Z. Hu and H. Li
Nagoya Math. J.
Vol. 179 (2005), 147–162

CLASSIFICATION OF MÖBIUS ISOPARAMETRIC HYPERSURFACES IN $\mathbb{S}^4$

ZEJUN HU* AND HAIZHONG LI†

Abstract. Let M^n be an immersed umbilic-free hypersurface in the $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$, then M^n is associated with a so-called Möbius metric g, a Möbius second fundamental form $\mathbf{B}$ and a Möbius form Φ which are invariants of M^n under the Möbius transformation group of $\mathbb{S}^{n+1}$. A classical theorem of Möbius geometry states that M^n $(n \geq 3)$ is in fact characterized by g and $\mathbf{B}$ up to Möbius equivalence. A Möbius isoparametric hypersurface is defined by satisfying two conditions: (1) $\Phi \equiv 0$; (2) All the eigenvalues of $\mathbf{B}$ with respect to g are constants. Note that Euclidean isoparametric hypersurfaces are automatically Möbius isoparametric, whereas the latter are Dupin hypersurfaces.

In this paper, we prove that a Möbius isoparametric hypersurface in $\mathbb{S}^4$ is either of parallel Möbius second fundamental form or Möbius equivalent to a tube of constant radius over a standard Veronese embedding of $\mathbb{R}P^2$ into $\mathbb{S}^4$. The classification of hypersurfaces in $\mathbb{S}^{n+1}$ $(n \geq 2)$ with parallel Möbius second fundamental form has been accomplished in our previous paper [6]. The present result is a counterpart of Pinkall's classification for Dupin hypersurfaces in $\mathbb{E}^4$ up to Lie equivalence.

§1. Introduction

Let $x : M^n \to \mathbb{S}^{n+1}$ be a hypersurface in the $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$ without umbilic point and $\{e_i\}$ be a local orthonormal basis with respect to the induced metric $I = dx \cdot dx$ with dual basis $\{\theta_i\}$. Let $II = \sum_{i,j} h_{ij}\theta_i \otimes \theta_j$ be the second fundamental form with length square $\|II\|^2 = \sum_{i,j}(h_{ij})^2$ and $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature of x, respectively. Define $\rho^2 = n/(n-1) \cdot (\|II\|^2 - nH^2)$, then the positive definite form $g = \rho^2 dx \cdot dx$ is a Möbius invariant and is called the Möbius metric of $x : M^n \to \mathbb{S}^{n+1}$. The Möbius second fundamental form $\mathbf{B}$, another basic Möbius invariant of x, together with g determine completely a hypersurface of $\mathbb{S}^{n+1}$ up to Möbius equivalence, see Theorem 2.2 below.

Received May 8, 2003.
2000 Mathematics Subject Classification: Primary 53A30; Secondary 53B25.
*Partially supported by grants of DAAD and SRF for ROCS, SEM.
†Partially supported by the Zhongdian grant No. 10131020 of NSFC.

An important class of hypersurfaces for Möbius differential geometry is the so-called Möbius isoparametric hypersurfaces in $\mathbb{S}^{n+1}$. They are first defined in [8] as hypersurfaces of $\mathbb{S}^{n+1}$ such that the Möbius invariant 1-form

$$\Phi = -\rho^{-1}\sum_i\Big\{e_i(H)+\sum_j(h_{ij}-H\delta_{ij})e_j(\log\rho)\Big\}\theta_i \tag{1.1}$$

vanishes and all eigenvalues w.r.t. g of the Möbius shape operator

$$\mathcal{S} := \rho^{-1}(\mathbf{S}-H\cdot id) \tag{1.2}$$

are constant, here $\mathbf{S}$ denotes the shape operator of $x : M^n \to \mathbb{S}^{n+1}$. This definition of Möbius isoparametric hypersurfaces is meaningful when we compared it with that of (Euclidean) isoparametric hypersurfaces in $\mathbb{S}^{n+1}$, as we see that the images of all hypersurfaces of the sphere with constant mean curvature and constant scalar curvature under Möbius transformation satisfy $\Phi\equiv 0$ and that the *Möbius invariant operator* $\mathcal{S}$ play the same role in Möbius geometry as $\mathbf{S}$ does in the Euclidean situation (see Theorem 2.2 below). Standard examples of Möbius isoparametric hypersurfaces are the images of (Euclidean) isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ under Möbius transformations. But there are other examples which cannot be obtained in this way, e.g., it occurs among our classification for hypersurfaces of $\mathbb{S}^{n+1}$ with parallel Möbius second fundamental form, this means that the Möbius second fundamental form is parallel with respect to the Levi-Civita connection of the Möbius metric g, see [6] and [8] for details. On the other hand, it was proved in [8] that any Möbius isoparametric hypersurface is in particular a Dupin hypersurface, which implies from [14] that for a compact Möbius isoparametric hypersurface embedded in $\mathbb{S}^{n+1}$, the number γ of distinct principal curvatures can only take the values $\gamma = 2,3,4,6$. In [8], the authors classified locally all Möbius isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ with $\gamma = 2$.

