

# SOME REMARKS IN THE FOURIER ANALYSIS

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To KIYOSHI NOSHIRO on his 60th Birthday

In this paper, I should like to add some remarks to my previous note titled "Some results in the Fourier analysis" (Nagoya Math. Journal, Vol. 27, 1966). At first, we shall show that the orthonormal set  $\exp [2 \pi i(m_1 x_1 + \cdots + m_n x_n)]$  is complete in the Hilbert space  $L^2$  over the unit cube  $E = \{(x_1, \dots, x_n); 0 \leq x_j \leq 1 (1 \leq j \leq n)\}$ , where the inner product  $\langle f, g \rangle$  for  $f, g \in L^2$  over  $E$  is defined by

$$\int \cdots \int_E f(x_1, \dots, x_n) \bar{g}(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

This means that if  $f \in L^2$  over  $E$  and

$$\langle f, \exp [2 \pi i(m_1 x_1 + \cdots + m_n x_n)] \rangle = 0$$

for any integral values  $m_1, \dots, m_n$ , then  $f = 0$  almost everywhere. To prove this, we define the set function

$$F(S) = \int \cdots \int_S f(u_1, \dots, u_n) du_1 \cdots du_n.$$

If  $S$  is any interval in  $E$ , then clearly  $F(S) = 0$ , in virtue of Lemma 2 in the paper cited above. Hence, if  $S$  is a closed set in  $E$ , accordingly if  $S$  is a measurable set in  $E$ , then  $F(S) = 0$ . Let  $E_n = \{(x_1, \dots, x_n); (x_1, \dots, x_n) \in E, f(x_1, \dots, x_n) \geq \frac{1}{n}\}$  and  $E'_n = \{(x_1, \dots, x_n); (x_1, \dots, x_n) \in E, f(x_1, \dots, x_n) \leq -\frac{1}{n}\}$ . Since  $0 = F(E_n) \geq \frac{1}{n} m(E_n)$  and  $0 = F(E'_n) \leq -\frac{1}{n} m(E'_n)$ , we obtain  $m(E_n) = m(E'_n) = 0$ . From this consideration, we have  $m(\{(x_1, \dots, x_n); (x_1, \dots, x_n) \in E, f(x_1, \dots, x_n) \neq 0\}) = 0$ . We get, therefore, in the usual way, the following

**THEOREM 3.** *Let  $f(x_1, \dots, x_n)$ ,  $g(x_1, \dots, x_n)$  be  $L^2$ -integrable in the unit cube  $E$  and set  $a(m_1, \dots, m_n) = \langle f, \exp [2 \pi i(m_1 x_1 + \cdots + m_n x_n)] \rangle$ ,  $b(m_1, \dots,$*

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$m_n) = \langle g, \exp [2 \pi i(m_1 x_1 + \dots + m_n x_n)] \rangle$ . Then we have

$$\int \cdots \int_{\mathbb{R}} f(x_1, \dots, x_n) \bar{g}(x_1, \dots, x_n) dx_1 \cdots dx_n = \sum_{m_1 \cdots m_n = -\infty}^{\infty} a(m_1, \dots, m_n) \bar{b}(m_1, \dots, m_n).$$

Next, we note that if

$$\frac{\partial^{p_1 + \dots + p_n}}{\partial^{p_1} x_1 \cdots \partial^{p_n} x_n} f(x_1, \dots, x_n) \quad (0 \leq p_j \leq 2)$$

are continuous and  $L$ -integrable over  $X$ , the whole  $n$ -dimensional Euclidean space, then

$$g(v_1, \dots, v_n) = \int \cdots \int_X e^{2 \pi i(v_1 u_1 + \dots + v_n u_n)} f(u_1, \dots, u_n) du_1 \cdots du_n$$

is  $L$ -integrable over  $X$ . Consequently, by Theorem 2 of the paper cited above, the Fourier-transform formula

$$f(x_1, \dots, x_n) = \int \cdots \int_X e^{-2 \pi i(v_1 x_1 + \dots + v_n x_n)} g(v_1, \dots, v_n) dv_1 \cdots dv_n$$

holds.

