F. Laytimi and W. Nahm
Nagoya Math. J.
Vol. 180 (2005), 35–43

A VANISHING THEOREM

F. LAYTIMI AND W. NAHM

Abstract. The main result is a general vanishing theorem for the Dolbeault cohomology of an ample vector bundle obtained as a tensor product of exterior powers of some vector bundles. It is also shown that the conditions for the vanishing given by this theorem are optimal for some parameter values.

§1. Introduction

The main result of the paper is a vanishing theorem, which is the strongest possible immediate generalization of the vanishing theorems of Sommese [13], Manivel [11], Le Potier [8] and Laytimi-Nahm [7].

Let E_i be vector bundles of ranks d_i, $i = 1, \dots, m$ over a compact complex manifold X of dimension n and L a line bundle on X.

Theorem 1.1. *If* $\bigotimes_{i=1}^m \bigwedge^{r_i} E_i \otimes L$ *is ample, then*

$$H^{p,q}\big(X, \textstyle\bigotimes_{i=1}^m \bigwedge^{r_i} E_i \otimes L\big) = 0 \quad \textit{for} \quad p + q - n > \displaystyle\sum_{i=1}^m r_i(d_i - r_i).$$

One might suppress the explicit mention of the line bundle L by including it among the E_i as a vector bundle of rank 1.

Sommese's result yields the vanishing with the same inequality for $p+q$ but with all E_i ample. The ampleness of E_i is not equivalent to that of $\bigwedge^{r_i} E_i$; for example the universal rank-r bundle Q on the Grassmannian $G(r, d)$ is not ample, but $\Lambda^r Q$ is. Moreover, as follows from our theorem, it suffices to assume only that the tensor product of several terms of the form $\bigwedge^{r_i} E_i$ is ample.

The result of Le Potier is a particular case of Sommese's theorem for $m = 1$.

For $p = n$ and all E_i, $i = 1, 2, \dots, m$ ample vector bundles, Ein-Lazarsfeld obtained less restrictive vanishing condition on q [4].

Received September 21, 2004.
2000 Mathematics Subject Classification: 14F17.

With our ampleness condition we conjecture that Theorem 1.1 is the best possible, as explained in Section 4.

A partition $R = (r_1, r_2, \ldots, r_m)$ is a sequence of decreasing positive integers r_i, its length is m and its weight is $|R| = \sum_{i=1}^{m} r_i$. Let $\tilde{R}$ be the transposed partition.

To each partition R corresponds the canonical irreducible $Gl(V)$-module $S_R(V)$. The functor S_R is called the Schur functor (for a precise definition see [5, p. 45]). In particular $S^k V = \mathcal{S}_{(k)} V$, and $\bigwedge^h V = \mathcal{S}_{\underbrace{(1,1,\ldots,1)}_{h \text{ times}}} V$.

We use the notation $\bigwedge_R = S_{\tilde{R}}$, where $\tilde{R}$ is the transpose of the partition R.

Schur functors were initially defined on the category of vector spaces and linear maps, but by functoriality the definition carries over to vector bundles on X.

Theorem 1.2. *Let E be a vector bundle of rank d over a compact complex manifold X of dimension n and L a line bundle on X, then for any partition $\tilde{R} = (r_1, r_2, \ldots, r_m)$, if $\bigwedge_R E \otimes L$ is ample, then*

$$H^{p,q}\big(X, \textstyle\bigwedge_R E \otimes L\big) = 0 \quad \textit{for} \quad p + q - n > \displaystyle\sum_{i=1}^{m} r_i(d - r_i).$$

This is Manivel's vanishing theorem with a weakened hypothesis; we do not assume the ampleness of E, as Manivel does. In fact Manivel's result is an immediate consequence of Sommese's.

It is crucial to have the optimal ampleness hypothesis as well as the optimal vanishing conditions in a vanishing theorem, specially for the geometrical applications like the study of degeneracy loci [6].

§2. Proof of Theorem 1.1

Let E_i be as above and $Y_i = Gr_i(E_i)$ the relative Grassmannian of subspaces of codimension r_i in the fibers of E_i. Let Q_i be the universal quotient bundles over Y_i.

