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A 7-LOCAL IDENTIFICATION OF THE MONSTER

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Abstract. We identify the monster from two of its 7-constrained maximal 7-local subgroups.

§1. Introduction

All groups considered in this article are finite. Suppose that X is a group and p is a prime. Then X is *p-constrained* if $C_X(O_p(X)) \leq O_p(X)$. For a group G, $S \in Syl_p(G)$, and T a non-trivial subgroup of S, $N_G(T)$ is called a *p-local subgroup*. We say that G has *local characteristic p* if every p-local subgroup is p-constrained and we say that G has *parabolic characteristic p* provided every p-local subgroup that contains S is p-constrained. A $\mathcal{K}$*-group* is a group which has all its composition factors from among the known simple groups and a group G is $\mathcal{K}$*-proper* if every proper subgroup of G is a $\mathcal{K}$-group.

It is expected that the programme to identify the $\mathcal{K}$-proper groups of local characteristic p orchestrated by Meierfrankenfeld, Stellmacher and Stroth (see [20]) will soon have a list of possible amalgams within such groups. Some of these amalgams will uniquely determine the target group via its p-local geometry (for example via the building if G is expected to be a Lie type group in characteristic p of rank at least 3). For other groups, where for example there are only two p-local subgroups containing a Sylow p-subgroup, other methods will be needed. In this paper we investigate one of these exceptional configurations. Our main theorem is as follows.

THEOREM 1.1. *Suppose that G is a $\mathcal{K}$-proper group, $S \in Syl_7(G)$, $Z_\beta = Z(S)$ has order 7, $Z_\alpha = Z_2(S)$ (the second centre of S) has order 49 and*

(a) $N_\beta = N_G(Z_\beta) \sim 7^{1+4}_+.2{\cdot}Alt(7).6$ *is 7-constrained; and*

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(b) $N_\alpha = N_G(Z_\alpha) \sim 7^{2+1+2}.GL_2(7)$ *is* 7*-constrained.*

Assume that $O_7(\langle N_\alpha, N_\beta\rangle) = 1$*. Then* G *is isomorphic to the monster, the largest sporadic simple group. In particular,* G *is of parabolic characteristic* 7 *but not of local characteristic* 7*.*

In Theorem 1.1 we have used the Atlas [5, page xx] notation for group extensions. We also use the Atlas [5] notation for the sporadic simple groups except that the baby monster is denoted by BM rather than B. Cyclic groups of order n will be denoted by n or $\mathbb{Z}_n$. To avoid confusion with Lie type groups of type A and D, we use $Alt(n)$ and $Sym(n)$ for the alternating and symmetric groups of degree n, respectively, and $Dih(n)$ for the dihedral group of order n. We use p_+^{1+2n} to denote the extraspecial group of order p^{1+2n} and exponent p when p is odd and, when $p = 2$, the extraspecial group of order 2^{1+2n} which has elementary abelian subgroups of order 2^{1+n}. We write $G \sim A.B.\dots.Z$ or say that G has *shape* $A.B.\dots.Z$ when G has a normal series with factors of shape $A, B, \dots, Z$. Thus, for example, $G \sim 7_+^{1+4}.2{}^{\cdot}Alt(7)$ indicates that G contains proper normal subgroups of order 7, 7^5, 2.7^5 and $2.7^5.|Alt(7)|$. It also indicates that the normal subgroup Q of order 7^5 is an extraspecial group of exponent 7 and that G/Q is isomorphic to a non-split extension of $Alt(7)$ by a cyclic group of order 2. The notation $G \sim 7^{2+1+2}.GL_2(7)$ indicates that G contains proper normal subgroups of orders 7^2, 7^3, and 7^5 but nothing further about the action of $G/O_7(G) \cong GL_2(7)$ on $O_7(G)$ is meant to be implicit in the notation. A p-element $x \in G$ is called *p-central* if $C_G(x)$ contains a Sylow p-subgroup of G.

The amalgam $\mathcal{A}$ consisting of the triple $(N_\alpha, N_\beta, N_\alpha \cap N_\beta)$ and the two inclusion mappings is an example of a *symplectic amalgam* and appears as $\mathcal{A}_{48}$ in [22, Table 1.8]. Notice that we have not assumed that $\mathcal{A}$ is isomorphic to the corresponding amalgam in M (though it may well be) nor have we assumed that G is generated by N_α and N_β.

Our strategy for identifying the monster from this 7-local information is to determine the structures of the centralizers of the involutions in G. Thus our objective is to find 2 involutions, one of which has 2-constrained centralizer of shape $2_+^{1+24}.Co_1$ and the other one has centralizer $2{}^{\cdot}BM$. Once this has been done, and after noting that the monster sporadic simple group possesses two 7-local subgroups as described in Theorem 1.1 (see [25] or [5, page 234]), the main result follows from [12]. As part of our proof we present some elementary results about $\mathcal{K}$-groups which characterize certain of the simple $\mathcal{K}$-groups by the structure of the centralizer of elements of order 7 or by the structure of their Sylow p-subgroups. These results are included in Section 2. Finally we mention that the monster has been characterized from the amalgam of its 2-local subgroups by Ivanov [15] and from the amalgam of its 3-local subgroups by Ivanov and Meierfrankenfeld [16]. These two results actually show that the universal completion of the amalgam under investigation is isomorphic to the monster.

There are other exotic amalgams related to the large sporadic simple groups. Four examples occur with $p = 5$ and are listed as $\mathcal{A}_{20}$, $\mathcal{A}_{21}$, $\mathcal{A}_{46}$ and $\mathcal{A}_{53}$ in [22, Table 1.8]. These four amalgams are related to HN, BM, Ly, and M, respectively, and are the subject of [21], [23], and [24]. A further remarkable example which occurs in the monster has $N_\beta \sim 13_+^{1+2}.12.Sym(4)$ and $N_\alpha \sim 13^2.4.L_2(13).2$. With respect to this amalgam the monster is of parabolic characteristic 13; unfortunately, as yet we have no idea how to characterize M from these two subgroups. The strategies used in [21], the present paper, [23] and [24] will not work in this particular case since the largest elementary abelian 13-subgroup in the centralizer of an involution in N_α and N_β has order 13. Thus the critical method of proof used in Propositions 5.6 and 6.5 to control $O_{p'}(C_G(t))$ for an involution t fails.

Dr. Corinna Wiedorn died on 25th February 2005 in Lingen, Germany. Her contribution to our mathematical community is greatly missed.

§2. Preliminaries

This section primarily contains characterizations of $\mathcal{K}$-groups by their 7-local structure. However, we begin with a result concerning the 4-dimensional $GF(7)$-module for $G = 2^{\cdot}Alt(7).2$ which, as we shall see, appears in N_β as $O_7(N_\beta)/Z_\beta$.

Firstly we note that the smallest $GF(7)$ representation of $Alt(7)$ is 6-dimensional (see [17, page 15]). So G contains a subgroup isomorphic to the non-split extension $2^{\cdot}Alt(7)$. Furthermore, as the multiplicative group of $GF(7)$ has no elements of order 4, we have $G/Z(G) \cong Sym(7)$. We note also before we start that the smallest dimension of a faithful $2^{\cdot}Alt(7).2$ module over $GF(7)$ is 4. Again this well-known fact can be read, for example, from [17, page 15].