In this paper, by relaxing the restriction of $\gamma = 2$, we will consider Möbius isoparametric hypersurfaces of $\mathbb{S}^4$ and finally determine all of them in explicit form. To state the result, we first recall that for the n-dimensional hyperbolic space of constant sectional curvature $-c<0$

$$\mathbb{H}^n(-c)=\big\{(y_0,y_1,\dots,y_n)\in L^{n+1}\;\big|\;-y_0^2+y_1^2+\cdots+y_n^2=-c^2,\ y_0\ge c\big\}$$

and the hemisphere $\mathbb{S}^n_+ = \big\{(x_1,\dots,x_{n+1})\in\mathbb{S}^n \;\big|\; \sum_i x_i^2 = 1,\ x_1>0\big\}$, we can define, respectively, the conformal diffeomorphism $\sigma:\mathbb{R}^n\to\mathbb{S}^n\setminus$

$\{(-1,0,\dots,0)\}$ and $\tau : \mathbb{H}^n(-1) \to \mathbb{S}^n_+$ by

$$\sigma(u) = \left(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\right), \quad u \in \mathbb{R}^n, \tag{1.3}$$

$$\tau(y) = \left(\frac{1}{y_0}, \frac{y'}{y_0}\right), \; y_0 \geq 1, \; y = (y_0, y_1, \dots, y_n) := (y_0, y') \in \mathbb{H}^n(-1). \tag{1.4}$$

Then we can state our main result as the following

Classification Theorem. *Let $x : M^3 \to \mathbb{S}^4$ be a Möbius isoparametric hypersurface. Then x is Möbius equivalent to an open part of one of the following hypersurfaces in $\mathbb{S}^4$:*

(1) *the torus $\mathbb{S}^k(a) \times \mathbb{S}^{3-k}(b)$ with $k = 1, 2$ and $a^2 + b^2 = 1$;*
(2) *the image of σ of the standard cylinder $\mathbb{S}^k(1) \times \mathbb{R}^{3-k} \subset \mathbb{R}^4$ with $k = 1, 2$;*
(3) *the image of τ of the standard hyperbolic cylinder*

$$\mathbb{S}^k(r) \times \mathbb{H}^{3-k}(-\sqrt{1+r^2}) \subset \mathbb{H}^4(-1)$$

with $k = 1, 2$ and $r > 0$;

(4) *the image of σ of the warped product embedding*

$$\tilde{x} : \mathbb{S}^1(a) \times \mathbb{S}^1(b) \times \mathbb{R}^+ \longrightarrow \mathbb{R}^4 \quad \textit{with} \quad a^2 + b^2 = 1$$

defined by

$$\tilde{x}(u, v, t) = (tu, tv), \quad u \in \mathbb{S}^1(a), \; v \in \mathbb{S}^1(b), \; t > 0;$$

(5) *a tube of constant radius over a standard Veronese embedding of $\mathbb{R}P^2$ into $\mathbb{S}^4$.*

Remark 1.1. From the fact that the Lie sphere transformation group contains the Möbius transformation group in $\mathbb{S}^{n+1}$ as a subgroup and the dimension difference is $n + 3$, we see that the Möbius differential geometry should be quite different from the Lie sphere geometry in many respects. Therefore it is remarkable to find that the classification theorem above can be regarded as a counterpart of Pinkall's classification theorem [13] for Dupin hypersurfaces in $\mathbb{E}^4$ under equivalence of Lie sphere transformation.

Remark 1.2. The classification in [8], done under the condition $\gamma = 2$ for all dimensions, covers for $n = 3$ the first three cases of the above classification theorem. We note that (1) and (5) are Euclidean isoparametric hypersurfaces in $\mathbb{S}^4$, whereas (2) and (3) are euclidean isoparametric hypersurfaces in $\mathbb{R}^4$ and $\mathbb{H}^4(-1)$, respectively.

Remark 1.3. Recall that a hypersurface M of $\mathbb{S}^{n+1}$ is said to be Möbius homogeneous if for any two points $p, q \in M$ there exists a Möbius transformation T of $\mathbb{S}^{n+1}$ such that $T(M) = M$ and $T(p) = q$. We note that in [16], by a very different method, it was proved that up to Möbius equivalence (4) and (5) are the only Möbius homogeneous hypersurfaces of $\mathbb{S}^4$ with different principal curvatures.

