FINITE DIMENSIONAL APPROXIMATION TO BAND LIMITED WHITE NOISE

TAKEYUKI HIDA and HISAO NOMOTO

To Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

1. Introduction. One of the authors discussed finite dimensional approximations to a white noise and a periodic Brownian motion with period 2π on the projective limit space of spheres ([2]). The group of unitary operators derived from the periodic white noise has *a pure point spectrum* which consists of all integers with countably infinite multiplicity. We also have much interest in the investigation of *a band limited white noise* which is another typical example having quite different spectral type. Indeed, the corresponding group of unitary operators has *a continuous spectrum* with countably infinite multiplicity.

A band limited white noise to the band from 0 to W is, as is well known, a Gaussian stationary stochastic process $X_W(t, \omega)$, $-\infty < t < \infty$, $\omega \in \Omega(P)$, which has the following spectral representation:

$$X_W(t) = \int_{-\pi W}^{\pi W} e^{it\lambda} dZ(\lambda), \tag{1}$$

where $dZ(\lambda)$ is a complex Gaussian random measure defined on $\mathscr{B}([-\pi W, \pi W])$, the smallest Borel field generated by all open subsets of $[-\pi W, \pi W]$, satisfying

$$EZ(\Delta) = 0, \ E|Z(\Delta)|^2 = |\Delta| \quad \text{(the Lebesgue measure of } \Delta) \tag{2}$$

and

$$Z(-\Delta) = \overline{Z(\Delta)}, \qquad \Delta \in \mathscr{B}([-\pi W, \pi W]).$$

The covariance function of $X_W(t)$ is given by the formula

$$\gamma(h) = E(X_W(t+h)\overline{X_W(t)}) = \frac{2}{|h|} \sin \pi |h| W. \tag{3}$$

For simplicity we always assume that $W = 1$ throughout this note.

In order to obtain a finite dimensional approximation to the process $X_W(t)$,

Received July 11, 1966.

we shall begin with the construction of a random measure $Z^{(n)}(\lambda)$ which approximates $dZ(\lambda)$ appeared in the expression (1). Our method is quite similar to what was used in the course of approximation to the periodic white noise (cf. [2, § 3]).

Having got the Fourier transform of $Z^{(n)}(\lambda)$

$$X^{(n)}(t) = \int_{-\pi}^{\pi} e^{it\lambda} Z^{(n)}(\lambda)\, d\lambda, \qquad -\infty < t \infty,$$

we shall show that the stochastic process $X^{(n)}(t)$ approches to a band limited white noise required to be approximated in the sense to be prescribed as follows: The process $X^{(n)}(t)$ determines a probability measure μ_n on the space of all continuous functions on R^1 with compact uniform topology. Appealing to Prokhorov's theorem [3], we shall prove that there exists a probability measure μ which is the weak limit of μ_n. This measure μ will turn out to be the same measure as the one derived from a band limited white noise to the band from 0 to 1.

2. The complex white noise with circular parameter

We shall first list some results obtained in [1] and [2] which will be needed for our present purpose.

Let S^n be the n-dimensional sphere with radius $\sqrt{n+1}$ and let $x^{(n+1)} = (x_1^{(n+1)}, \ldots, x_{n+1}^{(n+1)})$ be a point of S^n. Then $x^{(n+1)}$ can be expressed in the form

$$x_1^{(n+1)} = \sqrt{n+1} \prod_{i=1}^{n} \sin \theta_i,$$

$$x_k^{(n+1)} = \sqrt{n+1} \cos \theta_{k-1} \prod_{i=\kappa}^{n} \sin \theta_i, \quad 2 \le k \le n,$$

$$x_{n+1}^{(n+1)} = \sqrt{n+1} \cos \theta_n,$$

where $0 \le \theta_1 < 2\pi$, $0 \le \theta_i \le \pi$, $i = 2, 3, \ldots, n$. Let Ω_n be a subset of S^n defined by

$$\Omega_n = \{x^{(n+1)} \,;\, x^{(n+1)} \in S^n,\ 0 < \theta_i < \pi,\ i \ge 2\}$$

and let P_n be the restriction to $\mathscr{B}_n = \mathscr{B}(\Omega_n)$ of the uniform probability measure over S^n. Then we obtain a probability space $(\Omega, \mathscr{B}, P)$ as the projective limit of measure spaces $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$, $n = 1, 2, \ldots$ (see [1]).