Lemma 2.1. *Let $G \sim 2{}^{\cdot}Alt(7).2$, V be a faithful irreducible 4-dimensional $GF(7)G$-module, and denote by $\mathcal{P}(V) = \{W \leq V \mid \dim W = 1\}$ the set of one-dimensional subspaces of V.*

(1) *Let $x \in G$, $o(x) = 3$. If x projects to a 3-cycle in $G/Z(G) \cong Sym(7)$, then $C_V(x) = 0$. If x projects to a product of two disjoint 3-cycles in $G/Z(G)$, then $\dim C_V(x) = 2$.*
(2) *If $H \leq G$ with $H \cong SL_2(7)$, then H acts irreducibly on V.*
(3) *$G' \cong 2{}^{\cdot}Alt(7)$ has just two orbits on $\mathcal{P}(V)$. They have lengths 120 and 280 and stabilizers $2 \times 7 : 3$ and $2 \times 3 \times 3$, respectively.*
(4) *G' has just two orbits on $V^{\#}$. They have lengths 720 and 1680 and stabilizers of order 7 and 3, respectively.*
(5) *The orbits in (3) and (4) are preserved by G, the corresponding stabilizers being $2 \times 7 : 6$ and $2 \times 3 \times Sym(3)$ in (3) and $7 : 2$ and $Sym(3)$ in (4).*
(6) *Let $P = \langle x, y\rangle \in Syl_3(G)$ such that $V = C_V(x) \oplus C_V(y)$. Let $W \in \mathcal{P}(V)$, $W \leq C_V(x)$. Then $N_G(W)$ acts irreducibly on $C_V(y)$. In particular, if $W \leq W_1 \leq V$ such that W_1 is $N_G(W)$-invariant and $\dim W_1 = 3$ then $W_1 = W + C_V(y)$.*
(7) *For $S \in Syl_7(G)$, $\dim C_V(S) = \dim V/[V,S] = \dim[V,S]/[V,S,S] = 1$.*

Proof. First of all as V is a faithful module for G, we note that $\langle z\rangle = Z = Z(G)$ inverts each vector of V. This fact will be used without further comment below.

(1) Let x and y be elements of order 3 projecting to (567) and (123) in $G/Z(G)$, respectively, and suppose $W = C_V(x) \neq 0$. Let Q be the preimage in G of the subgroup $\langle(12)(34),(13)(24)\rangle \leq Alt(7)$. Then $Q \cong Q_8$ and $\langle Q, y\rangle \cong Q_8 : 3$. Since $[\langle Q, y\rangle, x] = 1$, $\langle Q, y\rangle$ normalizes W, and since $Z \leq Q$ we see that $\langle Q, y\rangle$ acts faithfully and irreducibly on W and on V/W and that $\dim W = \dim V/W = 2$. Since $\langle Q, y\rangle = \langle y^Q\rangle$, we have that $\dim C_W(y) = \dim C_{V/W}(y) = 1$. Set $X = \langle x, y\rangle$. Then $X \in Syl_3(G)$, $\dim C_V(X) = 1$, and consequently

$$V/C_V(X) = \bigoplus_{u \in X^{\#}} C_{V/C_V(X)}(u)$$

has dimension 3. Since $C_V(X)$ is normalized by $N_G(X)$, since x and y as well as xy and xy^{-1} are conjugate in $N_G(X)$, and since $\dim_{V/C_V(X)}(x) =$

1, we infer that $\dim C_{V/C_V(X)}(u) \geq 1$ for all $u \in X^\#$. This means that $\dim V/C_V(X) > 3$, a contradiction. Thus we have that $C_V(x) = 0$ and as a consequence we have $\dim C_V(xy) = 2$. Hence (1) holds.

(2) Suppose that (2) is false and let $W \leq V$ be such that W is normalized by H. Since $Z \leq H$, we have $\dim W = 2$ and H acts faithfully in its natural representation on W and V/W. Since 3-elements of $SL_2(7)$ act fixed point freely on the natural $SL_2(7)$-module, they act fixed point freely on V. But the only 3-elements of G which are conjugate into H are those which correspond to a product of two disjoint 3-cycles in $Alt(7)$ and these elements have fixed points by (1). Therefore, (2) holds.

(3) and (4) Let $P \in Syl_3(G')$. Then by (1) there are $x \in P^\#$ and $W \in \mathcal{P}(V)$ such that $[W, x] = 0 \neq [W, P]$ and such that W is normalized by P. Thus, $\langle z, P\rangle \leq N_{G'}(W)$, $\langle z, P\rangle \cong 2 \times 3 \times 3$, and $\langle z, P\rangle \cap C_{G'}(W) = \langle x\rangle$. In particular, $\langle z, P\rangle$ realizes the full automorphism group $\mathbb{Z}_6$ on W. Hence, if $N_{G'}(W) > \langle z, P\rangle$ then $C_{G'}(W) > \langle x\rangle$. Now any subgroup of $Alt(7)$ properly containing a Sylow 3-subgroup also contains an involution. But all involutions in $Alt(7)$ lift to elements of order 4 in $2^{\cdot}Alt(7)$, whence they square to z and so cannot be contained in $C_{G'}(W)$. It follows that W is a representative for an orbit of length 280 for G' on $\mathcal{P}(V)$ and that the structures of $N_{G'}(W)$, and $C_{G'}(W)$ are as claimed in (3) and (4).

Since G' has an orbit of length 280 on $\mathcal{P}(V)$, it follows that any other orbit of G' on V has length at most $400 - 280 = 120$.

Let $S \in Syl_7(G')$. Then there is $U \in \mathcal{P}(V)$ with $U \leq C_V(S)$. Certainly U is in a different G' orbit than W. Suppose that two distinct Sylow 7-subgroups of G' centralize U. Then U is centralized by a subgroup isomorphic to $SL_2(7)$ or by G'. This contradicts (2) or the fact that V is an irreducible G'-module. Hence we infer that U is centralized by a unique Sylow 7-subgroup and so $N_{G'}(U) \leq N_{G'}(S) \cong 2 \times 7 : 3$. Since $|G' : N_{G'}(S)| = 120$, we conclude that $N_{G'}(U) = N_{G'}(S)$. Clearly, $z \in N_{G'}(U) \setminus C_{G'}(U)$ and by (1) and the previous paragraph U is not centralized by a 3-element, either. So (3) and (4) are proved.

(5) Since 400 does not divide $|Sym(7)|$, we see that $G \sim 2^{\cdot}Sym(7)$ must preserve the orbits of G'. Since $|G : G'| = 2$ and, as we have seen that, for any $X \in \mathcal{P}(V)$, $N_{G'}(X)$ realizes $Aut(X)$ on X, it is easy to deduce from (3), (4), and the structure of a group of shape $2^{\cdot}Sym(7)$ that the structures of $N_G(X)$ and $C_G(X)$ are as stated.

(6) In the proofs of (3), (4), and (5) we have seen that $N_G(W) \cong 2 \times 3 \times Sym(3)$. Further, $N_G(W) \leq C_G(y)$ and so $N_G(W)$ acts on $C_V(y)$. Since x acts fixed point freely on $C_V(y)$ and $C_G(W) \cong Sym(3)$ we see that

$C_G(W)$ cannot normalize a 1-space in $C_V(y)$.