Remark 1.4. Comparing the above Classification Theorem with Theorem 2 of [16], we easily see that a umbilic free hypersurface of $\mathbb{S}^4$ being Möbius isoparametric is equivalent to that it is Möbius homogeneous. For higher dimensions, the following problem might be interesting: *Try to elaborate on the relation between "Möbius isoparametric hypersurface" and "Möbius homogeneous hypersurface"?* This will be studied in a separate paper.

This paper consists of three sections. In Section 2, we first review some elementary facts of Möbius geometry for hypersurfaces in $\mathbb{S}^{n+1}$, and then we present (without proof) a classification result for hypersurfaces of $\mathbb{S}^{n+1}$ with parallel Möbius second fundamental form which we accomplished in [6]. Section 3 is the key part of this paper, which is devoted to the proof of our classification theorem.

Acknowledgements. This work was done when both authors were having research stay at TU Berlin in 2002. We would like to express our gratitude to Professor U. Simon for his hospitality and many helps. We also express our thanks to Prof. C. P. Wang for helpful discussions with him about this paper. Finally, we are deeply indebted to the referee whose valuable comments improved our original version greatly.

§2. Möbius invariants for hypersurfaces in $\mathbb{S}^{n+1}$

In this section we define Möbius invariants and recall structure equations for hypersurfaces in $\mathbb{S}^{n+1}$. For more detail we refer to [15].

Let $\mathbb{L}^{n+3}$ be the Lorentz space, i.e., $\mathbb{R}^{n+3}$ with inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, w\rangle = -x_0w_0 + x_1w_1 + \cdots + x_{n+2}w_{n+2} \tag{2.1}$$

for $x = (x_0, x_1, \ldots, x_{n+2})$, $w = (w_0, w_1, \ldots, w_{n+2}) \in \mathbb{R}^{n+3}$.

Let $x : M^n \to \mathbb{S}^{n+1} \to \mathbb{R}^{n+2}$ be an immersed hypersurface of $\mathbb{S}^{n+1}$ without umbilics. We define the Möbius position vector $Y : M^n \to \mathbb{L}^{n+3}$ of x by

$$Y = \rho(1, x), \quad \rho^2 = \frac{n}{n-1}\big(\|II\|^2 - nH^2\big) > 0. \tag{2.2}$$

Then we have the following

THEOREM 2.1. ([15]) *Two hypersurfaces $x, \tilde{x} : M^n \to \mathbb{S}^{n+1}$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(n+2,1)$ in $\mathbb{L}^{n+3}$ such that $\tilde{Y} = YT$.*

It follows immediately from Theorem 2.1 that

$$g = \langle dY, dY\rangle = \rho^2 dx\cdot dx$$

is a Möbius invariant (cf. [15]).

Let Δ be the Laplacian with respect to g, we define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle\Delta Y, \Delta Y\rangle Y, \tag{2.3}$$

then we have (cf. [15])

$$\langle\Delta Y, Y\rangle = -n, \quad \langle\Delta Y, dY\rangle = 0, \quad \langle\Delta Y, \Delta Y\rangle = 1 + n^2\kappa, \tag{2.4}$$

$$\langle Y, Y\rangle = 0, \quad \langle N, Y\rangle = 1, \quad \langle N, N\rangle = 0, \tag{2.5}$$

where κ is the normalized scalar curvature of g and is called the normalized Möbius scalar curvature of x.

Let $\{E_1, \ldots, E_n\}$ be a local orthonormal basis for (M^n, g) with dual basis $\{\omega_1, \ldots, \omega_n\}$ and write $Y_i = E_i(Y)$, then we have from (2.2), (2.4) and (2.5) that

$$\langle Y_i, Y\rangle = \langle Y_i, N\rangle = 0, \quad \langle Y_i, Y_j\rangle = \delta_{ij}, \quad 1 \le i, j \le n. \tag{2.6}$$

Let V be the orthogonal complement to the subspace $Span\{Y, N, Y_1, \ldots, Y_n\}$ in $\mathbb{L}^{n+3}$, then along M we have the following orthogonal decomposition:

$$\mathbb{L}^{n+3} = Span\{Y, N\} \oplus Span\{Y_1, \ldots, Y_n\} \oplus V. \tag{2.7}$$

V is called the Möbius normal bundle of x. A local unit vector basis $E = E_{n+1}$ for V can be written as (cf. [15])

$$E = E_{n+1} := (H, Hx + e_{n+1}). \tag{2.8}$$

Then $\{Y, N, Y_1, \ldots, Y_n, E\}$ forms a moving frame in $\mathbb{L}^{n+3}$ along M^n. Unless otherwise stated, we will use the following range of indices throughout this paper: $1 \leq i, j, k, l, m \leq n$.

