_j+1}\right)^q \int_{[0,1)^{|S|}} M_p^q(\partial_{|S|}\tilde{f}_S, r_S)\prod_{k\in S}(1-r_k)^{\alpha_k+q}dr_k\\ &= \prod_{j\in S}\left(\frac{q}{\alpha_j+1}\right)^q \left\|\prod_{j\in S}(1-|z_j|)\partial_{|S|}\tilde{f}_S\right\|^q_{\mathcal{L}^{p,q}_{\alpha_S}}\\ &= \prod_{j\in S}\left(\frac{q}{\alpha_j+1}\right)^q \prod_{k\notin S}(\alpha_k+1)\\ &\quad\times \left\|\prod_{j\in S}(1-|z_j|)\frac{\partial^{|S|} f}{\prod_{j\in S}\partial z_j}(\chi_S(1)z_1, \dots, \chi_S(n)z_n)\right\|^q_{\mathcal{L}^{p,q}_\alpha}.\end{aligned}$$

Hence

$$\begin{aligned}&\|f(\chi_S(1)z_1,\ldots,\chi_S(n)z_n)\|_{\mathcal{A}^{p,q}_\alpha}\\&\quad\le\prod_{j\in S}\frac{q}{\alpha_j+1}\left\|\prod_{j\in S}(1-|z_j|)\frac{\partial^{|S|}f}{\prod_{j\in S}\partial z_j}(\chi_S(1)z_1,\ldots,\chi_S(n)z_n)\right\|_{\mathcal{L}^{p,q}_\alpha}.\end{aligned}$$

This gives the result in this case, that is,

$$\|f\|_{\mathcal{A}^{p,q}_\alpha}\le C'\|f\|_*,$$

where $C'>0$ is a constant depending only on α and q.

If $f(0,\ldots,0)\neq 0$ we write $f(z)=f(0,\ldots,0)+g(z)$, then $g(0,\ldots,0)=0$. We have

$$\begin{aligned}\|f\|_{\mathcal{A}^{p,q}_\alpha}&\le\|f(0,\ldots,0)\|_{\mathcal{A}^{p,q}_\alpha}+\|g\|_{\mathcal{A}^{p,q}_\alpha}=C(\alpha)^{1/q}|f(0,\ldots,0)|+\|g\|_{\mathcal{A}^{p,q}_\alpha}\\&\le C(\alpha)^{1/q}|f(0,\ldots,0)|+C'\|g\|_*\\&\le(C(\alpha)^{1/q}+C')(|f(0,\ldots,0)|+\|g\|_*)\\&=(C(\alpha)^{1/q}+C')\|f\|_*,\end{aligned}$$

where $C(\alpha)=1/\prod_{j=1}^n(\alpha_j+1)$, as desired. To remove the restriction of the finiteness of the integrals we consider the holomorphic function $f_\rho(z)=f(\rho z)$ with $\rho<1$. By the Monotone Convergence Theorem, when $\rho\to 1$, we obtain the result.

Necessity. The proof of this part of the theorem is a special case of the proof of Theorem 3 (a) in [3].

LEMMA 2. *Let $p>0$ and $\alpha_j>-1$, $j=1,\ldots,n$. Then for every $S\subseteq\{1,\ldots,n\}$, there exists a constant C independent of f such that*

$$\|(1-|z_k|)f\|_{\mathcal{L}^p_\alpha}\le C\left\|\prod_{j\in S}z_j(1-|z_k|)f\right\|_{\mathcal{L}^p_\alpha},$$

for every $f\in H(\mathbf{D}_n)$ and $k\in\{1,\ldots,n\}$.