(7) Suppose that S centralizes two subspaces $U, W \in \mathcal{P}(V)$. Then by (3) we have $W = U^g$ for some $g \in G' \setminus N_{G'}(S)$. So $S < \langle S, S^g \rangle \leq C_G(W)$ and this contradicts (3) and (4). Hence $\dim C_V(S) = 1$. It follows from [1, Lemma 5.5, Theorem 6.4] that as an S-module V embeds into the regular S-module of dimension 7. Since this module is uniserial the claims in (7) all hold. ∎

Corollary 2.2. *If $G \sim 2{}^{\cdot}Alt(7).2$ acts faithfully on a 4-dimensional vector space over $GF(7)$, then $G \cong 2^{-}Sym(7)$.*

Proof. By Lemma 2.1(5) there is an involution $x \in G \setminus G'$ such that x centralizes a non-zero vector in V and normalizes a Sylow 7-subgroup of G. The latter forces x to map onto a product of three disjoint transpositions in $Sym(7)$. In other words, the involutions of $Sym(7)$ which lift to involutions in G are the products of three disjoint transpositions whereas transpositions lift to elements of order 4 in G. By [5, page 236], this is just the definition of the group $2^{-}Sym(7)$ (in contrast to $2^{+}Sym(7)$). ∎

The irreducibility of both conjugacy classes of subgroups isomorphic to $SL_2(7)$ in $2{}^{\cdot}Alt(7)$ on V proved in Lemma 2.1(2) will be a key point later and distinguishes the situation considered in this paper significantly from the corresponding 5-local case, which can be seen in the sporadic simple group discovered by Lyons. Indeed, in Ly we have the 5-constrained 5-local subgroup $5^{1+4}_{+}.2{}^{\cdot}Alt(6).4$. So $2.Alt(6)$ is acting on a 4-dimensional $GF(5)$-space. Now $2{}^{\cdot}Alt(6)$ contains two conjugacy classes of subgroup $2{}^{\cdot}Alt(5) \cong SL_2(5)$. It turns out that the module restricted to one of these subgroups is irreducible and restricted to the other one it is an indecomposable extension of two natural modules for $SL_2(5)$ (see [21] for further details).

In the following theorem we determine all the simple $\mathcal{K}$-groups which have an extraspecial Sylow p-subgroup of order p^3 for some prime p. In this paper, of course, we will then just make use of the case $p = 7$ but we will use the case $p = 5$ in [24]. We remark that the simple groups with dihedral Sylow 2-subgroups have been determined by Gorenstein and Walter (see [10], [11]) independent from the $\mathcal{K}$-group hypothesis. A shorter proof has been established by Bender and Glauberman (see [2], [3]). It was shown by Brauer and Suzuki (see [4]) that there are no finite simple groups with quaternion Sylow 2-subgroups. It is also a pleasure at this stage to acknowledge the assistance of our referee who pointed out that our original proof of our next theorem could be dramatically shortened by citing [9, 4.10.3 (c)].

THEOREM 2.3. *Let G be a simple $\mathcal{K}$-group, p a prime, and $S \in Syl_p(G)$. Assume that S is extraspecial of order p^3. Then one of the following holds.*

(1) $G \cong L_3(p)$ *or p is odd and* $G \cong U_3(p)$.
(2) $p = 2$ *and* $G \cong Alt(7)$ *or* $G \cong L_2(q)$ *where* $q \equiv \pm 7 \pmod{16}$.
(3) $p = 3$ *and* $G \cong G_2(2)' \cong U_3(3)$ *or* $G \cong G_2(r^a)$ *for some prime r where* $r \equiv \pm 2, \pm 4 \pmod 9$, *$a$ is not divisible by* 3, *and* $a > 1$ *if* $r = 2$.
(4) $p = 3$ *and* $G \cong {}^2F_4(2)'$ *or* $G \cong {}^2F_4(2^a)$, *where* $a \geq 3$ *is odd and not divisible by* 3.
(5) *G is a sporadic group and*
   (i) $p = 3$ *and G is one of* M_{12}, M_{24}, J_2, J_4, He, Ru; *or*
   (ii) $p = 5$ *and G is one of* HS, Co_3, Co_2, McL, Ru, Th; *or*
   (iii) $p = 7$ *and G is one of* He, $O'N$, Fi'_{24}; *or*
   (iv) $p = 11$ *and* $G \cong J_4$; *or*
   (v) $p = 13$ *and* $G \cong M$.

Proof. For the case $p = 2$ we simply refer to [4] and to [10, Theorem 2] (see also [6, page 462]). So from now on we assume p is odd.

Suppose first that $G \cong Alt(m)$ for some m. Then $p > 3$ since there are no alternating groups with Sylow 3-subgroup of order 3^3. Further, as $|S| = p^3$, we must have $3p \leq m \leq 4p - 1$. But for such m the Sylow p-subgroups of $Alt(m)$ are elementary abelian, generated by three disjoint p-cycles. So G is not an alternating group.

If G is a Lie type group in characteristic p with associated root system Φ, then, since the order of S is at least $p^{a|\Phi|/2}$, we deduce that G is isomorphic to either $L_3(p)$ or $U_3(p)$.

Suppose now that G is a group of Lie type defined over a field of order r^a where r is a prime with $r \neq p$. By a result due to Huppert [8, 15.21] the exponent of S is p and so S contains $p+1$ maximal elementary abelian subgroups. Using [9, 4.10.3 (c)] we infer that $p = 3$ and that G is isomorphic to one of $A_2(r^a)$ with $r^a \equiv 1 \pmod 3$, ${}^2A_2(r^a)$ with $r^a \equiv -1 \pmod 3$, ${}^2F_4(2^a)'$, $G_2(r^a)'$, or ${}^3D_4(r^a)$. In the first two cases a Sylow 3-subgroup of the universal group of Lie type is contained in a subgroup $(r^a - 1) : Sym(3)$ when $G \cong A_2(r^a)$ and in a subgroup $(r^a + 1) : Sym(3)$ when $G \cong {}^2A_2(r^a)$.

Thus, a Sylow 3-subgroup of the simple group G (which is the universal group factored by its centre of order 3) has order either 3^2 or strictly greater than 3^3. So these cases do not occur. Also, by [7, 10-1(4)] we have that $G \not\cong {}^3D_4(r^a)$. Therefore, to conclude the proof of (3) and (4) we only have to establish the assertions about r and a. First let $G \cong {}^2F_4(2^a)$. Then a is odd since the groups ${}^2F_4(2^a)$ are only defined for odd a. To prove that a is not divisible by 3, we examine each of the possibilities $a \equiv 0, \pm 1 \pmod 3$ and look when

$$2^a \equiv -1 \pmod 3, \quad \text{but} \quad 2^a \not\equiv -1 \pmod 9.$$

If $a = 3b$ for some b then b must be odd (since a is odd). So

$$2^a = 2^{3b} \equiv 8^b \equiv (-1)^b = -1 \pmod 9.$$

Hence this case cannot occur. If $a = 3b+1$ then b is even. So

$$2^a = 2^{3b+1} \equiv 2 \cdot 8^b \equiv 2 \cdot (-1)^b = 2 \pmod 9$$

and

$$2^a = 2^{3b+1} \equiv 2 \cdot 2^b \equiv 2 \cdot (-1)^b = 2 \equiv -1 \pmod 3.$$

Similarly, if $a = 3b+2$ then b is odd. So

$$2^a = 2^{3b+2} \equiv 4 \cdot 8^b \equiv 4 \cdot (-1)^b = -4 \pmod 9$$

and

$$2^a = 2^{3b+2} \equiv 4 \cdot 2^b \equiv 1 \cdot (-1)^b = -1 \pmod 3.$$

So when 3 does not divide a, the Sylow 3-subgroups of ${}^2F_4(2^a)$ are at least of order 3^3. As ${}^2F_4(2^a) \geq {}^2F_4(2)' \geq L_3(3)$ (see [5, page 74]) they are seen to be extraspecial and (4) is shown.

The proof for $G_2(r^a)$ is an easy exercise and goes in the same way, distinguishing the cases $r \equiv \pm 1 \pmod 3$ and $ord_3(r^a) = 1, 2$. We leave the details for the reader and just mention that $G_2(r^a) \geq G_2(2)' \cong U_3(3)$ for all r (see [5, page 14] for $r = 2$ and [18] for r odd). This again shows that the Sylow 3-subgroups are extraspecial in the respective cases.

