

ON q -TH DERIVATIVE OF VECTOR BUNDLES

AKIKUNI KATO

To Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

(0.0) In the present note we shall be concerned with the improvement of fundamental definitions in higher order enumerative geometry which has been recently given by W. F. Pohl. Pohl's definition of q -th derivative of vector bundle is very complicated. We shall give a simpler and more reasonable definition of the q -th derivative of vector bundle in terms of sheaf theory and simplify the proofs in [P]. We shall also give a definition of higher order singularity of map.

1. The sheaf D_q

(1.1) Let (X, O_X) be an object of a category C of ringed spaces over ground field k . Ringed space over k is a pair (X, O_X) of topological space X and sheaf O_X of k -algebras over X . (cf. [G]).

The following are most important examples of the category C ;

- (i) the category of differentiable-manifolds; k is the field R of real numbers.
- (ii) the category of complex manifolds; k is the field C of complex numbers.
- (iii) the category of algebraic k -schemes (or varieties defined over k).

(1.2) Let F_1 be the sheaf of derivations of O_X over k , i.e.

$$F_1 = \{t \in \text{Hom}_k(O_X, O_X) \mid t(ab) = a \cdot t(b) + bt(a) \text{ for any } a, b \in O_X \\ \text{and } t(c) = 0 \text{ for any } c \in k\}$$

where $\text{Hom}_k(O_X, O_X)$ denotes the sheaf of k -homomorphisms of O_X into itself.

Then, F_1 is a sheaf of k -modules. Tensoring O_X with F_1 over k , we get a sheaf of O_X -modules $F = F_1 \otimes_k O_X$.

Although we may also consider F_1 as a sheaf of O_X -modules, we do not

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give O_X -modules structure to F_1 . Hence O_X -modules structure of F is given only by base extension.

(1.3) Let $T(F)$ be a tensor algebra of F over O_X , i.e. $T(F) = \bigoplus_{n \geq 0} F^{\otimes n}$. $T(F)$ is a sheaf of graded rings. Let p be a characteristic of a field k . Then, \mathfrak{A} is a sheaf of two-sided ideals of $T(F)$ generated by the elements:

$$t \otimes a - at \otimes 1 - t(a), (t_1 \otimes 1) \otimes (t_2 \otimes 1) - (t_2 \otimes 1) \otimes (t_1 \otimes 1) - [t_1, t_2] \otimes 1, (t \otimes 1)^{\otimes p} - t^p, \text{ where } t, t_1, t_2 \in F_1, a \in O_X \text{ and } [t_1, t_2] \text{ denotes Lie bracket of } t_1 \text{ and } t_2.$$

The quotient D of $T(F)$ by \mathfrak{A} is a sheaf of filtered rings. If we denote by D_q the q -th part of D , then $D_0 = O_X$, and we put $D_q/D_0 = F_q$ which is called the sheaf of germs of osculating vector fields of order q in [P]. We denote by $t_1 a_1 t_2 a_2 \cdots t_q a_q$ the canonical image in D of $(t_1 \otimes a_1) \otimes (t_2 \otimes a_2) \otimes \cdots \otimes (t_q \otimes a_q)$ and $t \cdot 1 = 1 \cdot t$ is also written simply by t .

(1.4) Let $S(F_1)$ be a symmetric algebra of F_1 over k , i.e. $S(F_1)$ is the quotient of tensor algebra $T(F_1)$ by the ideal $\{t \otimes u - u \otimes t, u \in F_1\}$. Then $S(F_1)$ is a sheaf of graded rings. We denote by $S^q(F_1)$ the q -th degree part of $S(F_1)$. There is a canonical homomorphism

$$S_q : F_1^{\otimes q} \rightarrow S^q(F_1).$$

A homomorphism $m : F_1^{\otimes q} \rightarrow D_q$ is defined by $m(t_1 \otimes t_2 \otimes \cdots \otimes t_q) = t_1 t_2 \cdots t_q$. Then for $q \geq 1$, we have $m(t_1 \otimes t_2 \otimes \cdots \otimes t_q - t_1 \otimes \cdots \otimes t_{i+1} \otimes t_i \otimes \cdots \otimes t_q) = t_1 \cdots t_{i-1} [t_i, t_{i+1}] t_{i+2} \cdots t_q D_{q-1}$, hence m induces a homomorphism

$$\varphi_q : S^q(F_1) \rightarrow D_q/D_{q-1} \text{ for } q \geq 1.$$

PROPOSITION (1.5) φ_q is an isomorphism, hence we have an exact sequence

$$(1.5.1) \quad 0 \rightarrow D_{q-1} \xrightarrow{I_q} D_q \rightarrow S^q(F_1) \rightarrow 0$$

where I_q is a canonical inclusion.