Lemma 2.1. *Let L be a line bundle on X,*

$$\pi_i : Gr_1(E_1) \times_X \cdots \times_X Gr_m(E_m) \longrightarrow Gr_i(E_i),$$
$$\pi \ : Gr_1(E_1) \times_X \cdots \times_X Gr_m(E_m) \longrightarrow X$$

the natural maps. Then

$$\textstyle\bigotimes_{i=1}^{m} \Lambda^{r_i} E_i \otimes L \ \textit{ample} \implies \textstyle\bigotimes_{i=1}^{m} \pi_i^*(\det Q_i) \otimes \pi^* L \ \textit{ample}$$

Proof. Let us introduce the following notations

$$\begin{aligned}
W &= \textstyle\bigotimes_{i=1}^m \Lambda^{r_i} E_i \otimes L \\
Y &= Gr_1(E_1) \times_X \cdots \times_X Gr_m(E_m) \\
Z &= \mathbb{P}(\Lambda^{r_1} E_1) \times \cdots \times \mathbb{P}(\Lambda^{r_m} E_m) \\
T &= \mathbb{P}(\Lambda^{r_1} E_1 \otimes \cdots \otimes \Lambda^{r_m} E_m) \\
Y_i &= Gr_i(E_i) \\
Z_i &= \mathbb{P}(\Lambda^{r_i} E_i) \\
\phi &: Y \longrightarrow Z \quad \text{the Plücker embedding} \\
\gamma &: Z \longrightarrow T \quad \text{the Segre embedding.}
\end{aligned}$$

The commutative diagram

$$\begin{array}{ccccc}
Y & \xhookrightarrow{\phi} & Z & \xhookrightarrow{\gamma} & T \\
{\scriptstyle \pi_i}\downarrow & & \downarrow{\scriptstyle p_i} & & \\
Y_i & \xhookrightarrow{\phi_i} & Z_i & &
\end{array}$$

yields

$$\begin{aligned}
\mathcal{O}_T(1)|_Y &\simeq \phi^*\big(\textstyle\bigotimes_{i=1}^m p_i^*(\mathcal{O}_{Z_i}(1))\big) \\
&\simeq \textstyle\bigotimes_{i=1}^m \pi_i^*(\phi_i^* \mathcal{O}_{Z_i}(1)) \simeq \bigotimes_{i=1}^m \pi_i^*(\det Q_i).
\end{aligned}$$

Hence

$$\mathcal{O}_{\mathbb{P}(W)}(1)|_Y \simeq \mathcal{O}_T(1)|_Y \otimes \pi^* L \simeq \textstyle\bigotimes_{i=1}^m \pi_i^*(\det Q_i) \otimes \pi^* L,$$

and we are done. $\square$

For the sequel we need to recall the Borel-Le Potier spectral sequence in our context.

The projection $\pi : Y \to X$ yields a filtration of the bundle Ω_Y^P of exterior differential forms of degree P on Y, namely

$$F^p(\Omega_Y^P) = \pi^* \Omega_X^p \wedge \Omega_Y^{P-p}.$$

The corresponding graded bundle is given by

$$F^p(\Omega_Y^P)/F^{p+1}(\Omega_Y^P) = \pi^* \Omega_X^p \otimes \Omega_{Y/X}^{P-p},$$

$\Omega^{P-p}_{Y/X}$ is the bundle of relative differential forms of degree $P-p$. For a given line bundle $\mathcal{L}$ over Y, the filtration on Ω^P_Y induces a filtration on $\Omega^P_Y \otimes \mathcal{L}$. This latter filtration yields the Borel-Le Potier spectral sequence, which abuts to $H^{P,q}(Y, \mathcal{L})$. It is given by the data X, Y, $\mathcal{L}$, P and will be denoted by ${}^P\mathcal{E}_B$. Its $\mathcal{E}_1$-terms

$$^P\mathcal{E}^{p,q-p}_{1,B} = H^q\big(Y, \pi^*(\Omega^p_X) \otimes \Omega^{P-p}_{Y/X} \otimes \mathcal{L}\big)$$

can be calculated as limit groups of the Leray spectral sequence ${}^{p,P}\mathcal{E}_L$ associated to the projection π, for which

$$^{p,P}\mathcal{E}^{q-j,j}_{2,L} = H^{p,q-j}\big(X, R^j\pi_*(\Omega^{P-p}_{Y/X} \otimes \mathcal{L})\big).$$

Now we start to prove the main theorem.