Finally, if G is a sporadic group the statement follows by inspection of [9, pages 262–287]. This concludes the proof. □

Corollary 2.4. *Assume that G is a simple $\mathcal{K}$-group and $S \in Syl_7(G)$. If $N_G(S) \sim 7_+^{1+2} : (3:6) \sim 7_+^{1+2} : (3 \times Sym(3))$, then $G \cong He$.*

Proof. Since S is extraspecial of order 7^3, $G \cong L_3(7)$, $U_3(7)$, He, $O'N$ or Fi'_{24} by Theorem 2.3. The groups $L_3(7)$ and $U_3(7)$ have $N_G(S)/S \cong 6 \times 2$ and $N_G(S)/S \cong 48$, respectively. So G is neither of these two groups. We now consult [9, Tables 5.3s and 5.3v] to see that the only possibility is that $G \cong He$. ∎

THEOREM 2.5. *Let G be a simple $\mathcal{K}$-group and $S \in Syl_7(G)$. Assume that $|S| = 7^2$ and, for any $x \in S^\#$, either*

(1) *$C_G(x) \cong 7 \times Alt(7)$, or*
(2) *$C_G(x) \cong 7 \times L_3(2)$ or $7 \times 2^6.L_3(2)$,*

and that S contains elements of type (1) *as well as of* (*one of the types in*) (2). *Then $G \cong Co_1$ and the possibility that $C_G(x) \sim 7 \times 2^6.L_3(2)$ does not occur.*

Proof. Again we examine each of the possibilities for G. If G is an alternating group $Alt(n)$, then, as $|S| = 7^2$, $n \geq 14$. Taking x to be a 7-cycle we have $C_G(x) \cong Alt(n-7) \times 7$ and so $n = 14$. Now take x to be a permutation of cycle shape 7^2. Then $C_G(x) = S$, a contradiction. Thus G is not an alternating group.

Suppose that G is a Lie type group defined in characteristic r. From [9, Theorem 4.9.6], for $x \in S$, the components of $C_G(x)$ are Lie type groups in characteristic r. Since $Alt(7)$ is not a Lie type group in any characteristic, we have a contradiction in this case.

Finally assume that G is a sporadic simple group. Since $|S| = 7^2$, by considering group orders we see that the only possibilities are $G \cong Co_1$, $G \cong Th$ or $G \cong BM$. Consulting [9, Tables 5.3 l, 5.3 x and 5.3 y] and using the fact that G has at least two conjugacy classes of cyclic subgroups of order 7, we immediately obtain $G \cong Co_1$. Since Co_1 does not possess a 7-element with centralizer $7 \times 2^6.L_3(2)$, the remaining part of the theorem follows. ∎

THEOREM 2.6. *Assume that G is a simple $\mathcal{K}$-group and $S \in Syl_7(G)$. If $|S| = 7^2$ and, for some $x \in S^\#$, $C_G(x) \cong 7 \times 2{}^{\cdot}L_3(4).2$, then $G \cong BM$.*

Proof. Once more we consider the possibilities for G. If $G \cong Alt(n)$ with $n \geq 5$, then, as the minimal permutation representation of $2{}^{\cdot}L_3(4).2$ has degree at least 21, we have $|G|_7 \geq 7^3$, a contradiction.

Suppose that G is a Lie type group defined in characteristic r. Since G is not of local characteristic 7, $r \neq 7$. From [9, Theorem 4.2.2 (ii)], for

$x \in S$, the components of $C_G(x)$ are Lie type groups in characteristic r. Since $2^{\cdot}L_3(4).2$ is not a Lie type group in any characteristic, we have a contradiction in this case.

Finally assume that G is a sporadic simple group. Then, as $|S| = 7^2$, looking at [9, Tables 5.3 l, 5.3 x and 5.3 y] we immediately see that $G \cong BM$. □

LEMMA 2.7. *Let F be a group and H be a non-trivial subgroup of F such that $H \leq R$ whenever R is a non-trivial subgroup of F which is normalized by H. Then F has a unique minimal normal subgroup N, N is simple, and either*

(1) *F embeds into $Aut(N)$, or*
(2) *$H = N$ $cong \mathbb{Z}_p$ for some prime p, $C_F(N)$ is cyclic of order p^k for some $k \geq 1$, and $F/C_F(N)$ embeds into $Aut(\mathbb{Z}_p)$, or*
(3) *$H = N \cong \mathbb{Z}_2$ and $F = C_F(N)$ is a quaternion group.*

Proof. If N and M are minimal normal subgroups of F, then $1 \neq H \leq N \cap M$ and so $N = M$. Therefore, F has a unique minimal normal subgroup N.

Since N is a minimal normal subgroup of F, N is a direct product of simple groups, say, $N = N_1 \times N_2 \times \cdots \times N_k$. But as $H \leq N \leq N_G(N_i)$ for $1 \leq i \leq k$, we get $H \leq \bigcap_{i=1}^k N_i$ and so we conclude that N is simple. If N is a non-abelian simple group then (1) holds. So assume $H = N \cong \mathbb{Z}_p$ for some prime p. Set $K = C_F(N)$. Of course, F/K embeds into $Aut(\mathbb{Z}_p)$. Let $S \in Syl_r(K)$ for some prime $r \neq p$. Then $H = N \leq C_F(S)$ and so $H \leq S$ by hypothesis, a contradiction. Hence K is a p-group. Finally, if $x \in K$ is an element of order p then again $H = N \leq C_F(\langle x \rangle)$ and so $H = N = \langle x \rangle$. This shows that K contains a unique subgroup of order p. Now by [6, Theorem 4.10] either N is cyclic and (2) holds or $p = 2$, $F = K$ is a quaternion group, and (3) holds. □

§3. Some general properties of the amalgam

In this section we set up some more notation and we prove some general properties of the amalgam $\mathcal{A} = (N_\alpha, N_\beta, N_\alpha \cap N_\beta)$ described in the hypothesis of Theorem 1.1. These results will be used without further reference in the remaining sections.

For $\gamma \in \{\alpha, \beta\}$ we set $L_\gamma = N_\gamma^\infty$ and $Q_\gamma = O_7(N_\gamma)$. Also, we usually write $N_{\alpha\beta} = N_\alpha \cap N_\beta$ and $L_{\alpha\beta} = L_\alpha \cap L_\beta$. The results in the next lemma follow from the information presented in the statement of Theorem 1.1 with a little help from Lemma 2.1.