Proof (cf. [P]. Th. 2.1)

COROLLARY (1.6) If F_1 is locally free, then D_q is also locally free.

Proof. If F_1 is locally free, then $S^q(F_1)$ is locally free and it is proved by induction on q .

2. The sheaf $\mathcal{A}_q M$

DEFINITION (2.1) Let M be a sheaf of O_X -modules. The sheaf $\mathcal{A}_q M = D_q \otimes_{O_X} M$

is called the q -th derivative of M .

It is easily verified that this definition of q -th derivative of M is the same as Pohl's one (cf. [P]. § III. 1.). Δ_q is a functor from the category of sheaves of O_X -modules into itself.

ASSUMPTION (2.2) Assume that in category C , F_1 is always locally free. This assumption is satisfied, for example, if C is (i), (ii) of (1.1) or the category of non-singular algebraic varieties.

PROPOSITION (2.3) Under the assumption (2.2), there is an exact sequence for $q \geq 1$,

$$(2.3.1) \quad 0 \rightarrow \Delta_{q-1}M \xrightarrow{I_q^M} \Delta_q M \rightarrow \mathbf{S}^q(F_1) \otimes_{O_X} M \rightarrow 0$$

where $I_q^M = I_q \otimes i_M$ and i_M is identity of M .

Proof. By (1.5.1), there is an exact sequence

$$0 \rightarrow D_{q-1} \rightarrow D_q \rightarrow \mathbf{S}^q(F_1) \rightarrow 0.$$

Tensoring M , we get an exact sequence

$$D_{q-1} \otimes M \rightarrow D_q \otimes M \rightarrow \mathbf{S}^q(F_1) \otimes_{O_X} M \rightarrow 0.$$

On the other hand, $\mathbf{S}^q(F_1)$ is locally free, and so is O_X -flat. Therefore (2.3.1) is exact (cf. [B] I, 2 Prop. 4).

PROPOSITION (2.4) Under the assumption (2.2), the exact sequence (2.3.1) is an exact functor of M . In particular Δ_q is an exact functor.

Proof. If F_1 is locally free, D_q and $\mathbf{S}^q(F_1)$ are also locally free.

(2.5) We define a homomorphism $\varphi_1 : F = F_1 \otimes_k O_X \rightarrow O_X$ by $\varphi_1(t \otimes a) = t(a)$, $t \in F_1$, $a \in O_X$. A homomorphism $\varphi_n : F^{\otimes n} \rightarrow O_X$ is defined to be $\varphi_n = \varphi_1 \circ (i_F \otimes \varphi_{n-1})$ where i_F is identity homomorphism of F . Let $\varphi_0 : O_X \rightarrow O_X$ be the identity. From $\{\varphi_n\}_{n=0}$, we get a homomorphism

$$\varphi : T(F) \rightarrow O_X.$$

It is clear that $\text{Ker } \varphi \supset \mathfrak{A}$. φ induces a homomorphism

$$\rho : D = T(F)/\mathfrak{A} \rightarrow O_X.$$

We denote by ρ_q the restriction of ρ on D_q , i.e. $\rho_q : D_q \rightarrow O_X$. Since $D_q \otimes_{O_X} O_X = D_q$, $\Delta_q O_X^N = D_q^N$.

Definition (2.5.1) We denote by ω_q the map $\rho_q^N = \rho_q + \cdots + \rho_q : \mathcal{A}_q O_X^N \rightarrow O_X^N$.
The restriction of ω_q on $\mathcal{A}_{q-1} O_X^N$ is ω_{q-1} , i.e.

$$(2.5.2) \quad \omega_{q-1} = \omega_q \circ I_q^N$$

because it holds $\rho_{q-1} = \rho_q \circ I_q$ (cf. [P]. Prop. 3.9_q)

Let $\omega_0 : \mathcal{A}_0 O_X^N = O_X^N \rightarrow O_X^N$ be the identity. We get also

$$(2.5.3) \quad \omega_1 \circ I_0^N = \text{identity},$$

i.e. ω_1 defines a splitting of (2.3.1) for $q=1$ (cf [P]. Prop. 3.8).