Proof. Without loss of generality we may assume that $n=2$, $k=1$, and $S=\{2\}$. Let $f\in H(\mathbf{D}_2)$, then

$$\begin{aligned}&\|(1-|z_1|)f\|^p_{\mathcal{L}^p_\alpha}\\&\quad=\int_0^{1/2}\int_0^{1/2}+\int_0^{1/2}\int_{1/2}^1+\int_{1/2}^1\int_0^{1/2}+\int_{1/2}^1\int_{1/2}^1 g(r_1,r_2)dr_1dr_2,\end{aligned}$$

where $g(r_1,r_2) = M_p^p(f,r_1,r_2)(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2}$. Now we estimate these four integrals, which we denote by I_i, $i=1,2,3,4$.

Since $f \in H(\mathbf{D}_2)$, the function f is holomorphic in each variable separately on $\mathbf{D}$ and consequently $M_p^p(f,r_1,r_2)$ is nondecreasing in r_1 and r_2. Let

$$C_{\alpha_i} = \int_0^{1/2}(1-r_i)^{\alpha_i+\delta_1^i}dr_i\Big/\int_{1/2}^1(1-r_i)^{\alpha_i+\delta_1^i}dr_i, \quad i=1,2,$$

where $\delta_1^1 = p$ and $\delta_1^2 = 0$.

Note that C_{α_i}, $i=1,2$, are well defined and finite numbers since $(1-r_i)^{\alpha_i+\delta_1^i}$ are positive integrable functions on $(0,1)$.

Using the above mentioned facts and definitions we have

$$\begin{aligned}
(6)\quad I_1 &\le M_p^p(f,1/2,1/2)\int_0^{1/2}\int_0^{1/2}(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2}dr_1dr_2\\
&= C_{\alpha_1}C_{\alpha_2}M_p^p(f,1/2,1/2)\int_{1/2}^1\int_{1/2}^1(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2}dr_1dr_2\\
&\le C_{\alpha_1}C_{\alpha_2}\int_{1/2}^1\int_{1/2}^1 M_p^p(f,r_1,r_2)(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2}dr_1dr_2\\
&\le 2^pC_{\alpha_1}C_{\alpha_2}\int_{1/2}^1\int_{1/2}^1 M_p^p(f,r_1,r_2)(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2}r_2^pdr_1dr_2\\
&\le 2^pC_{\alpha_1}C_{\alpha_2}\|z_2(1-|z_1|)f\|_{\mathcal{L}_\alpha^p}^p.
\end{aligned}$$

$$\begin{aligned}
(7)\quad I_2 &\le \int_{1/2}^1 M_p^p(f,1/2,r_2)(1-r_2)^{\alpha_2}dr_2\int_0^{1/2}(1-r_1)^{\alpha_1+p}dr_1\\
&= C_{\alpha_1}\int_{1/2}^1\int_{1/2}^1 M_p^p(f,1/2,r_2)(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2}dr_1dr_2\\
&\le C_{\alpha_1}\int_{1/2}^1\int_{1/2}^1 M_p^p(f,r_1,r_2)(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2}dr_1dr_2\\
&\le 2^pC_{\alpha_1}\int_{1/2}^1\int_{1/2}^1 M_p^p(f,r_1,r_2)(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2}r_2^pdr_1dr_2\\
&\le 2^pC_{\alpha_1}\|z_2(1-|z_1|)f\|_{\mathcal{L}_\alpha^p}^p.
\end{aligned}$$

Similarly

$$
\begin{aligned}
(8) \qquad I_3 &\leq 2^p C_{\alpha_2} \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, r_1, r_2)(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2} r_2^p dr_1 dr_2 \\
&\leq 2^p C_{\alpha_2} \|z_2(1-|z_1|)f\|^p_{\mathcal{L}^p_\alpha}.
\end{aligned}
$$

Finally, it is clear that

$$
\begin{aligned}
(9) \qquad I_4 &\leq 2^p \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, r_1, r_2)(1-r_1)^{\alpha_1+p}(1-r_2)^{\alpha_2} r_2^p dr_1 dr_2 \\
&\leq 2^p \|z_2(1-|z_1|)f\|^p_{\mathcal{L}^p_\alpha}.
\end{aligned}
$$

From (6)–(9) we obtain

$$
\|(1-|z_1|)f\|^p_{\mathcal{L}^p_\alpha} \leq 2^p(C_{\alpha_1}+1)(C_{\alpha_2}+1)\|z_2(1-|z_1|)f\|^p_{\mathcal{L}^p_\alpha},
$$

as desired.