Denote the fibers of E_i at a point $x \in X$ by V_i, and consider the line bundle on Y $\mathcal{L} = \bigotimes_{i=1}^m \pi_i^*(\det Q_i) \otimes \pi^* L$. We have

$$\begin{aligned} &R^j\pi_*\big(\Omega^{P-p}_{Y/X} \otimes \mathcal{L}\big) \\ &\quad \simeq H^j\big(Gr_1(V_1) \times \cdots \times Gr_m(V_m), \Omega^{P-p} \textstyle\bigotimes_{i=1}^m \det Q_i\big) \otimes L. \end{aligned}$$

Using

$$H^{p,q}(Gr(V), \det Q) = 0 \quad \text{if} \quad (p,q) \neq (0,0),$$

and the Künneth formula, we get the degeneracy at the first step of both Leray and Borel spectral sequences. Now

$$H^0(Gr_i(V_i), \det Q_i) \simeq \textstyle\bigwedge^{r_i} V_i,$$

thus

$$H^{p,q}(Y, \mathcal{L}) \simeq H^{p,q}\big(X, \textstyle\bigotimes_{i=1}^m \bigwedge^{r_i} E_i \otimes L\big).$$

The result follows from Lemma 2.1 and Kodaira-Akizuki-Nakano vanishing theorem [1]. $\square$

§3. Proof of the Theorem 1.2

Let's state this particular case of Theorem 1.1 as a

COROLLARY 3.1. *Let E be vector bundles of ranks d over a compact complex manifold X of dimension n and L a line bundle on X, then if $\bigotimes_{i=1}^m \bigwedge^{r_i} E \otimes L$ is ample, then*

$$H^{p,q}\big(X, \textstyle\bigotimes_{i=1}^m \bigwedge^{r_i} E \otimes L\big) = 0 \quad \textit{for} \quad p+q-n > \displaystyle\sum_{i=1}^m r_i(d-r_i).$$

Actually we prove below

$$(*) \qquad \text{Corollary 3.1} \iff \text{Theorem 1.2}$$

Recall the definition of the dominance partial order for arbitrary partitions [7].

Definition 3.2. Let $I = (i_1, i_2, \dots)$, $J = (j_1, j_2, \dots)$ be any partitions of arbitrary weights. We define the dominance relation

$$I \preceq J \quad \text{if for all } l \ \ \frac{i_1 + i_2 + \cdots + i_l}{|I|} \le \frac{j_1 + j_2 + \cdots + i_l}{|J|}.$$

We write

$$I \sim J \quad \text{if} \quad I \succeq J \text{ and } I \preceq J.$$

Modulo this equivalence, this dominance relation is a partial order.

We recall from [9] that

$$\text{If} \quad |I| = |J|, \quad \text{then} \quad (I \succeq J \iff \tilde{I} \preceq \tilde{J}).$$

We have [7]:

Theorem 3.3. *Let I, J be any partitions and E a vector bundle.*

If $I \succeq J$, then $\mathcal{S}_I E$ ample (resp. nef) $\Longrightarrow$ $\mathcal{S}_J E$ ample (resp. nef).

In particular, if $I \sim J$, then

$$\mathcal{S}_I E \textit{ ample (resp. nef)} \iff \mathcal{S}_J E \textit{ ample (resp. nef)}.$$

This theorem gives in particular

$$\mathcal{S}^k E \text{ ample (resp. nef)} \iff E \text{ ample (resp. nef)},$$

and for any $k \geq 0$

$$\bigwedge^m E \text{ ample (resp. nef)} \Longrightarrow \bigwedge^{m+k} E \text{ ample (resp. nef)}.$$

Lemma 3.4. *Let $\tilde{R} = (r_1, \dots, r_m)$ be a partition, then*

$$\bigotimes_{i=1}^m \bigwedge^{r_i} E \textit{ ample (resp. nef)} \iff \bigwedge_R E \textit{ ample (resp. nef)}.$$

Proof. For the direct implication, the vector bundle $\bigwedge_R E$ is a direct summand of $\bigotimes_{i=1}^m \bigwedge^{r_i} E$, by the Littlewood-Richardson rules. For the opposite direction, all direct summands $S_\lambda E$ appearing in $\bigotimes_{i=1}^m \bigwedge^{r_i} E$ satisfy by also Littlewood-Richardson rules $\lambda \preceq \tilde{R}$. Hence the result follows from Theorem 3.3. ∎

Lemma 3.4 and the fact that the vector bundle $\bigwedge_R E$ is a direct summand of $\bigotimes_{i=1}^m \bigwedge^{r_i} E$, yields Theorem 1.2.