3. Higher order singularities of maps

(3.1) Let $G_{n,r}$ be the Grassmanian manifold of $(r-1)$ -dimensional projective subspaces in n -dimensional projective space. Given a suitable structure sheaf O_G on $G_{n,r}$, $G_{n,r}$ may be considered as an object of category (i), (ii) or (iii) of (1.1). $G_{n,r}$ is imbedded in some projective space by Plücker coordinates $p = (\dots, p_{i_1 \dots i_r}, \dots)$. Plücker coordinates $p_{i_1 \dots i_r}$ satisfy the well-known relations

$$(3.1.1) \quad \sum_{k=1}^{r+1} (-1)^k p_{i_1 \dots i_{r-1} j_k} p_{j_1 \dots \widehat{j_k} \dots j_{r+1}} = 0$$

for any $i_1, \dots, i_{r-1}, j_1, \dots, j_{r+1}$ such that $1 \leq i_1 < \dots < i_{r-1} \leq n$, $1 \leq j_1 < \dots < j_{r+1} \leq n$.

(3.2) Let $V_{i_1 \dots i_r}$ be the set of points $p = (\dots, p_{i_1 \dots i_r}, \dots)$ of $G_{n,r}$ such that $p_{i_1 \dots i_r} \neq 0$. Then $V_{i_1 \dots i_r}$ is an (affine) open subset of $G_{n,r}$, and $\{V_{i_1 \dots i_r}\}$ is an open covering of $G_{n,r}$. Let $\mathcal{A}_{i_1 \dots i_r}$ be the restriction of the sheaf O_G^r on $V_{i_1 \dots i_r}$. Let $v_1^{(i)}, \dots, v_r^{(i)}$ and $v_1^{(j)}, \dots, v_r^{(j)}$ be the canonical basis of $\mathcal{A}_{i_1 \dots i_r}$ and $\mathcal{A}_{j_1 \dots j_r}$ respectively. Now we define a homomorphism

$$(3.2.1) \quad f_{(j)}^{(i)} = f_{j_1 \dots j_r}^{i_1 \dots i_r} : \mathcal{A}_{i_1 \dots i_r} | V_{i_1 \dots i_r} \cap V_{j_1 \dots j_r} \rightarrow \mathcal{A}_{j_1 \dots j_r} | V_{i_1 \dots i_r} \cap V_{j_1 \dots j_r},$$

by $f_{(j)}^{(i)}(v_\mu^{(i)}) = \sum_{\nu=1}^r a_{\nu\mu}^{(i)(j)} v_\nu^{(j)}$, where $a_{\nu\mu}^{(i)(j)} = (-1)^{r-\nu} \frac{p_{i_1 \dots \widehat{i_\nu} \dots i_r j_\mu}}{p_{i_1 \dots i_r}} \in \Gamma(V_{i_1 \dots i_r}, O_G)$.

We shall show $f_{(k)}^{(j)} f_{(j)}^{(i)} = f_{(k)}^{(i)}$ on $V_{i_1 \dots i_r} \cap V_{j_1 \dots j_r} \cap V_{k_1 \dots k_r}$ for any $i_1, \dots, i_r, j_1, \dots, j_r, k_1, \dots, k_r$.

$$\begin{aligned} f_{(k)}^{(j)} f_{(j)}^{(i)}(v_\nu^{(i)}) &= \sum_{\mu=1}^r a_{\nu\mu}^{(i)(j)} f_{(k)}^{(j)}(v_\mu^{(j)}) \\ &= \sum_{\lambda=1}^r a_{\nu\lambda}^{(i)(j)} \left(\sum_{\mu=1}^r a_{\mu\lambda}^{(j)(k)} v_\mu^{(k)} \right) = \sum_{\lambda=1}^r \left(\sum_{\mu=1}^r a_{\nu\mu}^{(i)(j)} a_{\mu\lambda}^{(j)(k)} \right) v_\lambda^{(k)} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\lambda=1}^r \frac{(-1)^{r-\nu}}{\hat{p}_{i_1 \dots i_r} \cdot \hat{p}_{j_1 \dots j_r}} (\hat{p}_{i_1 \dots \hat{i}_\nu \dots i_r j_1} \cdot (-1)^{r-1} \\
 &\quad \hat{p}_{j_2 \dots j_r k_\lambda} + \hat{p}_{i_1 \dots \hat{i}_\nu \dots i_r j_2} \cdot (-1)^{r-2} \hat{p}_{j_1 j_3 \dots j_r k_\lambda} + \\
 &\quad \cdot \cdot \cdot + \hat{p}_{i_1 \dots \hat{i}_\nu \dots i_r j_r} \cdot \hat{p}_{j_1 \dots j_{r-1} k_\lambda}) v_\lambda^{(k)} \\
 &= \sum_{\lambda=1}^r \frac{(-1)^{r-\nu} \hat{p}_{i_1 \dots \hat{i}_\nu \dots i_r k_\lambda} \cdot \hat{p}_{j_1 \dots j_r}}{\hat{p}_{i_1 \dots i_r} \cdot \hat{p}_{j_1 \dots j_r}} v_\lambda^{(k)} \quad (\text{by (3.1.1)}) \\
 &= \sum_{\lambda=1}^r a_{\nu\mu}^{(i)(k)} v_\lambda^{(k)} = f_{(k)}^{(i)}(v_{(\nu)}^{(i)}).
 \end{aligned}$$