We can write the structure equations as follows:

$$dY = \sum_i Y_i \omega_i, \tag{2.9}$$

$$dN = \sum_{i,j} A_{ij}\omega_j Y_i + \sum_i C_i \omega_i E, \tag{2.10}$$

$$dY_i = -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_i B_{ij}\omega_j E, \tag{2.11}$$

$$dE = -\sum_i C_i \omega_i Y - \sum_{i,j} B_{ij}\omega_j Y_i, \tag{2.12}$$

where ω_{ij} is the connection form of the Möbius metric g, $\mathbf{A} = \sum_{i,j} A_{ij}\omega_i \otimes \omega_j$, $\Phi = \sum_i C_i\omega_i$ and $\mathbf{B} = \sum_{i,j} B_{ij}\omega_i \otimes \omega_j$ are called the Blaschke tensor, the Möbius form and the Möbius second fundamental form of x, respectively. The relations between Φ, $\mathbf{B}$, $\mathbf{A}$ and the Euclidean invariants of x are given by (cf. [15])

$$C_i = -\rho^{-2}\Big[e_i(H) + \sum_j (h_{ij} - H\delta_{ij}) e_j(\log\rho)\Big], \tag{2.13}$$

$$B_{ij} = \rho^{-1}(h_{ij} - H\delta_{ij}), \tag{2.14}$$

$$\begin{aligned} A_{ij} = {} & -\rho^{-2}\big[\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}\big] \\ & - \frac{1}{2}\rho^{-2}\big(\|\nabla \log\rho\|^2 - 1 + H^2\big)\delta_{ij}, \end{aligned} \tag{2.15}$$

where Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to $dx \cdot dx$.

The covariant derivative of C_i, A_{ij}, B_{ij} are defined by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2.16}$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2.17}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \tag{2.18}$$

The integrability conditions for the structure equations (2.9)–(2.12) are given by (cf. [15])

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k, \tag{2.19}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{kj}A_{ki}), \tag{2.20}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.21}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{2.22}$$

$$R_{ij} := \sum_k R_{ikjk} = -\sum_k B_{ik}B_{jk} + (\operatorname{tr}\mathbf{A})\delta_{ij} + (n-2)A_{ij}, \tag{2.23}$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j} (B_{ij})^2 = \frac{n-1}{n}, \quad \operatorname{tr}\mathbf{A} = \sum_i A_{ii} = \frac{1}{2n}(1 + n^2\kappa), \tag{2.24}$$

where R_{ijkl} denote the curvature tensor of g and $\kappa = \frac{1}{n(n-1)}\sum_{i,j} R_{ijij}$ is the normalized Möbius scalar curvature of $x : M^n \to \mathbb{S}^{n+1}$.

The second covariant derivative of B_{ij} are defined by

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k} + \sum_l B_{lj,k}\omega_{li} + \sum_l B_{il,k}\omega_{lj} + \sum_l B_{ij,l}\omega_{lk}. \tag{2.25}$$

By exterior differentiation of (2.18), we have the following Ricci identities

$$B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj}R_{mikl} + \sum_m B_{im}R_{mjkl}. \tag{2.26}$$

We get from (2.14) that

$$\mathcal{S} := \rho^{-1}(\mathbf{S} - H \cdot id) = \sum_{i,j} B_{ij}\omega_i E_j, \tag{2.27}$$

where $\mathbf{S}$ is the Weingarten operator for $x : M^n \to \mathbb{S}^{n+1}$, $\mathcal{S}$ is called the Möbius shape operator of x. We can show that all coefficients in (2.9)–(2.12) are determined by $\{g, \mathcal{S}\}$ and thus we obtain

Theorem 2.2. ([15], or [1], [2]) *Two hypersurfaces $x : M^n \to \mathbb{S}^{n+1}$ and $\tilde{x} : \tilde{M}^n \to \mathbb{S}^{n+1}$ $(n \geq 3)$ are Möbius equivalent if and only if there exists a diffeomorphism $\sigma : M^n \to \tilde{M}^n$ which preserves the Möbius metric and the Möbius shape operator.*

Recall that a umbilic free hypersurface $x : M^n \to \mathbb{S}^{n+1}$ is said to have parallel Möbius second fundamental form if $B_{ij,k} \equiv 0$ for all i, j, k. In [6], we obtained a complete classification for hypersurfaces of $\mathbb{S}^{n+1}$ with parallel Möbius second fundamental form. This classification will be used in this paper, so we state it as followings:

