PERFECT PICARD SET OF POSITIVE CAPACITY

KIKUJI MATSUMOTO

To Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

1. In our previous paper [1], we proved that any Cantor set E with successive ratios ξ_n satisfying the condition

$$\xi_{n+1} = o(\xi_n^2) \tag{1}$$

is a Picard set, that is, each single-valued meromorphic function with E as the set of essential singularities takes on every value infinitely often in any neighborhood of each singularity with the possible exception of two values. But the capacity of this Cantor set is zero since the condition (1) implies the condition

$$\sum \frac{\log \xi_n^{-1}}{2^n} = +\infty, \tag{2}$$

which is necessary and sufficient for E to be of capacity zero. Then there remains an interesting problem: Is there a perfect Picard set of positive capacity? The aim of this paper is to give the positive answer to this problem. We shall prove in the below the following theorem, an amelioration of the above result.

THEOREM. *If the successive ratios ξ_n of a Cantor set E satisfy the conditions*

$$\xi_{n+1} = o(\xi_n^\lambda) \quad \textit{with } \lambda \geqq \sqrt{2} \textit{ and } n = 1, 2, \dots, \tag{3}$$

and

$$\xi_{pq+1} = o(\xi_{pq}^2) \quad \textit{with odd } p \textit{ and } q = 1, 2, \dots, \tag{4}$$

then it is a Picard set.

It is easy to see that there exists a Cantor set whose successive ratios ξ_n satisfy the conditions of the theorem and further the condition

$$\sum \frac{\log \xi_n^{-1}}{2^n} < +\infty.$$

Received February 28, 1966.

Such a Cantor set is a perfect Picard set of positive capacity.

2. We shall consider the Riemann sphere $\sum$ with radius 1/2 touching the w-plane at the origin. For any two points w and w' in the w-plane we denote by $[w, w']$ the chordal distance between them, that is,

$$[w, w'] = \begin{cases} \dfrac{|w-w'|}{\sqrt{(1+|w|^2)(1+|w'|^2)}} & \text{if } w \neq \infty \text{ and } w' \neq \infty \\ \dfrac{1}{\sqrt{1+|w|^2}} & \text{if } w' = \infty. \end{cases}$$

Further we denote by $C(w; \delta)$ $(\delta > 0)$ the spherical disc with center w and with chordal radius δ. For the proof of our theorem we shall need the following lemmas which were given in [1].

LEMMA 1. *Let $w = f(z)$ be a single-valued meromorphic function on an annulus $1 \leqq |z| \leqq e^{\mu}$ $(\mu > 0)$. If $f(z)$ takes there no value in a spherical disc $C(w_0; \delta)$, then there exists a positive constant A_δ depending only on δ such that the diameter of the image of $|z| = e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $A_\delta e^{-\mu/2}$ for sufficently large μ.*

In particular, if δ is sufficiently close to 1, that is, the complementary spherical disc $C(-1/w_0; d)$ of $C(w_0; \delta)$, $d = \sqrt{1-\delta^2}$, has a radius sufficiently small, we have

$$A_\delta < Bd,$$

where B is a positive constant.

COROLLARY. *Let $w = f(z)$ be a single-valued regular function in an annulus $1 < |z| < e^{\mu}$ omitting two values 0 and 1. Then there exists a positive constant A not depending on μ and $f(z)$ such that the diameter of the image of $|z| = e^{\mu/2}$ by $f(z)$ with respect to the chordal distance is dominated by $Ae^{-\mu/2}$ for sufficiently large μ.*

Let E be a Cantor set on the closed interval I_0: $[-1/2, 1/2]$ on the real axis of the z-plane with successive ratios ξ_n, $0 < \xi_n = 2\ell_n < 2/3$. Defining the Cantor set E, we repeat successively to exclude an open segment from the middle of another segment and there remain 2^n segments of equal length $\Pi_{k=1}^n \ell_k$ after we repeat n times beginning with the interval I_0. We denote these segments by $I_{n,k}$ $(n = 1, 2, \ldots;\ k = 1, 2, \ldots, 2^n)$ and denote by $S_{n,k}$

($n=1, 2, \ldots ; k=1, 2, \ldots, 2^n$) the following annuli on the complementary domain Ω of E:

$$S_{n,k}=\{z;(\Pi_{k=1}^{n}\ell_k)(1-\ell_{n+1})<|z-z_{n,k}|<(\Pi_{k=1}^{n-1}\ell_k)(1-\ell_n)/2\},$$

where $z_{n,k}$ is the middle point of $I_{n,k}$. The harmonic modulus μ_n of $S_{n,k}$ is greater than $\log(2/3\xi_n)$. We map $S_{n,k}$ conformally onto the annulus $1<|\eta|<e^{\mu_n}$ and consider the inverse image $\Gamma_{n,k}$ of the circle $|\eta|=e^{\mu_n/2}$ on $S_{n,k}$. Supposing that $S_{n,k}$ encloses $S_{n+1,2k-1}$ and $S_{n+1,2k}$, we denote by $\Delta_{n,k}$ the triply connected domain bounded by three curves $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$.