Now we can introduce a flow $\{T_\lambda^{(2n)} \,;\, \lambda \text{ real}\}$ on $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$ defined by

$$(4)\qquad T_\lambda^{(2n)}(x^{(2n+1)})=\begin{bmatrix}1 & & & \\ & A_1(\lambda) & & 0\\ & & \ddots & \\ & 0 & & A_n(\lambda)\end{bmatrix}x^{(2n+1)},\quad x^{(2n+1)}=\begin{bmatrix}x_1^{(2n+1)}\\ \cdot\\ \cdot\\ \cdot\\ x_{2n+1}^{(2n+1)}\end{bmatrix},$$

where $A_k(\lambda)$'s are given by

$$A_k(\lambda)=\begin{bmatrix}\cos k\lambda & -\sin k\lambda\\ \sin k\lambda & \cos k\lambda\end{bmatrix},\ k=1,2,\ldots.$$

Since the flows $\{T_\lambda^{(2n)}\}$, $n=1,2,\ldots$, form a system of consistent flows, we can uniquely determine a flow $\{T_\lambda;\ \lambda \text{ real}\}$ (see [2]). The flow $\{T_\lambda\}$ is obviously a periodic flow with period 2π.

We are now in a position to define a finite dimensional approximation $Z^{(2n)}(\lambda,\ x^{(2n+1)})$ to the complex white noise $dZ(\lambda, x)$. Let us define unitary groups $\{U_\lambda;\ \lambda \text{ real}\}$ and $\{U_\lambda^{(2n)};\ \lambda \text{ real}\}$ by

$$(5)\qquad U_\lambda f(x)=f(T_\lambda x),\ \text{for } f\in L^2(\Omega,\ \mathscr{B},\ P),\ -\infty<\lambda<\infty,$$

and

$$(5')\quad U_\lambda^{(2n)}f(x^{(2n+1)})=f(T_\lambda^{(2n+1)}x^{(2n+1)}),\ \text{for } f\in L^2(\Omega_{2n},\ \mathscr{B}_{2n},\ P_{2n}),\ -\infty<\lambda<\infty,$$

respectively. Then it can be proved that U_λ and $U_\lambda^{(2n)}$ are strongly continuous in λ, λ real, and that both of them are periodic:

$$U_{\lambda+2\pi}=U_\lambda,\ \ U_{\lambda+2\pi}^{(2n)}=U_\lambda^{(2n)}.$$

Since $T_\lambda^{(2n)}x^{(2n+1)}$ together with $x^{(2n+1)}$ may be regarded as $(2n+1)$-dimensional vectors, we may consider scalar products such as $(x^{(2n+1)},\ a)$, $(T_\lambda^{(2n)}x^{(2n+1)},\ b)$, etc., where a and b are $(2n+1)$-dernensional vectors. Now let us take a particular $(2n+1)$-dimensional vector a such as

$$a=\left(\frac{1}{2\pi},\ \frac{1}{\pi},\ 0,\ \frac{1}{\pi},\ 0,\ \ldots,\ \frac{1}{\pi},\ 0\right).$$

A functional $f_a(x^{(2n+1)})$ defined by

$$f_a(x^{(2n+1)})=\frac{1+i}{2}(x^{(2n+1)},\ a)$$

belongs to $L^2(\Omega_{2n},\ \mathscr{B}_{2n},\ P_{2n})$. We can therefore apply $U_\lambda^{(2n)}$ to f_a. Define $Z^{(2n)}(\lambda)$ by

$$Z^{(2n)}(\lambda) = U_\lambda f_a(x^{(2n+1)}) + U_{-\lambda}\bar{f}_a(x^{(2n+1)}) \tag{6}$$