Conversely it is easy to see that Theorem 1.2 implies the special case of Theorem 1.1 for which $E_i = E$, $i = 1, \dots, m$. Indeed for any $\bigwedge_\mu E$ which is a direct summand of $\bigotimes_{i=1}^m \bigwedge^{r_i} E$, we have

$$\mu \succeq R, \tag{$**$}$$

where the partitions $R = (r_1, r_2, \dots, r_m)$, $\mu = (\mu_1, \mu_2, \dots, \mu_l)$, $l \le m$ have the same weight.

To get the wanted vanishing condition, we need to show that

$$\sum_{i=1}^m r_i(d - r_i) \ge \sum_{i=1}^m \mu_i(d - \mu_i), \quad \text{with } \mu_i = 0 \text{ if } i > l$$

or equivalently

$$\sum_{i=1}^m \mu_i^2 \ge \sum_{i=1}^m r_i^2.$$

When we write

$$\mu_i = r_i + (\alpha_i - \alpha_{i-1})$$

with $\alpha_0 = 0$, the inequality $(**)$ implies that the α_i are non-negative. With

$$\sum_{i=1}^m \mu_i^2 = \sum_{i=1}^m r_i^2 + \sum_{i=1}^m \bigl(2\alpha_i(r_i - r_{i+1}) + (\alpha_i - \alpha_{i-1})^2\bigr)$$

we are done. ∎

Remark 3.5. This gives a very short proof of our result in [7]. It still uses Theorem 3.2 proved in [7].

The following examples show that the numerical condition obtained in Theorem 1.1 is optimal for certain triples (p, q, n).

§4. Example

Let V be a complex vector space, $G_d(V)$ is the Grassmannian of codimension-d subspaces of V, Q and S the universal quotient bundle and the universal subbundle on $G_d(V)$.

For the sequel we need to recall Proposition 3 in [10] which is a generalization of a method developed in [12].

Proposition 4.1. *Let r, d, l be integers such that $0 \le r < d$, and*

$$P(d,l,r) = (l-1)\frac{d(d+1)}{2} - dl + l + rd - \binom{r+1}{2}.$$

Then if $p \ge P(d,l,r)$,

$$H^{p,q}\big(G_d(V), \textstyle\bigwedge^r Q \otimes (\det Q)^l\big) = \bigoplus_{\alpha=0}^{l-1} \bigoplus_{\varepsilon \in \{0,1\}^d_r} \delta_{p,p(d,l,\varepsilon,\alpha)} \delta_{q,q(d,l,\varepsilon,\alpha)} \mathcal{S}_\beta V.$$

Here $\{0,1\}^d_r$ is the set of sequences $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$, $\varepsilon_i \in \{0,1\}$ with $\sum_{i=1}^d \varepsilon_i = r$,

$$p(d,l,\varepsilon,\alpha) = \sum_{i=1}^{d} (d+1-i)\varepsilon_i - d\alpha + (l-1)\frac{d(d+1)}{2},$$
$$q(d,l,\varepsilon,\alpha) = p(d,l,\varepsilon,\alpha) - d(l-1) - r + \alpha, \quad \text{and}$$
$$\beta = (\alpha+1, 1, \dots, 1), \quad |\beta| = dl + r.$$

Note that here we have removed the first condition of the Proposition 3 in [10] which is not necessary.

Let $a = (\underbrace{1,\dots,1}_{r \text{ times}}, \underbrace{0,\dots,0}_{d-r \text{ times}})$. We have $p(d,l,\varepsilon,\alpha) \le p(d,l,a,0)$, with equality only for $\varepsilon = a$, $\alpha = 0$. The latter case gives the following

Example 4.2. Let V be a vector space of dimension $r+d$, with $r \le d$. If $p = dr - \binom{r}{2}$, and $q = p - r$, then

$$p + q - \dim G_d(V) = r(d-r)$$

and

$$H^{p,q}\big(G_d(V), \textstyle\bigwedge^r Q \otimes \det Q\big) = \det V.$$

Moreover $\det Q \otimes \bigwedge^r Q$ is ample.