LEMMA 2. *Let the successive ratios ξ_n satisfy the condition*

$$\lim_{n\to\infty}\xi_n=0 \tag{5}$$

and let $w=f(z)$ be a single-valued regular function in Ω which omits two values 0 and 1 and has E as the set of essential singularities. Then for any small $\delta>0$, there exists an infinite number of $\Delta_{n,k}$ such that the images of the three boundary components $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$ are contained completely in the three discs $C(0;\delta)$, $C(1;\delta)$ and $C(\infty;\delta)$ one by one, where we assume that these three discs are disjoint by pairs, and $f(z)$ takes on each ualue outside the union of these three discs once and only once in $\Delta_{n,k}$.

In the proof of this lemma it is shown that holds good the following fact, which we state as a lemma.

LEMMA 2′. *Let $C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ be the discs containing the images of $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. If their radii are less than $\delta/3$, then either* (1) *$C_{n,k}$, $C_{n+1,2k-1}$ and $C_{n+1,2k}$ contain the origin, the point $w=1$ and the point at infinity one by one, so that they are contained in $C(0;\delta)$, $C(1;\delta)$ and $C(\infty;\delta)$ one by one and $f(z)$ takes on each value outside the union of $C(0;\delta)$, $C(1;\delta)$ and $C(\infty;\delta)$ once and only once in $\Delta_{n,k}$, or* (2) *any one of them cannot be disjoint from the union of the other two, so that there is a disc with radius at most the sum of their radii which contains the image of $\Delta_{n,k}$.*

LEMMA 3. *For any doubly connected domain contained in $\Delta_{n,k}\cup\Gamma_{n+1,2k-1}\cup\Delta_{n+1,2k-1}$ such that one connected component of its complement contains the circles $\Gamma_{n,k}$ and $\Gamma_{n+1,2k}$ and the other contains the circles $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$, its harmonic modulus is dominated by* $\log(32/\xi_{n+1})$.

3. *Proof of the theorem.* Contrary to our assertion, let us suppose that there exists a single-valued meromorphic function $f(z)$ in the complementary domain Ω of E which has E as the set of essential singularities and has three exceptional values at an essential singularity $\zeta \in E$, where we may assume that these values are 0, 1 and ∞. Since our argument given in the below is applicable locally, it will not bring any loss of generality if we give a contradiction under the stronger assumption that $f(z)$ omits the values 0, 1 and ∞ in Ω.

Now we take $\delta > 0$ so small that the discs $C(0; 2\delta)$, $C(1; 2\delta)$ and $C(\infty; 2\delta)$ are disjoint by pairs and n_0 so large that $Ae^{-\mu_n/2} < \delta/3$ for any $n \geqq n_0$, where A is the constant in Corollary of Lemma 1. By Lemma 2 there is a $\Delta_{n,k} (n \geqq n_0)$ whose three boundary components are mapped into $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; \delta)$ one by one, where we may assume that the boundary curve $\Gamma_{n+1,2k-1}$ of $\Delta_{n,k}$ is mapped into $C(\infty; \delta)$. We consider the quadruply connected domain $D = \Delta_{n,k} \cup \Gamma_{n+1,2k-1} \cup \Delta_{n+1,2k-1}$. The images of the boundary curves $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$ of D are contained in some spherical discs C and C' with radius $Ae^{-\mu_{n+1}/2} < \delta/3$ respectively, and we see that C and C' are contained in $C(\infty, 2\delta)$. Thus we can find a positive $d \leqq 2\delta$ such that the disc $C(\infty; d)$ contains C and C' and $f(z)$ takes on each value outside the union of the three discs $C(0; \delta)$, $C(1; \delta)$ and $C(\infty; d)$ once and only once in D. Hence we see that the ring domain in D which is mapped onto the annulus R: $2 < |w| < \sqrt{1-d^2}/d$ under $w = f(z)$, separates the boundary curves $\Gamma_{n,k}$ and $\Gamma_{n+1,2k}$ of D from the boundary curves $\Gamma_{n+2,4k-3}$ and $\Gamma_{n+2,4k-2}$. By Lemma 3 we have

$$\text{har. mod. of } R = \log(\sqrt{1-d^2}/2d) \leqq \log(32/\xi_{n+1}).$$

Since $d \leqq 2\delta < \pi/6$, we have the estimate that

$$d \geqq (\sqrt{1-(\pi/6)^2}/64)\xi_{n+1} = \xi_{n+1}/M. \tag{6}$$

This implies that C or C', say C' must intersect the disc $[w, \infty] \geqq \xi_{n+1}/M = m$[1].

