HYPERBOLIC MOTIONS

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To Professor KIYOSHI NOSHIRO for his 60 birthday

The observation by Poincaré that Möbius transformations in the complex plane can be lifted to a half-space raises the need to be able to handle motions in hyperbolic space of more than two dimensions by means of an analytic apparatus of not too forbidding complexity. In my experience the best way to do so is to be guided by analogies with the familiar twodimensional case. The purpose of this little paper is to collect a few formulas that the writer has found useful when working with certain hyperbolically invariant operators.

1. Vectors in $\mathbf{R}^n$ will be denoted by $x=(x_1, \ldots, x_n)$, the inner product by xy, and the norm by $|x|$. The reflection of x in the unit sphere S^{n-1} is denoted by $x^* = x/|x|^2$. We use the notation B for the unit ball in $\mathbf{R}^n$, and B^* for its exterior. The latter should rightly include ∞, but since we shall be concerned mainly with transformations that leave B invariant we need not pay any attention to this compactification.

The full group $\hat{M}$ of hyperbolic motions and reflections is generated by the reflections in spheres or planes orthogonal to S^{n-1}. The subgroup M obtained by an even number of reflections is the group of hyperbolic motions.

We use the notation $A(x)$ for the image of x under $A \in M$. We write $A'(x)$ for the Jacobian matrix at x and $|A'(x)|$ for the linear magnification. In other words, $dA(x) = A'(x)dx$ and $|dA(x)| = |A'(x)|\,|dx|$, the ratio being the same in all directions. Observe that $|\det A'(x)| = |A'(x)|^n$. The conformality implies furthermore that $|A'(x)|^{-1}A'(x)$ is an orthogonal matrix, and consequently ${}^tA'A' = |A'|^2$ (${}^tA'$ is the transpose, and the unit matrix is denoted by 1).

If 0 is a fixed point, $A(0)=0$, A' is a constant orthogonal matrix.

2. We consider a fixed y with $|y|>1$ and determine the reflection in the

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orthogonal sphere with center y. It is given by

$$A_y(x) = y + \frac{|y|^2 - 1}{|x-y|^2}(x-y), \tag{1}$$

and we observe that $A_y(0) = y^*$.

As y recedes to ∞ in a fixed direction the sphere flattens to a plane and the reflection sends x to $x - 2(xy)y^*$. We prefer to write this as a matrix multiplication

$$x \to (1 - 2\,Q(y))x \tag{2}$$

where

$$Q(y)_{ij} = \frac{y_i y_j}{|y|^2}. \tag{3}$$

Note that

$$Q(y)^2 = Q(y) \tag{4}$$

and

$$(1 - 2\,Q(y))^2 = 1. \tag{5}$$

We observe further that $|x|^{-2}(1 - 2\,Q(x))$ is the Jacobian of the reflection $x \to x^*$.

3. We shall use the notation (1) even when $|y| < 1$, although the geometric meaning is then not quite so obvious. There are two rather remarkable identities, namely

$$(1 - 2\,Q(y))A_y(x) = -A_{y^*}(x^*) \tag{6}$$

and

$$(1 - 2\,Q(y-x))A_y(x) = A_x(y). \tag{7}$$

They can be verified by direct computation. Curiously enough, the geometric interpretation is rather involved.

As our basic notation we shall introduce

$$T_y(x) = (1 - 2\,Q(y))A_{y^*}(x) = -A_y(x^*). \tag{8}$$

For $|y| < 1$ this represents a transformation in M which carries y to 0. If S has the same property $T_y S^{-1}$ leaves 0 fixed and is hence an orthogonal transformation. Thus the most general transformation in M (or $\hat{M}$) which carries y to 0 is of the form UT_y with orthogonal U.

4. From (6) and (8) we obtain

$$T_x(y) = -(1-2\,Q(y^*-x))(1-2\,Q(y))\,T_y(x), \tag{9}$$

and by symmetry we are led to the identity

$$(1-2\,Q(y^*-x))(1-2\,Q(y)) = (1-2\,Q(x))(1-2\,Q(x^*-y)). \tag{10}$$

A trivial consequence of (9) is the relation

$$|T_x y| = |T_y x|. \tag{11}$$

5. From (1) we obtain

$$A_y'(x) = \frac{|y|^2-1}{|x-y|^2}(1-2\,Q(x-y)) \tag{12}$$

and from (8)

$$T_y'(x) = \frac{1-|y|^2}{|y|^2|x-y^*|^2}(1-2\,Q(y))(1-2\,Q(x-y^*)). \tag{13}$$

In particular,

$$|T_y'(x)| = \frac{1-|y|^2}{|y|^2|x-y^*|^2}. \tag{14}$$

It should be observed that $|y|\,|x-y^*| = |x|\,|y-x^*|$ is symmetric (in two dimensions it corresponds to $|1-x\bar{y}|$ in complex notation). By (10) and (13) we have hence

$$\frac{T_y'(x)}{1-|y|^2} = \frac{{}^tT_x'(y)}{1-|x|^2}. \tag{15}$$

From (9) and (13) we can also derive the relation

$${}^tT_y'(x)\,T_y(x) = -|T_y'(x)|\,T_x(y). \tag{16}$$

6. It is useful to take advantage of the equation $|a^*-b^*| = |a-b|/|a|\,|b|$. With its aid we obtain from (1)

$$|A_y(u)-A_y(v)| = \frac{(|y|^2-1)|u-v|}{|u-y|\,|v-y|}.$$

and since reflection in a plane does not alter distances we have also

$$|T_y(u)-T_y(v)| = \frac{(1-|y|^2)|u-v|}{|y|^2|u-y^*|\,|v-y^*|}. \tag{17}$$

For $y = \infty$ this specializes to

$$|T_y(x)| = \frac{|x-y|}{|y|\,|x-y^*|}. \tag{18}$$

Another version of (18) is

$$1-|T_y(x)|^2 = \frac{(1-|x|^2)(1-|y|^2)}{|y|^2|x-y^*|^2} \tag{19}$$

and together with (14) we find

$$\frac{|T'_y(x)|}{1-|T_y(x)|^2} = \frac{1}{1-|x|^2}. \tag{20}$$

This is nothing else than the invariance of the hyperbolic metric.