It is clear that $f_{(i)}^{(i)}$ = identity, hence $f_{(j)}^{(i)} = (f_{(i)}^{(j)})^{-1}$ on $V_{i_1 \dots i_r} \cap V_{j_1 \dots j_r}$, i.e. the homomorphism $f_{(j)}^{(i)}$ of (3.2.1) is an isomorphism. Then we can patch $\mathcal{A}_{i_1 \dots i_r}$ together with the isomorphism (3.2.1) and we obtain a sheaf of O_G -modules $\mathcal{C}_{n,r}$ over $G_{n,r}$ (cf [G] Chap. 0, (3.3.1)). $\mathcal{C}_{n,r}$ is locally free of rank r .

Remark (3.2.2) $\mathcal{C}_{n,r}$ is the sheaf of germs of cross-sections of the universal r -plane bundle over $G_{n,r}$.

(3.3) We shall define a homomorphism σ of $\mathcal{C}_{n,r}$ into O_G^{n+1} . Firstly we define a homomorphism

$$\sigma_{i_1 \dots i_r} : \mathcal{A}_{i_1 \dots i_r} \simeq \mathcal{C}_{n,r} | V_{i_1 \dots i_r} \rightarrow O_G^{n+1} | V_{i_1 \dots i_r}.$$

Let u_0, \dots, u_n be the canonical basis of O_G^{n+1} . Then $\sigma_{i_1 \dots i_r}$ is defined by $\sigma_{i_1 \dots i_r}(v_\nu^{(i)}) = \sum_{\mu=0}^n b_{\nu\mu} \cdot u_\mu$ where $b_{\nu\mu} = (-1)^{r-\nu} \frac{\hat{p}_{i_1 \dots \hat{i}_\nu \dots i_r}}{\hat{p}_{i_1 \dots i_r}} \in \Gamma(V_{i_1 \dots i_r}, O_G)$.

By virtue of (3.1.1), it is easily verified that

$$\sigma_{i_1 \dots i_r} | V_{i_1 \dots i_r} \cap V_{j_1 \dots j_r} = \sigma_{j_1 \dots j_r} | V_{i_1 \dots i_r} \cap V_{j_1 \dots j_r}.$$

Patching $\sigma_{i_1 \dots i_r}$ together, we get a homomorphism

$$(3.3.1) \quad \sigma : \mathcal{C}_{n,r} \rightarrow O_G^{n+1}.$$

(3.4) Let f be a morphism of X into $G_{n,r}$. By virtue of the homomorphism σ in (3.3.1), we obtain a homomorphism

$$f^*(\sigma) : f^*(\mathcal{C}_{n,r}) \rightarrow O_X^{n+1} = f^*(O_G^{n+1})$$

(where f^* denotes the reciprocal image) and consequently a homomorphism

$$(3.4.1) \quad \Delta_q f^*(\sigma) : \Delta_q f^*(\mathcal{C}_{n,r}) \rightarrow \Delta_q(O_X^{n+1}).$$

DEFINITION (3.5) We say that the morphism $f : X \rightarrow G_{n,r}$ is *non-singular of order q at $x \in X$* , if the homomorphism

$$\omega_q \circ \Delta_q f^*(\sigma) : \Delta_q f^*(\mathcal{C}_{n,r}) \rightarrow O_X^{n+1},$$

which is the composition of $\Delta_q f^*(\sigma)$ of (3.4.1) and ω_q of (2.5.1), is injective at X .

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