*Proof.* We take  $a_m$  such that

$$a_1 < a_2 < \dots < a_m \rightarrow \infty \quad (\text{as } m \rightarrow \infty),$$

and define

$$l_m(t) = a_m^2 - t)^{2n} \left( t - \frac{a_m^2}{2} \right)^{2n}$$

$$k_m(r) = \frac{1}{C} \int_{a^2 m/2}^r l_m(t) dt, \text{ where } C = \int_{a^2 m/2}^{a^2 m} l_m(t) dt,$$

$$h_m(x_1, \dots, x_n) = \begin{cases} 1 & x_1^2 + \dots + x_n^2 \leq \frac{a_m^2}{2} \\ 1 - k_m(x_1^2 + \dots + x_n^2) & \frac{a_m^2}{2} \leq x_1^2 + \dots + x_n^2 \leq a_m^2 \\ 0 & a_m^2 \leq x_1^2 + \dots + x_n^2, \end{cases}$$

and set

$$f_m(x_1, \dots, x_n) = f(x_1, \dots, x_n) h_m(x_1, \dots, x_n).$$

By partial integration,

$$\begin{aligned}
 g_m(v_1, \dots, v_n) &= \int \dots \int_X e^{2\pi i(v_1 u_1 + \dots + v_n u_n)} f_m(u_1, \dots, u_n) du_1 \dots du_n \\
 &= \int_{-a_m}^{a_m} \dots \int_{-a_m}^{a_m} \left\{ \left[ \frac{\exp [2\pi i(v_1 u_1 + \dots + v_n u_n)]}{2\pi i v_1} f_m(u_1, \dots, u_n) \right]_{-a_m}^{a_m} \right. \\
 &\quad \left. - \int_{-a_m}^{a_m} \frac{\exp [2\pi i(v_1 u_1 + \dots + v_n u_n)]}{2\pi i v_1} \frac{\partial}{\partial u_1} f_m(u_1, \dots, u_n) du_2 \dots du_n \right\} \\
 &= \dots \\
 &= \frac{(-1)^{q_1 + \dots + q_n}}{(2\pi i)^{q_1 + \dots + q_n}} \int_{-a_m}^{a_m} \dots \int_{-a_m}^{a_m} \frac{\exp [2\pi i(v_1 u_1 + \dots + v_n u_n)]}{v_1^{q_1} \dots v_n^{q_n}} \\
 &\quad \frac{\partial^{q_1 + \dots + q_n}}{\partial u_1^{q_1} \dots \partial u_n^{q_n}} f_m(u_1, \dots, u_n) du_1 \dots du_n,
 \end{aligned}$$

where  $q_j$  are taken such that  $q_j = 2$  if  $|v_j| \geq 1$  and  $q_j = 0$  if  $|v_j| < 1$ . Since

$$\frac{\partial^{p_1 + \dots + p_n}}{\partial x_1^{p_1} \dots \partial x_n^{p_n}} h_m(x_1, \dots, x_n)$$

are uniformly bounded for all  $m$ , we obtain

$$g_m(v_1, \dots, v_n) = O\left(\frac{1}{|v_1|^{q_1} \dots |v_n|^{q_n}}\right)$$

By the Lebesgue dominated convergence theorem, sending  $m \rightarrow \infty$ ,

$$g(v_1, \dots, v_n) = O\left\{\text{Min}\left(1, \frac{1}{v_1^2}\right) \dots \text{Min}\left(1, \frac{1}{v_n^2}\right)\right\},$$

from which we can infer that  $g(v_1, \dots, v_n)$  is  $L$ -integrable over  $X$ .

Finally I should like to express my thanks to the authors who presented the above problems, especially to Bochner, Hecke, Siegel, Takagi and Weyl.

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