Theorem 2.3. ([6]) *Let $x : M^n \to \mathbb{S}^{n+1}$ $(n \geq 2)$ be an immersed umbilic free hypersurface with parallel Möbius second fundamental form. Then x is Möbius equivalent to an open part of one of the following hypersurfaces in $\mathbb{S}^{n+1}$:*

(1) *the torus $\mathbb{S}^k(a) \times \mathbb{S}^{n-k}(b)$ with $1 \leq k \leq n-1$ and $a^2 + b^2 = 1$;*

(2) *the image of σ of the standard cylinder $\mathbb{S}^k(1) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ with $1 \leq k \leq n-1$;*

(3) *the image of τ of the standard hyperbolic cylinder $\mathbb{S}^k(r) \times \mathbb{H}^{n-k}(-c) \subset \mathbb{H}^{n+1}(-1)$ with $1 \leq k \leq n-1$ and $c = \sqrt{1+r^2}$ for $r > 0$;*

(4) *the image of σ of the warped product embedding*

$$\tilde{x} : \mathbb{S}^p(a) \times \mathbb{S}^q(b) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \longrightarrow \mathbb{R}^{n+1}$$

with $a^2 + b^2 = 1$, $p \geq 1$, $q \geq 1$, $p + q \leq n-1$, defined by

$$\tilde{x}(u', u'', t, u''') = (tu', tu'', u'''),$$
$$u' \in \mathbb{S}^p(a),\ u'' \in \mathbb{S}^q(b),\ t \in \mathbb{R}^+,\ u''' \in \mathbb{R}^{n-p-q-1}.$$

§3. Proof of the classification theorem

In this section, we first prove that for a Möbius isoparametric hypersurface of $\mathbb{S}^4$, if its Möbius second fundamental form is not parallel, then it must be Möbius equivalent to an (Euclidean) isoparametric hypersurface of

$\mathbb{S}^4$ with three distinct principal curvatures. According to Cartan's result in [3], the latter is in fact a tube of constant radius over a standard Veronese embedding of $\mathbb{R}P^2$ into $\mathbb{S}^4$. Then our classification theorem follows from this and Theorem 2.3.

Hereafter we assume that $n = 3$ and $x : M^3 \to \mathbb{S}^4$ is a Möbius isoparametric hypersurface, i.e. we have $\Phi = 0$ and the Möbius shape operator $\mathcal{S}$ has constant eigenvalues. For our choice of the frame $\{E_i\}$, $\mathcal{S}$ has constant eigenvalues being equivalent to that the matrix (B_{ij}) has constant eigenvalues. From $\Phi = 0$ and (2.20), we get

$$\sum_k \big(B_{ik}A_{kj} - A_{ik}B_{kj}\big) = 0, \quad \text{for all } i,\, j, \tag{3.1}$$

this implies that we can choose the local orthonormal basis $\{E_i\}$ to diagonalize (A_{ij}) and (B_{ij}) simultaneously. Let us write

$$(B_{ij}) = \operatorname{diag}(b_1, b_2, b_3), \quad (A_{ij}) = \operatorname{diag}(a_1, a_2, a_3), \tag{3.2}$$

where $\{b_i\}$ are all constants and from (2.24) they satisfy

$$b_1 + b_2 + b_3 = 0, \quad b_1^2 + b_2^2 + b_3^2 = \frac{2}{3}. \tag{3.3}$$

We also note that from $\Phi = 0$, (2.19) and (2.21), both $B_{ij,k}$ and $A_{ij,k}$ are totally symmetric tensors. As usual we define

$$\omega_{ij} = \sum_k \Gamma^i_{kj}\omega_k, \quad \Gamma^i_{kj} = -\Gamma^j_{ki}. \tag{3.4}$$

From (2.18), (3.2) and $\{b_i\}$ being constants, we get

$$B_{ij,k} = (b_i - b_j)\Gamma^i_{kj}, \quad \text{for all } i,\, j,\, k, \tag{3.5}$$

and thus

$$B_{ij,i} = B_{ii,j} = 0, \quad \text{for all } i,\, j. \tag{3.6}$$

If locally $B_{12,3} \equiv 0$, combining it with (3.6) we have $B_{ij,k} \equiv 0$, for any i, j, k and thus $x : M^3 \to \mathbb{S}^4$ has parallel Möbius second fundamental form. Then from Theorem 2.3 our classification theorem is valid.