LEMMA 3.1. (1) *$S = Q_\alpha Q_\beta$, $Z_\beta \le Z_\alpha \le Q_\beta$.*

(2) *Q_β is extraspecial and Q_β/Z_β is an irreducible module for $L_\beta/Q_\beta \cong 2^{\cdot} Alt(7)$.*

(3) *$Q_\alpha \cap Q_\beta = C_{Q_\beta}(Z_\alpha) = [Q_\alpha, Q_\beta]$ and $Z_\alpha/Z_\beta = C_{Q_\beta/Z_\beta}(S)$.*

(4) *Z_α is elementary abelian, $|\Phi(Q_\alpha)| = 7^3$, $\Phi(Q_\alpha) \le Q_\beta$, and Z_α and $Q_\alpha/\Phi(Q_\alpha)$ are natural modules for $N_\alpha/Q_\alpha \cong GL_2(7)$; in particular, all elements in $Z_\alpha^\#$ are conjugate in N_α.*

(5) *$N_{N_\beta}(Z_\alpha) = N_{\alpha\beta} = N_{N_\alpha}(Z_\beta)$ and $N_{\alpha\beta}/Q_\alpha \cong 7:6\times 6 \cong N_{\alpha\beta}/Q_\beta$.*

(6) *$L_\beta = C_{N_\beta}(Z_\beta)$; in particular, N_β realizes the full automorphism group $\mathbb{Z}_6$ on Z_β.*

(7) *$N_\beta = L_\beta N_{N_\alpha}(Z_\beta) = L_\beta N_{\alpha\beta}$ and $N_\beta/Q_\beta \sim 2^{\cdot}(3\times Sym(7))$.*

(8) *$N_\alpha = L_\alpha N_{N_\beta}(Z_\alpha) = L_\alpha N_{\alpha\beta}$.*

(9) *Q_β and Q_α are both characteristic subgroups of S.*

(10) *Q_β contains representatives of at most two G-conjugacy classes of cyclic subgroups of order 7.*

Proof. (1) Since $O_7(\langle N_\alpha, N_\beta\rangle) = 1$, the statements $S = Q_\alpha Q_\beta$ and $Z_\beta \le Z_\alpha$ are immediate from the definitions. Furthermore, as $Z_\alpha = Z_2(S)$ and Q_β is normal in S, $Z_\alpha \le Q_\beta$.

(2) Since, by [17], the smallest dimension of a faithful representation of $2^{\cdot} Alt(7)$ over $GF(7)$ is 4, (2) holds.

(3) Observing that $[Q_\alpha, Q_\beta] \le Q_\alpha \cap Q_\beta$ and that the latter is of order 7^4 the statement in (3) follows from (1), the fact that N_β is 7-constrained and Lemma 2.1(7).

(4) If $N_\alpha \le N_\beta$, then $Q_\beta \le O_7(\langle N_\alpha, N_\beta\rangle) = 1$, which is a contradiction. Thus N_α does not normalize Z_β. In particular, Z_α is not cyclic, for otherwise Z_β would be the unique cyclic subgroup of Z_α of order 7 and as such would be normal in N_α. Furthermore, we see that $N_\alpha/Q_\alpha \cong GL_2(7)$ acts as $GL_2(7)$ on Z_α. Since $\Phi(Q_\alpha) \ge [Q_\beta \cap Q_\alpha, Q_\alpha] = [Q_\beta, Q_\alpha, Q_\alpha]$ which has order 7^3 by Lemma 2.1(7) and $Q_\alpha/\Phi(Q_\alpha)$ admits N_α/Q_α faithfully, the remaining parts of (4) follow easily.

(5) The first claim follows directly from the definition of N_α and N_β and (4) and the structure of $GL_2(7)$ imply the structure of $N_{\alpha\beta}/Q_\alpha$. The structure of $N_{\alpha\beta}/Q_\beta$ can be infered from the action of $GL_2(7)$ on the natural module $Q_\alpha/\Phi(Q_\alpha)$ involved in Q_α and (1).

(6) A cyclic group of order 6 which normalizes Z_β and acts faithfully on it can be observed in the quotient $N_\alpha/Q_\alpha \cong GL_2(7)$.

(7) The first statement follows from parts (5) and (6). Further, since $Aut(Alt(7)) \cong Sym(7)$ either $N_\beta/Q_\beta \sim 2{}^{\cdot}(6 \times Alt(7))$ or N_β/Q_β is as claimed. But the first possibility obviously contradicts the structure of $N_{\alpha\beta}/Q_\beta$ stated in (5).

(8) To see this we note that $N_\alpha = L_\alpha N_{N_\alpha}(S) = L_\alpha N_{N_\alpha}(Z_\beta)$ by a Frattini argument.

(9) Because $|S| = 7^6$, $Z_\beta = Z(S)$, and $Z_\alpha/Z_\beta = C_{Q_\beta/Z_\beta}(S)$, we have that Q_β is the unique extraspecial subgroup of order 7^5 in S. To see that Q_α is characteristic in S we simply note that Z_α is characteristic in S by its definition and that $Q_\alpha = C_S(Z_\alpha)$.

(10) By Lemma 2.1(3), N_β has two orbits on the cyclic subgroups of Q_β/Z_β. Since for each subgroup $F \le Q_\beta$ of order 49 containing Z_β, Q_β has two orbits on the cyclic subgroups of F, we infer that there are 3 N_β-conjugacy classes of cyclic subgroups in Q_β. One of these conjugacy classes consists of Z_β and we have seen in (4) that Z_β is N_α-conjugate to a cyclic subgroup of Q_β not contained in Z_β. Hence (10) is true. ∎

We now exploit the previous lemma to deduce some information about the structures of N_α and N_β. We have that $N_{\alpha\beta} \sim S : (6 \times 6)$. Let T be a fixed complement to S in $N_{\alpha\beta}$. So $N_{\alpha\beta} = ST$ and $T \cong 6 \times 6$. Let t_1, t_2, t_3 be the three involutions in T. From the structure of N_α and N_β we may assume that $t_1 Q_\beta \in Z(N_\beta/Q_\beta)$ and $t_2 Q_\alpha \in Z(N_\alpha/Q_\alpha)$. We set $J_\beta = C_{N_\beta}(t_1)$, $K_\beta = J_\beta^\infty$ and, similarly, $J_\alpha = C_{N_\alpha}(t_2)$, $K_\alpha = J_\alpha^\infty$. So

$$J_\beta \sim (7 \times 2{}^{\cdot}Alt(7)) : 6, \qquad K_\beta \cong 2{}^{\cdot}Alt(7)$$
$$J_\alpha \sim (7 \times SL_2(7)) : 6 \quad \text{and} \quad K_\alpha \cong SL_2(7).$$

Lemma 3.2. *We have $t_2, t_3 \in N_\beta \setminus L_\beta$, $t_1, t_3 \in N_\alpha \setminus L_\alpha$. In particular, $\langle L_\beta, t_2\rangle = \langle L_\beta, t_3\rangle \cong 2^- Sym(7)$.*

Proof. Since the involutions in $Alt(7)$ lift to elements of order 4 in $2{}^{\cdot}Alt(7)$, L_β does not contain an elementary abelian group of order 4. Therefore the first statement holds for N_β. By Lemma 3.1(7) and Corollary 2.2 we have that $\langle L_\beta, t_2\rangle = \langle L_\beta, t_3\rangle \cong 2^- Sym(7)$. Since $SL_2(7)$ contains a single involution, the lemma also holds for N_α. ∎

§4. The centralizer of a non 7-central element

In this section we pick a certain subgroup of S of order 7 and show that it is not contained in the center of any Sylow 7-subgroup of G. We prove that its centralizer in G is isomorphic to $7 \times He$ and so is not 7-constrained. In particular, we note that G is not of local characteristic 7. The existence of the subgroup $7 \times He$ in G will be exploited in Sections 5 and 6 to obtain information about the 2-local structure of G.