Then we have the following simple expression

$$\begin{aligned} Z^{(2n)}(\lambda) &= \frac{1}{2\pi} x_1^{(2n+1)} + \sum_{k=1}^{n} \frac{\cos k\lambda}{\pi} x_{2k}^{(2n+1)} - i \sum_{k=1}^{n} \frac{\sin k\lambda}{\pi} x_{2k+1}^{(2n+1)} \\ &= Z_1^{(2n)}(\lambda) - iZ_2^{(2n)}(\lambda), \qquad Z_1^{(2n)}(\lambda),\ Z_2^{(2n)}(\lambda) \text{ real.} \end{aligned} \tag{6'}$$

Note that $Z^{(2n)}(\lambda)$ and $Z_i^{(2n)}(\lambda)$, $i=1, 2$, can be regarded as random variables not only on $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$ but also on $(\Omega, \mathscr{B}, P)$.

PROPOSITION 1. *i) For any* $f \in L^2([-\pi, \pi])$

$$Z_i^{(2n)}(f) = \int_{-\pi}^{\pi} Z_i^{(2n)}(\lambda) f(\lambda)\, d\lambda, \quad i=1, 2,$$

belong to real $L^2(\Omega, \mathscr{B}, P)$, *and they converge to Gaussian random variables which we denote by* $Z_i(f)$, $i=1, 2$, *in* $L^2(\Omega, \mathscr{B}, P)$.

ii) For almost all $x \in \Omega$, *both* $Z_1(\varphi, x)$ *and* $Z_2(\varphi, x)$, $\varphi \in (\mathscr{D})_{[-\pi, \pi]}$, *are continuous linear functionals on* $(\mathscr{D})_{[-\pi, \pi]}$.

This proposition can be proved in a similar way to the discussions in [2, § 3] and the proof is omited.

Define $Z^{(2n)}(\Delta) = \int_\Delta Z^{(2n)}(\lambda)\, d\lambda$, then

$$EZ^{(2n)}(\Delta) = 0, \quad E(Z^{(2n)}(\Delta_1)\overline{Z^{(2n)}(\Delta_2)}) \to |\Delta_1 \cap \Delta_2| \qquad (n \to \infty) \tag{7}$$

and

$$Z^{(2n)}(-\Delta) = \overline{Z^{(2n)}(\Delta)}.$$

3. Approximation to a band limited white noise

Consider the Fourier transform of $Z^{(2n)}(\lambda)$, $-\pi \le \lambda \le \pi$:

$$X^{(2n)}(t, x^{(2n+1)}) = \int_{-\pi}^{\pi} e^{it\lambda} Z^{(2n)}(\lambda, x^{(2n+1)})\, d\lambda, \quad -\infty < t < \infty. \tag{8}$$

Since the relation (7) holds, $\{X^{(2n)}(t)\ ;\ t \text{ real}\}$ is a real valued second order stochastic process defined on $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$ (hence, on $(\Omega, \mathscr{B}, P)$).

PROPOSITION 2. *For any* t, $X^{(2n)}(t)$ *approaches to a random variable* $\tilde{X}(t)$ *of a band limited white noise in the sense of both mean square in* $L^2(\Omega, \mathscr{B}, P)$ *and almost sure* (P) *convergence.*

Proof. As was proved in [1], we can show that

$$(9)\qquad \lim_{n\to\infty} y_k^{(2n+1)} = \zeta_k, \qquad y_k^{(2n+1)} = x_{k-n-1}^{(2n+1)}, \qquad k=1, 2, \ldots .$$

exists almost surely. The collection $\{\zeta_k\}$ forms a system of independent Gaussian random variables with mean 0 and variance 1. Since $\sum_{k=-\infty}^{\infty} \left| \frac{\sin (t+k)\pi}{t+k} \right|^2 < \infty$ for every t, we can also prove that

$$(10)\qquad \lim_{n\to\infty} X^{(2n)}(t, x^{(2n+1)}) = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{\sin (t+k)\pi}{t+k} \zeta_k, \text{ a.e. } (P),$$

in a similar manner to [2, § 4].