In the sequel, we assume that $B_{12,3} \neq 0$. From (3.5), we see that b_1, b_2 and b_3 are distinct, and further from $0 = B_{ii,j} = B_{ij,i} = (b_i - b_j)\Gamma^i_{ij}$ we get

$$\Gamma^i_{ij} = 0, \quad \text{for all distinct } i,\, j. \tag{3.7}$$

LEMMA 3.1. $B_{12,3} = const.$

Proof. From (2.25) and (3.6), we have

$$\sum_k B_{12,3k}\omega_k = dB_{12,3}+\sum_k \big(B_{k2,3}\omega_{k1}+B_{1k,3}\omega_{k2}+B_{12,k}\omega_{k3}\big) = dB_{12,3}. \tag{3.8}$$

To calculate $B_{12,3k}$ $(k = 1, 2, 3)$, we first use the Ricci identity (2.26) and (3.2) to obtain

$$B_{ij,kl} = B_{ij,lk} + \sum_m \big(B_{mj}R_{mikl} + B_{im}R_{mjkl}\big) = B_{ij,lk} + (b_i - b_j)R_{ijkl}. \tag{3.9}$$

On the other hand, the Gauss equation (2.22) and (3.2) imply that

$$R_{ijkl} = 0, \quad \text{if three of } i,\ j,\ k,\ l \text{ are either the same or distinct.} \tag{3.10}$$

From (3.9) and (3.10), we get

$$B_{ij,kl} = B_{ij,lk}, \quad \text{if three of } i,\ j,\ k,\ l \text{ are distinct.} \tag{3.11}$$

Applying (2.25) again for distinct i, j and using (3.7), we get

$$\begin{aligned}\sum_k B_{ii,jk}\omega_k &= dB_{ii,j} + \sum_k \big(B_{ki,j}\omega_{ki} + B_{ik,j}\omega_{ki} + B_{ii,k}\omega_{kj}\big)\\ &= 2B_{il,j}\omega_{li} = 2B_{il,j}\sum_k \Gamma^l_{ki}\omega_k = 2B_{il,j}\Gamma^l_{ji}\omega_j,\end{aligned} \tag{3.12}$$

where i, j, l are distinct. From (3.12) we obtain, for distinct i, j, l,

$$B_{ii,jj} = 2B_{ij,l}\Gamma^l_{ji}, \quad B_{ii,jl} = 0. \tag{3.13}$$

From (3.11), (3.13) and $B_{ij,k}$ being totally symmetric, we get

$$B_{12,31} = B_{11,23} = 0, \quad B_{12,32} = B_{22,13} = 0, \quad B_{12,33} = B_{33,12} = 0. \tag{3.14}$$

Substituting (3.14) into (3.8), we obtain $dB_{12,3} = 0$. This proves that $B_{12,3}$ is constant. ∎

LEMMA 3.2. *a_1, a_2, a_3 are constants.*

Proof. From Lemma 3.1, (3.5) and the symmetricity of $B_{ij,k}$, we get

$$B_{12,3} = (b_1 - b_2)\Gamma^1_{32} = (b_2 - b_3)\Gamma^2_{13} = (b_3 - b_1)\Gamma^3_{21} = \mathit{const.} := c \neq 0, \tag{3.15}$$

which implies that $\{\Gamma^k_{ij}\}$ consists of constants and they satisfy

$$\Gamma^i_{kj} = \frac{c}{b_i - b_j} \neq 0, \quad \text{for distinct } i,\ j,\ k. \tag{3.16}$$

Applying the above facts to the definition of Riemannian curvature tensor, we easily have the calculation

$$R_{ijij} = \Gamma^i_{kj}\Gamma^j_{ik} + \Gamma^i_{jk}\Gamma^j_{ik} - \Gamma^i_{kj}\Gamma^i_{jk} = \frac{2c^2}{(b_k - b_j)(b_k - b_i)} \neq 0 \tag{3.17}$$

for distinct i, j, k. This shows that R_{ijij} is constant.

On the other hand, from the Gauss equation (2.22), we find that

$$R_{ijij} = b_i b_j + a_i + a_j, \quad i, j = 1, 2, 3 \text{ and } i \neq j. \tag{3.18}$$

From (3.18) we obtain, for distinct i, j, k,

$$a_k = \frac{1}{2}\big(R_{jkjk} + R_{kiki} - R_{ijij} - b_j b_k - b_k b_i + b_i b_j\big). \tag{3.19}$$

We have proved that $\{a_k\}$ consists of constants. ∎

Lemma 3.3. *There exist constants λ and μ, such that*

$$a_1 + \lambda b_1 + \mu = a_2 + \lambda b_2 + \mu = a_3 + \lambda b_3 + \mu = 0. \tag{3.20}$$