LEMMA 4.1. *Let $U_\alpha = C_{Q_\alpha}(t_2)$. Then $|U_\alpha| = 7$, $U_\alpha \leq Q_\beta$, and U_α is not 7-central in G.*

Proof. As Q_α involves two natural modules for N_α/Q_α and t_2 acts fixed point freely on such modules and L_α centralizes $\Phi(Q_\alpha)/Z_\alpha$, which by Lemma 3.1(4) is of order 7 and contained in Q_β, we have $|U_\alpha| = 7$ and $U_\alpha \leq Q_\beta$. Suppose $U_\alpha = Z_\beta^g$ for some $g \in G$. Then $C_G(U_\alpha) \leq N_\beta^g$. But $Z_\alpha U_\alpha K_\alpha \leq C_G(U_\alpha)$ and K_α acts irreducibly on Z_α. This shows that $Z_\alpha \leq Q_\beta^g$ and that $K_\alpha \cong K_\alpha Q_\beta^g/Q_\beta^g \cong SL_2(7)$ does not act irreducibly on $Q_\beta^g/Z_\beta^g = Q_\beta^g/U_\alpha$, a contradiction to Lemma 2.1(2). ∎

For $\gamma \in \{\alpha, \beta\}$ let $X_\gamma = C_{N_\gamma}(U_\alpha)$ and $Y_\gamma = N_{N_\gamma}(U_\alpha)$. Let $X = \langle X_\alpha, X_\beta\rangle$, $Y = \langle Y_\alpha, Y_\beta\rangle$, and let $X_0 = C_G(U_\alpha)$, $Y_0 = N_G(U_\alpha)$. Obviously,

$$(*) \qquad X \leq X_0 \cap Y \leq X_0 Y \leq Y_0.$$

LEMMA 4.2. *Let $C = C_{Q_\beta}(U_\alpha)$. Then $C \in Syl_7(A)$ for $A \in \{X, Y, X_0, Y_0\}$; in particular, the 7-part of the order of A is 7^4.*

Proof. We first note that $|C| = 7^4$ by Lemma 4.1 and as Q_β is extraspecial. Also, since $C \leq X$, $(*)$ shows that it suffices to prove the statement for $A = Y_0$. Let $R \in Syl_7(Y_0)$. We may assume that $C \leq R$, so $R \cap Q_\beta = C$. Since U_α is not 7-central by Lemma 4.1, we have $|R| \leq 7^5$, so $|R : C| \leq 7$ and $C \trianglelefteq R$. Hence also $Z_\beta = Q_\beta' = C' \trianglelefteq R$ and therefore $R \leq N_G(Z_\beta) = N_\beta$. If $|R| = 7^5$, then $R \not\leq Q_\beta$ and so $C \trianglelefteq Q_\beta R = R_1 \in Syl_7(N_\beta)$. But then $U_\alpha Z_\beta = Z(C) \trianglelefteq R_1$ and Lemma 2.1 applied to Q_β/Z_β implies that $Z(C)^g = Z_\alpha$ for $g \in N_\beta$ with $R_1^g = S$. As all elements in $Z_\alpha^\#$ are 7-central we get a contradiction to Lemma 4.1. So $R = C$ and we are done. ∎

LEMMA 4.3. (1) *$X_\alpha = U_\alpha \times Z_\alpha K_\alpha \cong 7 \times 7^2 : SL_2(7)$ and $Y_\alpha = U_\alpha Z_\alpha J_\alpha \sim (7 \times 7^2 : SL_2(7)) : 6$.*

(2) $X_\beta \sim 7 \times 7_+^{1+2}.(Sym(3) \times 3)$ *and* $Y_\beta \sim (7 \times 7_+^{1+2}).(Sym(3) \times 3 \times 6)$; *in particular,* $O_{7'}(X_\beta) = 1$.

(3) X_β *acts irreducibly on* $O_7(X_\beta/U_\alpha Z_\beta)$ *and* $|X_\beta : X_\beta \cap L_\beta| = 6$.

Proof. (1) is a straightforward consequence of the definition of U_α and (2) follows easily from Lemmas 4.1, 4.2, and 2.1. (Note that $t_1, t_3 \in (Y_\alpha \cap Y_\beta) \setminus (X_\alpha \cup X_\beta)$.) Finally, the first statement of (3) holds by Lemma 2.1(6), the second one by (2) and Lemma 2.1(4). ∎

Lemma 4.4. $N_{X_0}(C) = X_\beta = N_X(C)$.

Proof. Let $N = N_{X_0}(C)$. As $C \leq Q_\beta$ and $|C| = 7^4$, $C' = Z_\beta$ and therefore

$$N \leq N_{X_0}(C') = N_{X_0}(Z_\beta) \leq N_G(Z_\beta) = N_\beta.$$

Since $U_\alpha \leq Z(X_0)$ we get $N \leq C_{N_\beta}(U_\alpha) = X_\beta$ and so $N_{X_0}(C) = X_\beta = N_X(C)$. ∎

Proposition 4.5. $C_G(U_\alpha) = X \cong 7 \times He$ *and* $N_G(U_\alpha) = Y \sim (7 : 3 \times He) : 2$; *in particular,* $N_G(U_\alpha)$ *is not* 7*-constrained and* G *is not of local characteristic* 7.

Proof. With $C_G(U_\alpha) = X_0$ as above, by Lemma 4.2 we have $C = C_{Q_\beta}(U_\alpha) \in Syl_7(X_0)$. We are going to apply Corollary 2.4 and Lemma 2.7.

Let $K > U_\alpha$ be a subgroup of X_0 which is normalized by X. Suppose first $K = U_\alpha \times O$, where $O = O_{7'}(K)$. Then

$$O = \langle C_O(z) \mid z \in Z_\alpha^\# \rangle.$$

Since all elements in Z_α are conjugate in X into Z_β, we may assume that $C_O(Z_\beta) \neq 1$. But $C_O(Z_\beta) \trianglelefteq X_\beta$ and so $C_O(Z_\beta) \leq O_{7'}(X_\beta) = 1$, a contradiction.

Hence 7^2 divides $|K|$ and $K \cap C > U_\alpha$. Since $Z(C)$ is not normal in X_α we have $K \neq Z(C)$ and then Lemma 4.3(3) shows that $C \leq K$. Now X_α is generated by its Sylow 7-subgroups and so $X_\alpha \leq K$, too. Finally, $X_\alpha \cap X_\beta \sim 7 \times 7_+^{1+2}.(2 \times 3)$ which shows that X_β is generated by its conjugates of $X_\alpha \cap X_\beta$. We thus get $X = \langle X_\alpha, X_\beta \rangle \leq K$.

This shows that $F = X_0/U_\alpha$ and $H = X/U_\alpha$ satisfy the hypothesis of Lemma 2.7. So Lemma 2.7 tells us that X_0 has a unique normal subgroup N which is minimal with respect to properly containing U_α, that N/U_α is simple, and as N/U_α is obviously not cyclic, that X_0/U_α embeds into

$Aut(N/U_\alpha)$. Furthermore, as $N/U_\alpha \geq X/U_\alpha$, Lemma 4.4 and Corollary 2.4 imply that $N/U_\alpha \cong He$. Since the Schur multiplier of He is trivial (see [5, page 104]) this implies $N \cong 7 \times He$. Moreover, [5, page 104] reveals that X_α and X_β are both maximal subgroups of N and so $N = \langle X_\alpha, X_\beta \rangle = X$. Finally, $X_0 = NN_{X_0}(C) = NN_X(C) = X$ by a Frattini argument and Lemma 4.4.

From the definition of Y and the structures of Y_α, Y_β we further see that $X \trianglelefteq Y$ and that $Y = \langle X, x \rangle$ for some element x of order 6 whose cube does not centralize X'. Together with the fact that $|Out(He)| = 2$ this shows that the structure of Y is also as stated and that $Y = Y_0$. (Otherwise we would get $X_0 > X$.) ∎

Comparing Proposition 4.5 with the list of p-local subgroups of the monster M in [5, page 234] we see that Y is isomorphic to a $7A$-normalizer in M.

§5. 2-Central involutions

The main goal of this section is to prove that G contains a unique conjugacy class of 2-central involutions and to show that the centralizer of such an involution is of shape $2_+^{1+24}.Co_1$. As a consequence, we also get that the 2-part of $|G|$ is 2^{46}.