We denote by $\tilde{X}(t)$ the right hand side of (10). Then $\tilde{X}(t)$, $-\infty < t < \infty$, is obviously a Gaussian process. On the other hand, the band limited white noise $X_1(t)$ ($W=1$) introduced by the formula (1) can be expressed in the form

$$(11)\qquad X_1(t) = \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{\sin (t+k)\pi}{t+k} \xi_k,$$

where $\{\xi_k\}$ is a system of independent standard Gaussian random variables. This shows that $\{X_1(t)\}$ and $\{\tilde{X}(t)\}$ are the same process. Consequently, almost sure convergence is proved.

The fact that $X^{(2n)}(t)$ converges to $\tilde{X}(t)$ strongly in $L^2(\Omega, \mathscr{B}, P)$ follows easily from Proposition 1, *i*).

COROLLARY. *Any finite dimensional distribution of the stochastic process* $\{X^{(2n)}(t)\}$ *converges to the finite dimensional distribution of* $\{X_1(t)\}$.

Under these preparations we shall finally show much stronger convergence of $X^{(n)}(t)$ to $X_1(t)$. By the expression (8) we see that $X^{(2n)}(t, x^{(2n+1)})$ is continuous in t for all $x^{(2n+1)} \in \Omega_{2n}$, which means $X^{(2n)}(t)$ determines a probability measure μ_n on the measurable space $(\mathbf{C}, \mathscr{B}_{\mathbf{C}})$, where $\mathbf{C}$ is the space of all continuous functions on R^1 and $\mathscr{B}_{\mathbf{C}}$ is the topological Borel field. The situation is the same for $X_1(t)$ and we denote by μ the derived probability measure from $X_1(t)$. Now we can state

THEOREM. *The measure* μ_n *converges to* μ *weakly.*

Proof. We have already proved that $\mu_n(E)$ tends to $\mu(E)$, as $n\to\infty$, for any cylinder set E of $\mathbf{C}$ (Corollary of Proposition 2). We shall now apply Prokhorov's theorem [3, Chapt. 2] to our discussions. We have

$$E|X^{(2n)}(t) - X^{(2n)}(s)|^2 = \frac{2}{\pi} \sum_{k=-n}^{n} \left| \frac{\sin (t+k)\pi}{t+k} - \frac{\sin (s+k)\pi}{s+k} \right|^2$$

since the system $\{y_k^{(2n+1)};\ -n\leq k\leq n\}$ forms an orthonormal basis of $(\Omega_{2n}, \mathscr{B}_{2n}, P_{2n})$. Observing the Fourier coefficients of $e^{it\lambda}-e^{is\lambda}$, we obtain

$$E|X^{(2n)}(t)-X^{(2n)}(s)|^2\leq C\int_{-\pi}^{\pi}|e^{it\lambda}-e^{is\lambda}|^2 d\lambda\leq C'|t-s|^2,$$

where C and C' are constants being independent of n, t, and s. Thus the assumptions of Prokhorov's theorem are satisfied, and hence our theorem is proved.

References

[1] T. Hida and H. Nomoto, Gaussian measure on the projective limit space of spheres. Proc. Japan Acad. 40, no. 5 (1964), 301-304.
[2] T. Hida, Finite dimensional approximations to white noise and Brownian motion. J. Math. Mech., (to appear).
[3] Yu. V. Prokhorov, Convergence of random processes and limit theorems in probability theory. Theory of Prob. and Appl. I, no. 2 (1956), 157-214 (English translation).