Proof. From (2.17), (3.2), (3.4) and Lemma 3.2, we have

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k \big(A_{kj}\omega_{ki} + A_{ik}\omega_{kj}\big) = (a_i - a_j)\sum_k \Gamma^i_{kj}\omega_k,$$

which gives that

$$A_{ij,k} = (a_i - a_j)\Gamma^i_{kj}, \quad \text{for all } i,\ j,\ k. \tag{3.21}$$

In particular, we have

$$\begin{cases} A_{12,3} = (a_1 - a_2)\Gamma^1_{32} = (a_2 - a_3)\Gamma^2_{13} = (a_3 - a_1)\Gamma^3_{21} = \mathit{const.} := \tilde{c}, \\ A_{ii,j} = 0, \quad \text{for all } i,\ j. \end{cases} \tag{3.22}$$

Comparing (3.22) with (3.15), we arrive at

$$\frac{a_1-a_2}{b_1-b_2}=\frac{a_2-a_3}{b_2-b_3}=\frac{a_3-a_1}{b_3-b_1}=\frac{\tilde{c}}{c}=const.:=-\lambda. \tag{3.23}$$

From (3.23) we find that

$$a_1+\lambda b_1=a_2+\lambda b_2=a_3+\lambda b_3=const.:=-\mu,$$

then (3.20) follows. ∎

LEMMA 3.4. *The constants λ and μ in* (3.20) *satisfy*

$$\lambda^2+2\mu<0. \tag{3.24}$$

Proof. To prove (3.24), we first express λ and μ explicitly in terms of $\{b_i\}$.

From (3.17) and (3.19), we get

$$2a_k=-\frac{4c^2}{(b_k-b_i)(b_k-b_j)}-b_kb_i-b_kb_j+b_ib_j,\quad i,j,k\neq, \tag{3.25}$$

$$a_i-a_k=\frac{2c^2(2b_j-b_i-b_k)}{(b_i-b_j)(b_j-b_k)(b_k-b_i)}-b_j(b_i-b_k),\quad i,j,k\neq. \tag{3.26}$$

From (3.3), (3.23) and (3.26), we get

$$(\lambda-b_1)(b_2-b_3)=\frac{6b_1c^2}{(b_1-b_2)(b_2-b_3)(b_3-b_1)}, \tag{3.27}$$

$$(\lambda-b_2)(b_3-b_1)=\frac{6b_2c^2}{(b_2-b_3)(b_3-b_1)(b_1-b_2)}. \tag{3.28}$$

If $b_2=0$, then from (3.28) we get $\lambda=0$. If $b_2\neq 0$, then $\lambda\neq b_2$, (3.27) and (3.28) imply that

$$\frac{(\lambda-b_1)(b_2-b_3)}{(\lambda-b_2)(b_3-b_1)}=\frac{b_1}{b_2}. \tag{3.29}$$

Solving λ from (3.29), with using (3.3), we obtain

$$\lambda=-\frac{3b_1b_2b_3}{b_1^2+b_2^2+b_3^2}=-\frac{9}{2}b_1b_2b_3,$$

which is also valid for $b_2 = \lambda = 0$. Thus for any value of b_2, it holds

$$\lambda = -\frac{9}{2} b_1 b_2 b_3. \tag{3.30}$$

Because of (3.3) and the fact that $b_1, b_2, b_3 \neq$, without loss of generality, we can assume that

$$b_1 < b_2 < b_3, \quad b_1 < 0, \quad b_3 > 0, \tag{3.31}$$

then from (3.3) again we can write

$$b_2 = -\frac{b_1}{2} - \sqrt{\frac{1}{3} - \frac{3}{4} b_1^2}, \quad b_3 = -\frac{b_1}{2} + \sqrt{\frac{1}{3} - \frac{3}{4} b_1^2}, \tag{3.32}$$

which together with $b_1 < b_2$ further imply that $-\frac{2}{3} < b_1 < -\frac{1}{3}$.

From (3.25), (3.27), (3.28) and (3.30), we have the following calculation to express a_1 in terms of b_1:

$$\begin{aligned} 2a_1 &= \frac{4c^2(b_2 - b_3)}{(b_1 - b_2)(b_2 - b_3)(b_3 - b_1)} - b_1 b_2 - b_1 b_3 + b_2 b_3 \\ &= \frac{2(\lambda - b_1)(b_2 - b_3)^2}{3b_1} + b_1^2 + b_2 b_3 = -\frac{2}{9} + \frac{3}{4} b_1^2 + \frac{9}{4} b_1^4. \end{aligned} \tag{3.33}$$

Finally, from (3.20) and (3.30)–(3.33), we arrive at

$$\begin{cases} \lambda = -\frac{9}{2} b_1 \Big(b_1^2 - \frac{1}{3} \Big), \\ 2\mu = -2a_1 - 2\lambda b_1 = \frac{2}{9} - \frac{15}{4} b_1^2 + \frac{27}{4} b_1^4, \end{cases} \tag{3.34}$$

which gives that

$$\lambda^2 + 2\mu = \frac{1}{36}(9b_1^2 - 1)(9b_1^2 - 4)(9b_1^2 + 2) < 0,$$

where we have used the fact $-\frac{2}{3} < b_1 < -\frac{1}{3}$.