§3. Proof of the main result

We are now in a position to prove the main result in this paper.

Proof of Theorem 1. Fix $f \in \mathcal{A}^p_\alpha$. Let $C^{\vec{0}}(f)(z) = F(z)$. We prove the result in the case $n = 2$. The proof for $n \geq 3$ is only technically complicated. First, we prove that $(1-|z_1|)|\partial F/\partial z_1(z_1, 0)| \in \mathcal{L}^p_\alpha$. In fact we prove the equivalent result, (here we use Lemma 2), that $z_1(1-|z_1|)|\partial F/\partial z_1(z_1, 0)| \in \mathcal{L}^p_\alpha$.

In view of formula (2) we have

$$
\begin{aligned}
\frac{\partial F}{\partial z_1}(z_1, z_2) = &-\frac{1}{z_1^2 z_2} \int_0^{z_1} \int_0^{z_2} \frac{f(\omega_1, \omega_2)}{(1-\omega_1)(1-\omega_2)} d\omega_1 d\omega_2 \\
&+ \frac{1}{z_1 z_2} \int_0^{z_2} \frac{f(z_1, \omega_2)}{(1-z_1)(1-\omega_2)} d\omega_2,
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\frac{\partial F}{\partial z_1}(z_1, z_2) = &-\frac{1}{z_1} \int_0^1 \int_0^1 \frac{f(t_1 z_1, t_2 z_2)}{(1-t_1 z_1)(1-t_2 z_2)} dt_1 dt_2 \\
&+ \frac{1}{z_1} \int_0^1 \frac{f(z_1, t_2 z_2)}{(1-z_1)(1-t_2 z_2)} dt_2.
\end{aligned}
$$

Hence

$$|z_1|\left|\frac{\partial F}{\partial z_1}(z_1,0)\right| \le \int_0^1\int_0^1 \frac{|f(t_1z_1,0)|}{|1-t_1z_1|}dt_1dt_2 + \int_0^1 \frac{|f(z_1,0)|}{|1-z_1|}dt_2,$$

which implies

$$\begin{aligned} |z_1|(1-|z_1|)&\left|\frac{\partial F}{\partial z_1}(z_1,0)\right| \\ &\le \int_0^1\int_0^1 |f(t_1z_1,0)|dt_1dt_2 + \int_0^1 |f(z_1,0)|dt_2 \\ &= \int_0^1\int_0^1 |f(t_1z_1,0)|dt_1dt_2 + \int_0^1\int_0^1 |f(z_1,0)|dt_1dt_2. \end{aligned} \tag{10}$$

Let $h(z_1,z_2) = z_1(1-|z_1|)\partial F/\partial z_1(z_1,0)$. Taking (10) to the p-th degree, integrating obtained inequality over $[0,1)^2\times[0,2\pi]^2$ with respect the measure $\frac{d\theta_1 d\theta_2}{(2\pi)^2}(1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2}dr_1dr_2$, then using Minkowski's inequality and finally using the monotonicity of the integral means $M_p(f,r_1,r_2)$ in both variables, we obtain

$$\begin{aligned} \|h\|_{\mathcal{L}^p_\alpha} \le& \int_0^1\int_0^1\bigg(\frac{1}{(2\pi)^2}\int_{[0,1)^2}\int_{[0,2\pi]^2}|f(t_1r_1e^{i\theta_1},0)|^p d\theta_1 d\theta_2 \\ &\qquad\qquad \times (1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2}\bigg)^{1/p}dt_1dt_2 \\ &+\int_0^1\int_0^1\bigg(\frac{1}{(2\pi)^2}\int_{[0,1)^2}\int_{[0,2\pi]^2}|f(r_1e^{i\theta_1},0)|^p d\theta_1 d\theta_2 \\ &\qquad\qquad \times (1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2}\bigg)^{1/p}dt_1dt_2 \\ \le& \, 2\|f\|_{\mathcal{A}^p_\alpha}. \end{aligned} \tag{11}$$