Recall our fixed complement $T \cong 6 \times 6$ of S in $N_{\alpha\beta}$ and its three involutions t_1, t_2, t_3.

Lemma 5.1. *t_1, t_2, and t_3 are all conjugate in G.*

Proof. By Lemma 3.2 and Corollary 2.2, t_2 and t_3 both project to a permutation conjugate to $(12)(34)(56)$ in $\langle L_\beta, t_2 \rangle / O_{7,2}(\langle L_\beta, t_2 \rangle)$. In particular, they are conjugate in N_β. Further, it follows from the structure of $GL_2(7)$ that there is some $g \in N_{N_\alpha}(T) \setminus N_{\alpha\beta}$ such that $Z_\alpha = Z_\beta Z_\beta^g$. As $[t_1, Z_\beta] = 1 \neq [t_1, Z_\alpha]$ we get $[t_1, Z_\beta^g] \neq 1$ and so $t_1 \neq t_1^g$. Since g normalizes $\langle t_1, t_2 \rangle = \langle t_1, t_3 \rangle$ and $t_2^g = t_2$ we must have $t_1^g = t_3$ and the lemma follows. ∎

Set $E = C_{Q_\beta}(t_2)$.

Lemma 5.2. *$|E| = 7^2$ and $C_{N_\beta}(t_2)$ has two orbits $\{U_i \mid i = 0, 1, 2, 3\}$ and $\{Z_i \mid i = 0, 1, 2, 3\}$ on the set of cyclic subgroups of order 7 in E, where $U_0 = U_\alpha$ and the Z_i are conjugates of Z_β.*

Proof. We have

$$Q_\beta/Z_\beta = C_{Q_\beta/Z_\beta}(t_2) \oplus [Q_\beta/Z_\beta, t_2]$$

as a decomposition invariant under $C_{N_\beta}(t_2)$. Since

$$C_{\langle L_\beta, t_2\rangle}(t_2)Q_\beta/Q_\beta \sim 2 \times 2^{\cdot}Sym(4)$$

and t_1 inverts Q_β/Z_β we conclude that both factors must be 2-dimensional. So $|E| = 7^2$ as $Z_\beta \not\leq E$.

Clearly $U_0 = U_\alpha \leq E$. Now let $g \in N_\alpha$ with $t_1^g = t_3$ as in the proof of Lemma 5.1 and choose $h \in \langle L_\beta, t_2\rangle$ with $t_3^h = t_2$. Then $t_1^{gh} = t_2$ and so t_2 centralizes $Z_0 = Z_\beta^{gh}$. Since

$$Z_0 = Z_\beta^{gh} \leq Z_\alpha^{gh} = Z_\alpha^h \leq Q_\beta^h = Q_\beta$$

we see that $Z_0 \leq E$. Finally, the length of the orbits follows from the facts that Z_0 and U_0 are not conjugate in G by Proposition 4.5, that E contains precisely eight cyclic subgroups of order 7, and that $t_1 \in N'$ for any subgroup $N \leq C_{N_\beta}(t_2)$ with $|C_{N_\beta}(t_2) : N| < 4$. ∎

Set $N_0 = N_G(Z_0)$, $L_0 = C_G(Z_0)$, $Q_0 = O_7(N_0)$, $J_0 = C_{N_0}(t_2)$, and $K_0 = J_0^\infty$. Then $J_0 \sim (7 \times 2^{\cdot}Alt(7)) : 6$ and $K_0 \cong 2^{\cdot}Alt(7)$.

Let $K = C_G(t_2)$.

Lemma 5.3. *We have* $X \cap K = C_X(t_2) \sim 7 \times 2_+^{1+6}.L_3(2)$,

$$Y \cap K = C_Y(t_2) \sim (7 : 3 \times 2_+^{1+6}.L_3(2)).2,$$

and $O_2(X \cap K)/\langle t_2\rangle$ *is irreducible as module for* $(Y \cap K)/O_2(Y \cap K)$.

Proof. We know that $X \cap K \geq K_\alpha \cong SL_2(7)$ and so [5, page 104] shows that $X \cap K$ is as stated (since the other involution centralizer $2^2.L_3(4).2 \leq He$ does not contain $SL_2(7)$). The structure of $Y \cap K$ follows immediately. ∎

Lemma 5.4. $E \in Syl_7(K)$.

Proof. We have $E \leq K$ by the definition of E and K. Let $E \leq E_1 \leq E_0 \in Syl_7(K)$ such that $E \trianglelefteq E_1$. Then $C_E(E_1) \neq 1$ and by Lemma 5.2 we may assume that either $Z_0 \leq Z(E_1)$ or $U_\alpha \leq Z(E_1)$. In the second case we get $E = E_1 = E_0$ from Lemma 5.3. In the first case, as $t_2Q_0 \in Z(L_0/Q_0)$, t_2 inverts Q_0/Z_0 whence $E_1 \cap Q_0 = Z_0$ and $E_1Q_0 = EQ_0 \in Syl_7(L_0)$. So $E = E_1 = E_0$ as $|S : Q_\beta| = 7$. ∎

Corollary 5.5. $O_7(K) = 1$.

Proof. This can easily be deduced from Lemmas 5.2, 5.3, and the structure of J_0. ∎

Proposition 5.6. $K \sim 2_+^{1+24}.Co_1$.

Proof. Let $R = O_{7'}(K)$ and $\overline{K} = K/R$. Let $\overline{N}$ be a minimal normal subgroup of $\overline{K}$ and N its full preimage in K. Then 7 divides $|\overline{N}|$. Now Lemmas 5.2 and 5.4 immediately imply that $E \leq N$ and that $\overline{N}$ is simple. From the structures of $X \cap K$ and K_0 we infer further that

$$O^7(\overline{K}) = \overline{N} \geq \langle X \cap K, K_0 \rangle R/R.$$

In particular, $C_{\overline{N}}(\overline{Z_i}) \cong 7 \times Alt(7)$ and $C_{\overline{N}}(\overline{U_i}) \sim 7 \times 2^6.L_3(2)$ or $C_{\overline{N}}(\overline{U_i}) \cong 7 \times L_3(2)$ for $i = 0, \ldots, 3$. Applying Theorem 2.5 to $\overline{N}$ now shows that $\overline{N} \cong Co_1$ and also that $C_{\overline{N}}(U_i) \cong 7 \times L_3(2)$ must hold for $i = 0, \ldots, 3$. In particular, $R \neq \langle t_2 \rangle$.

On the other hand, as $R = \langle C_R(e) \mid e \in E^{\#} \rangle$ and we also see that $C_R(Z_i) = \langle t_2 \rangle$ and $C_R(U_i) \cong 2_+^{1+6}$ for $i = 0, \ldots, 3$ we conclude that R is a 2-group of order $|R| = 2^{25}$. Also, $t_2 \in R'$. Since $R/\langle t_2 \rangle$ admits $\overline{N}$ faithfully and the dimension of the smallest nontrivial representation of Co_1 over $GF(2)$ is 24, we get that R is extraspecial with $R' = \langle t_2 \rangle$ and R/R' is irreducible as $\overline{N}$-module. Moreover, by [13] the 24-dimensional representation of Co_1 over $GF(2)$ is absolutely irreducible and uniquely determined. So our representation of $\overline{N}$ on $R/\langle t_2 \rangle$ must be identical to the known one which embeds Co_1 into $O_{24}^+(2)$ and R must be extraspecial of $+$-type. Finally, we get $K = N$ since the outer automorphism group of Co_1 is trivial (see [5, page 180]). ∎

Corollary 5.7. *Let* $P \in Syl_2(K)$. *Then* $Z(P) = \langle t_2 \rangle$ *and* $P \in Syl_2(G)$; *in particular,* $\{t_2^g \mid g \in G\}$ *is the unique conjugacy class of* 2-*central involutions in* G.