This proves Lemma 3.4. ∎

Completion of proof of the Classification Theorem. From (2.9)–(2.11) and (3.20), we see that for the constants λ and μ in Lemma 3.3, it holds

$$dN - \lambda dE + \mu dY = 0,$$

so we can find a constant vector $\mathbf{C} \in \mathbb{L}^6$ such that

$$N - \lambda E + \mu Y = \mathbf{C}. \tag{3.35}$$

It follows from (3.35) and (2.5) that

$$\langle \mathbf{C}, \mathbf{C} \rangle = \lambda^2 + 2\mu = const. := -r^2 < 0, \quad \langle Y, \mathbf{C} \rangle = 1, \tag{3.36}$$

as it says that $\mathbf{C}$ is a constant timelike vector, we can find a Lorentz transformation $T \in O(5,1)$ in $\mathbb{L}^6$ such that

$$\tilde{\mathbf{C}} := (-r, 0) = \mathbf{C}T = (N - \lambda E + \mu Y)T. \tag{3.37}$$

Let $\tilde{x} : M^3 \to \mathbb{S}^4$ be a hypersurface which is Möbius equivalent to x by taking $\tilde{Y} = YT$. Then we have $\tilde{N} = NT$, $\tilde{E} = ET$ and $\tilde{\mathbf{C}} = (-r, 0) = \tilde{N} - \lambda\tilde{E} + \mu\tilde{Y}$.

Writting $\tilde{Y} = \tilde{\rho}(1, \tilde{x})$, from $\langle \tilde{Y}, \tilde{\mathbf{C}} \rangle = \langle YT, \mathbf{C}T \rangle = \langle Y, \mathbf{C} \rangle = 1$ and $\langle \tilde{Y}, \tilde{\mathbf{C}} \rangle = \tilde{\rho}\langle (1, \tilde{x}), (-r, 0) \rangle = r\tilde{\rho}$ we get

$$\tilde{\rho} = \frac{1}{r} = const. \tag{3.38}$$

Now we consider the Möbius isoparametric hypersurface $\tilde{x} : M^3 \to \mathbb{S}^4$. From (3.38) and the fact that $\tilde{x}$ has constant normalized Möbius scalar curvature $\tilde{\kappa} = \kappa = \frac{1}{6}\sum_{i,j} R_{ijij}$, cf. (3.17), we know that $\tilde{x}$ has constant norm square of second fundamental form and the Euclidean induced metric $d\tilde{x} \cdot d\tilde{x}$ has constant scalar curvature. Then from the Euclidean Gauss equation of $\tilde{x}$, we find that $\tilde{x} : M^3 \to \mathbb{S}^4$ also has constant mean curvature. Applying (2.27) to the Möbius isoparametric hypersurface $\tilde{x} : M^3 \to \mathbb{S}^4$, then the above facts imply that it is in fact an Euclidean isoparametric hypersurface with three distinct principal curvatures. By Cartan's result in [3], we know that $\tilde{x} : M^3 \to \mathbb{S}^4$ is a tube of constant radius over a standard Veronese embedding of $\mathbb{R}P^2$ into $\mathbb{S}^4$.

This completes the proof of our Classification Theorem. ∎

Remark 3.1. Our proof of Lemma 3.4 is motivated by [10], where for a Möbius hypersurface $x : M^n \to \mathbb{S}^{n+1}$, H. Li and C. P. Wang considered the case that the Blaschke tensor $\mathbf{A}$ and the Möbius second fundamental form $\mathbf{B}$ are related by $\mathbf{A} + \lambda\mathbf{B} + \mu g = 0$ for some smooth functions λ, μ and obtained the characterization of such hypersurfaces under the condition of $\Phi = 0$. We also note that $\Phi = 0$ is a natural condition under which a series of interesting results have been established in recent years, see references [5]–[12] for details.

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Zejun Hu
Department of Mathematics
Zhengzhou University
450052 Zhengzhou
People's Republic of China
huzj@zzu.edu.cn

Haizhong Li
Department of Mathematical Sciences
Tsinghua University
100084 Beijing
People's Republic of China
hli@math.tsinghua.edu.cn