Similarly we can say that

$$\left\| z_2(1-|z_2|)\frac{\partial F}{\partial z_2}(0,z_2)\right\|_{\mathcal{L}^p_\alpha} \le 2\|f\|_{\mathcal{A}^p_\alpha}. \tag{12}$$

Now we prove that

$$\left\| z_1z_2(1-|z_1|)(1-|z_2|)\frac{\partial^2 F}{\partial z_1\partial z_2}(z_1,z_2)\right\|_{\mathcal{L}^p_\alpha} \le 4\|f\|_{\mathcal{A}^p_\alpha}. \tag{13}$$

We have

$$\begin{aligned}\frac{\partial^2 F}{\partial z_1 \partial z_2}(z_1, z_2) &= \frac{1}{z_1^2 z_2^2}\int_0^{z_1}\int_0^{z_2} \frac{f(\omega_1,\omega_2)}{(1-\omega_1)(1-\omega_2)} d\omega_1 d\omega_2 \\ &- \frac{1}{z_1^2 z_2}\int_0^{z_1} \frac{f(\omega_1, z_2)}{(1-\omega_1)(1-z_2)} d\omega_1 - \frac{1}{z_1 z_2^2}\int_0^{z_2} \frac{f(z_1,\omega_2)}{(1-z_1)(1-\omega_2)} d\omega_2 \\ &+ \frac{1}{z_1 z_2}\frac{f(z_1,z_2)}{(1-z_1)(1-z_2)},\end{aligned}$$

from which it follows that

$$\begin{aligned}&|z_1 z_2|(1-|z_1|)(1-|z_2|)\left|\frac{\partial^2 F}{\partial z_1 \partial z_2}(z_1,z_2)\right| \\ &\leq \int_0^1\int_0^1 (\,|f(t_1 z_1, t_2 z_2)| + |f(t_1 z_1, z_2)| + |f(z_1, t_2 z_2)| + |f(z_1, z_2)|\,) dt_1 dt_2.\end{aligned}$$

The rest of the proof is similar to that for the function $z_1(1-|z_1|)\partial F/\partial z_1(z_1, 0)$ and will be omitted.

Note that by (1)

$$|F(0,0)| = |f(0,0)| \leq [(\alpha_1+1)(\alpha_2+1)]^{1/p}\|f\|_{\mathcal{A}^p_\alpha}. \tag{14}$$

Using (11)–(14) and Lemma 2 we have that

$$\|F\|_* \leq C\|f\|_{\mathcal{A}^p_\alpha},$$

where C is a positive constant depending only on p and α. From this and Theorem 2 the result follows.

Remark 2. We would like to point out that in the case $n \geq 3$ the result is proved in a similar way. It can be shown that

$$\left\|\prod_{j\in S} z_j(1-|z_j|)\frac{\partial^{|S|}F}{\prod_{j\in S}\partial z_j}\right\|_{\mathcal{L}^p_\alpha} \leq 2^{|s|}\|f\|_{\mathcal{A}^p_\alpha}$$

for every $f \in \mathcal{A}^p_\alpha$ and every $S \subseteq \{1,\ldots,n\}$, $S \neq \emptyset$, since the function

$$\prod_{j\in S} z_j(1-|z_j|)\left|\frac{\partial^{|S|}F}{\prod_{j\in S}\partial z_j}\right|$$

is estimated by $2^{|S|}$ integrals which are similar to the integrals in (10).

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