Proof. Since $K/O_2(K)$ acts faithfully on $O_2(K)$ and $O_2(K) \leq P$ we have $Z(P) \leq C_K(O_2(K)) \leq O_2(K)$. So $Z(P) \leq Z(O_2(K)) = \langle t_2 \rangle$, which is the first assertion. Now let $P \trianglelefteq P_1 \leq P_0 \in Syl_2(G)$. Then $P_1 \leq N_G(Z(P)) = N_G(\langle t_2 \rangle) = C_G(t_2) = K$ and so $P = P_1 = P_0$. ∎

§6. The centralizer of a non 2-central involution and the proof of Theorem 1.1

To establish the isomorphism $G \cong M$ we also need to find an involution with centralizer isomorphic to $2{\cdot}BM$ in G. By [13], as a $K/O_2(K)$-module, $O_2(K)/\langle t_2\rangle$ is isomorphic to the Leech lattice reduced mod 2. Let $s \in O_2(K)$ be an element such that $s\langle t_2\rangle/\langle t_2\rangle$ corresponds to a vector of type 2. Then from the Atlas [5, page 180], we have that s has order 2, $C_K(s) \sim 2^{1+1+22}.Co_2$ and $C_K(s)/\langle s\rangle \sim 2_+^{1+22}.Co_2$ is perfect.

Lemma 6.1. *s and t_2 are not conjugate in G.*

Proof. Suppose that s is conjugate to t_2. Then $C_K(s) = K \cap C_G(s)$ and

$$C_K(s)O_2(C_G(s))/O_2(C_G(s)) \cong Co_2,$$

as Co_2 is a maximal subgroup in Co_1. Thus $O_2(C_K(s)) \le O_2(K) \cap O_2(C_G(s))$. But

$$[O_2(K) \cap O_2(C_G(s)), O_2(K) \cap O_2(C_G(s))] \le \langle t_2\rangle \cap \langle s\rangle = 1$$

so $O_2(K) \cap O_2(C_G(s))$ is abelian. Since $O_2(C_K(s))$ is not abelian, we have a contradiction. Thus s is not conjugate to t_2. □

Lemma 6.2. *The elements of order 7 in $C_G(s)$ are not conjugate into Z_β. In particular, we can choose $s \in K = C_G(t_2)$ such that $s \in X$ and $C_X(s) \sim 7 \times 2^2{\cdot}L_3(4).2$*

Proof. Since all the involutions in $C_G(Z_\beta) = L_\beta \sim 7_+^{1+4}.2{\cdot}Alt(7)$ are conjugate to t_2, the first claim follows at once from Lemma 6.1. Consequently, since 7 divides $|Co_2|$, Lemmas 5.2 and 5.4 imply that we may choose $s \in O_2(C_K(t_2))$ so that $[s, U_\alpha] = 1$. Now $C_X(s) \sim 7 \times 2^2{\cdot}L_3(4).2$ follows from [5, page 104] and Lemmas 5.3 and 6.1. □

Set $L = C_G(s)$, let $F_0 \in Syl_7(X \cap L)$ and $F_0 \le F \in Syl_7(L)$. Observe that $|F_0| = 7^2$.

Lemma 6.3. *$F = F_0$ and all the elements of F are conjugate in L to elements of U_α.*

Proof. We have already seen that F does not contain any 7-central element. In particular, the same holds for F_0. Since by Lemma 4.2 $C_{Q_\beta}(U_\alpha) \in Syl_7(X)$ and as by Lemma 3.1 all elements in Q_β are either 7-central or conjugate into U_α, we conclude that all elements in F_0 must be conjugate into U_α. Thus, Sylow 7-subgroups of $C_G(f)$ are isomorphic to $C_{Q_\beta}(U_\alpha) \cong 7 \times 7_+^{1+2}$ for any $f \in F_0^\#$. We hence conclude that any 7-group containing F_0 properly contains 7-central elements and that $F = F_0$.

Finally, let $f \in F^\#$ and $g \in G$ with $f \in U_\alpha^g$. Then $s, s^{g^{-1}} \in C_G(U_\alpha) = X$. Since by [5, page 104] He just contains two conjugacy classes of involutions and as s is not conjugate to t_2 there must be some $h \in X$ such $s^h = s^{g^{-1}}$. Then $hg \in L$ and $U_\alpha^{hg} = U_\alpha^g = \langle f \rangle$. ∎

Lemma 6.4. *$C_X(s) \cap C_K(s)$ contains a Sylow 2-subgroup V of $C_X(s)$.*

Proof. From the identification of $O_2(K)$ in Proposition 5.6, we have that $C_{O_2(K)}(U_\alpha) \sim 2_+^{1+6}$ and so $C_{O_2(C_K(s))}(U_\alpha)$ has order 2^6. Now, by [5, page 154], the centralizer of a Sylow 7-subgroup in $C_K(s)/O_2(C_K(s)) \cong Co_2$ is $7 \times Dih(8)$. Thus we infer that $C_{C_K(s)}(U_\alpha) \sim 7 \times V$ where V has order 2^9. By Lemma 6.2, $|C_X(s)|_2 = 2^9$. This proves that $V \in Syl_2(C_X(s))$. ∎

Proposition 6.5. *$L \cong 2{\cdot}BM$.*

Proof. Clearly, $O_7(L) = 1$. Let $U = O_{7'}(L)$, $\overline{L} = L/U$, let $\overline{N}$ be a minimal normal subgroup of $\overline{L}$, and N its full preimage in L. Since 7 dives $|\overline{N}|$, $F \cap N \neq 1$ and so $F \leq N$, as all cyclic 7-subgroups of F are conjugate in L by Lemma 6.3. A similar argument shows that $\overline{N}$ must be simple.

By Lemmas 6.2 and 6.3 we now have $C_{\overline{N}}(\overline{f}) \sim 7 \times 2.L_3(4).2$ or $7 \times L_3(4).2$ for any $f \in F^\#$.

In the first case, Theorem 2.6 gives $\overline{N} \cong BM$. Furthermore, $L \cap C_G(F) \leq N$ and so L/N embeds into $Out(\overline{N})$. Since $Out(BM) = 1$ by [5, page 219] we get $L = N$. Also, as $U = \langle C_U(f) \mid f \in F^\# \rangle$ and now $C_U(f) = \langle s \rangle$ for any $f \in F^\#$ we get $U = \langle s \rangle$. Hence $L \cong 2{\cdot}BM$ since $s \in C_X(s)' \leq L'$.

It remains to exclude the second case, that is, that $2^2 \cong O_2(X \cap L) = U$. Now $U \leq V \leq C_K(s)$ where $V \in Syl_2(C_X(s))$ as in Lemma 6.4. As U is normalized by $C_K(s)$ we infer that $U = \langle s, t_2 \rangle$. On the other hand, by [5, page 104] all involutions in $O_2(X \cap L)$ are conjugate in X. This contradiction to Lemma 6.1 completes the proof. ∎

Proof of Theorem 1.1. This now follows from Propositions 5.6, 6.5, [12], and [25]. ∎

We remark that in [25] Wilson proved that M contains just one conjugacy class of maximal 7-local subgroups which is not mentioned in [5]. This additional 7-local subgroup was discovered by Chat Yin Ho [14] it is isomorphic to $7^2 : SL_2(7)$ and so does not contain a Sylow 7-subgroup of M.

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