A generalization of this example to products of Grassmannians by the use of the Künneth formula gives

Example 4.3. Let $G = \times_{i=1}^m G_{d_i}(V_i)$, where V_i are vector spaces of dimension $r_i + d_i$. Let $p_0 = \sum_{i=1}^m p_i$, $q_0 = \sum_{i=1}^m q_i$, where $p_i = r_i d_i - \binom{r_i}{2}$ and $q_i = r_i d_i - \binom{r_i+1}{2}$. Then

$$H^{p_0,q_0}\big(G, \bigotimes_{i=1}^m (\bigwedge^{r_i} Q_i \otimes \det Q_i)\big) = \bigotimes_{i=1}^m \det V_i$$

and $p_0 + q_0 - \dim G = \sum_{i=1}^m r_i(d_i - r_i)$.

In the formula we have suppressed the symbol for the obvious pull back of the universal quotient bundles Q_i to the product G.

It is well known that the Kodaira-Akizuki-Nakano vanishing theorem is optimal. In particular, for any triple p', q', n' with $p' + q' - n' \leq 0$ one can find a Cartesian product of n' curves X and an ample line bundle L on X such that $H^{p',q'}(X, L) \neq 0$. Taking the Cartesian product of X with the Grassmannian product G of the previous example, one finds examples with

$$H^{p,q}\big(X \times G, L \bigotimes_{i=1}^m (\bigwedge^{r_i} Q_i \otimes \det Q_i)\big) \neq 0$$

for any triple (p, q, n) such that $p \geq q_0$, $q \geq q_0$ and $p + q - n \leq \sum_{i=1}^m r_i(d_i - r_i)$, with p_0, q_0 as above and $n = \dim X \times G$.

Finally one has for $\dim V = d + 1$,

$$H^{d,d-1}(G_d V, \Lambda^{d-1} Q^* \otimes (\det Q)^2) = \det V,$$

since $\Lambda^{d-1} Q^* \otimes \det Q = Q$. Note that this example is legitimate, since we only demand that $\Lambda^{d-1} Q^* \otimes (\det Q)^2$ is ample, without any assumption on Q^*.

These examples are sufficiently diverse such that we conjecture that Theorem 1.1 is optimal in the full non-trivial parameter range of the triples (p, q, n).

References

[1] Y. Akizuki and S. Nakano, *Note on Kodaira-Spencer's proof of Lefschetz theorems*, Proc. Jap. Acad., **30** (1954), 266–272.

[2] R. Bott, *Homogeneous vector bundles*, Ann. Math., **66** (1957), 203–248.

[3] B. Demazure, *A very simple proof of Bott's theorem*, Invent. Math., **33** (1976), 271–220.

[4] L. Ein and R. Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math., **111** (1993), 51–67.
[5] W. Fulton and J. Harris, Representation theory, a first course, Graduate texts in Mathematics, Springer Verlag, 1991.
[6] F. Laytimi, *On degeneracy loci*, International Journal of Mathematics, **6** vol. 7 (1998), 203–220.
[7] F. Laytimi and W. Nahm, *A generalization of Le Potier's vanishing theorem*, Manuscripta math., **113** (2004), 165–189.
[8] Le Potier, *Cohomologie de la Grassmannienne à valeurs dans les puissances extérieures et symetriques du fibré universel*, Math. Ann., **226** (1977), 257–270.
[9] I. G. Macdonald, Symmetrics Functions and Hall polynomials, Claredon Press, Oxford, 1976.
[10] L. Manivel, *Un théorème d'annulation pour les puissances extérieures d'un fibré ample*, J. reine angew. Math., **422** (1991), 91–116.
[11] L. Manivel, *théorèmes d'annulation pour les fibrés associés à un fibré ample*, Scuola superiore Pisa (1992), 515–565.
[12] D. Snow, *Cohomology of twisted holomorphic forms on Grassmann manifolds and quadric hypersurfaces*, Math. Ann., **276** (1986), 159–176.
[13] A. J. Sommese, *Submanifold of Abelian Varieties*, Math. Ann., **233** (1978), 229–256.

F. Laytimi
Mathématiques - bât. M2
Université Lille 1
F-59655 Villeneuve d'Ascq Cedex
France
laytimi@agat.univ-lille1.fr

W. Nahm
Dublin Institute for Advanced Studies
10 Burlington Road
Dublin 4
Ireland
wnahm@stp.dias.ie