4. By the conditions (3) and (4) we may assume that n is so large that

$$(1 + 3A'M)B\xi_{n+q}^{(\lambda-1)/2} < 3AM \quad \text{with } q = 1, 2, \ldots, \tag{7}$$

$$(12A'M)^2 \xi_{n+q+1} \leqq \xi_{n+q}^{\lambda} \quad \text{with } q = 1, 2, \ldots \tag{8}$$

and

[1] To this place the argument is entirely the same as that in the proof given in [1], and so we have given only a sketch. For details see the proof in [1].

$$(12A'M)^2\xi_{n+q+1} \leqq \xi^2_{n+q} \quad \text{if } n+q=pr \text{ for some } r, \tag{9}$$

where $A'=\sqrt{3}/2A$ and B is the constant of Lemma 1. Here it does not bring any loss of generality to assume that $6A'M>1$. Then from (8) we have

$$Ae^{-\mu_{n+q+1}/2} < A'\sqrt{\xi}_{n+q+1} \leqq (1/12A'M)^{1+\lambda+\cdots+\lambda^{q-1}}\sqrt{\xi^{\lambda^q}_{n+1}}/12M, \tag{10}$$

so that the image of any $\Gamma_{n+q+1,j}$ with $q \geqq 2$ is contained in some spherical disc $C_{n+q+1,j}$ with radius less than $(1/2)^{q-1}\xi_{n+1}/12M=(1/2)^{q-1}m/12$.

Suppose that $n+1=pr$ for some r. Then from (9) we have

$$\text{the radius of } C' = Ae^{-\mu_{n+2}/2} < A'\sqrt{\xi}_{n+2} \leqq m/12. \tag{11}$$

Suppose that $n+1 \neq pr$ but the domain $\Delta_{n+2,4k-2}$ has the second property of Lemma 2′. Then by (10) the radius of the disc containing its image is less than $A'(\sqrt{\xi}_{n+2}+2\sqrt{\xi}_{n+3}) \leqq \sqrt{\xi}_{n+1}/6M$. Since the image of $\Delta_{n+1,2k-1}$ is contained in a disc with radius less than $A'(\sqrt{\xi}_{n+1}+2\sqrt{\xi}_{n+2}) \leqq (A'+1/6M)\sqrt{\xi}_{n+1}$ and since $\Delta_{n+1,2k-1}$ and $\Delta_{n+2,4k-2}$ have $\Gamma_{n+2,4k-2}$ as the common boundary, we see that the image of $\Delta_{n+1,2k-1} \cup \Gamma_{n+2,4k-2} \cup \Delta_{n+2,4k-2}$, consequently that of its subdomain $S_{n+2,4k-2}$ is contained in a disc with radius less than $(A'+1/3M)\sqrt{\xi}_{n+1}$. Applying Lemma 1 to $S_{n+2,4k-2}$ and $\Gamma_{n+2,4k-2}$ we have

$$\text{the radius of } C' \leqq ((1+3A'M)B\sqrt{\xi}_{n+1}/3M)e^{-\mu_{n+2}/2}, \tag{12}$$

so that by (7) and (8) the radius of C' is less than $m/12$. Thus in both cases the radius of C' is less than $m/12$. Then C' cannot contain any one of the origin, the point $w=1$ and the point at infinity and hence the domain $\Delta_{n+2,4k-2}$ must have the second property of Lemma 2′. Since the radius of any $C_{n+q+1,j}$ with $q \geqq 2$ is less than $(1/2)^{q-1}m/12$, the image of $\Delta_{n+2,4k-2}$ is covered by a disc with radius less than $m/12+2(m/24) \leqq m/4$. This disc intersects the disc $[w, \infty] \geqq m$, so that it must lie outside $C(\infty\,;\, m/2)$. Next consider the domains $\Delta_{n+3,8k-5}$ and $\Delta_{n+3,8k-4}$. Then by the same reason their images are covered by discs with radii less than $m/8$ and lying outside $C(\infty\,;\, m/4)$. By induction we can conclude that the image of any domain $\Delta_{n+q+1,j}$ lying inside of the simple closed curve $\Gamma_{n+2,4k-2}$ is contained in a spherical disc with radius less than $m/2^{q+1}$ and lying outside $C(\infty\,;\, m/2^q)$. It follows that, in the interior of $\Gamma_{n+2,4k-2}$, $f(z)$ takes on values only in a spherical disc with radius less than

$$\sum_{q=1}^{\infty} m/2^{q+1} < m/2. \tag{13}$$

By means of a linear transformation we can consider that $f(z)$ is bounded there; this contradicts our assumption that $f(z)$ has an essential singularity at every point of E.

Now we note that there remains only the possible case where $n+1 \neq pr$ and the domain $\Delta_{n+2,4k-2}$ has the first property of Lemma 2′. Then this $\Delta_{n+2,4k-2}$ satisfies the same condition imposed on $\Delta_{n,k}$ at the first part of our proof. Hence on applying the above argument to this $\Delta_{n+2,4k-2}$ we see that $n+3 \neq pr$. Repeat this consideration again and again. Then we observe that $n \equiv pr$ (mod. 2) for any sufficiently large r, but this is absurd because p is odd. Thus the assumption that $f(z)$ omits the values 0, 1 and ∞ in Ω denies all possible cases, and we must conclude that $f(z)$ cannot omit all these three values. The theorem is now established.

REFERENCES

[1] K. Matsumoto: Existence of perfect Picard sets, Nagoya Math. J. 27 (1966), 213-222.

Mathematical Institute
Nagoya University