CLASSES OF EQUATIONS OF THE TYPE $y^2 = x^3 + k$ HAVING NO RATIONAL SOLUTIONS

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The equation $y^2 = x^3 + k$, k an integer, has been discussed by many authors. Mordell [1] has found many classes of k values for which the equation has no integral solutions. Fueter [2], Mordell [3] and Chang [4] have found classes of k values for which the equation has no rational solutions. The following two theorems exhibit two more sets of conditions which give rise to classes of k values for which the corresponding equations have no rational solutions.

THEOREM 1. *The equation $y^2 = x^3 + k$ has no rational solutions if k is a square free positive integer and*

(1) $k \equiv 2$ *or* $3 \pmod 4$, $k \equiv -3 \pmod 9$,
i.e., $k \equiv 6$ *or* $15 \pmod{36}$,

(2) $3 \nmid H$, *H the class number of* $R(\sqrt{k})$,

(3) $U \equiv 3$ *or* $6 \pmod 9$ *where* (T, U) *is the fundamental solution of the Pellian equation*
$Y^2 - kX^2 = 1$,

(4) $3 \nmid h$, *h the class number of* $R\left(\sqrt{-\frac{1}{3}k}\right)$.

(5) *the integer solutions of* $p^2 + \frac{k}{3}q^2 = 3^{2h}$ *when* $h \equiv 1 \pmod 3$, *do not satisfy* $q \equiv \pm 1 \pmod 9$, *and when* $h \equiv -1 \pmod 3$, *do not satisfy* $q \equiv \pm 2\left(\frac{k}{3}\right)^2$ $\pmod 9$.

THEOREM 2. *The equation $y^2 = x^3 + k$ has no rational solutions if k is a square free positive integer and*

(1′) $k \equiv 5 \pmod 8$ and $k \equiv -3 \pmod 9$,
i.e., $k \equiv -3 \pmod{72}$,

(2′) $3 \nmid H$, *H the class number of* $R(\sqrt{k})$,

(3′) $U \equiv 3$ *or* $6 \pmod 9$, *U the least positive value of q satisfying the Pellian equation*

Received November 24, 1965.

$p^2 - kq^2 = +4,$

(4′) $3 \nmid h$, *h the class number of* $R\left(\sqrt{-\frac{1}{3}k}\right)$.

(5′) α, β *and* γ, δ *the respective integer solutions of the equations* $\frac{1}{4}\left(\alpha^2 + \frac{1}{3}k\beta^2\right) = 2^h$, $\frac{1}{4}\left(\gamma^2 + \frac{1}{3}k\delta^2\right) = 3^{2h}$ *satisfy the conditions:*

(a) (i) $\alpha \not\equiv 0 \pmod 9$ *when h is odd,*

(ii) $\beta \not\equiv 0 \pmod 9$ *when h is even;*

(b) *when* $h = 3n+1$, $\left\{\alpha \pm \beta \frac{1}{3}k^2\right\}\delta \not\equiv \pm 2 \pmod 9$ *and* $\delta \not\equiv \pm 2 \pmod 9$,

when $h = 3n-1$, $\left\{\alpha\left(\frac{1}{3}k\right) \pm \beta\right\}\delta \not\equiv \pm 2 \pmod 9$ *and* $\delta \not\equiv \pm 4\left(\frac{k}{3}\right)^2 \pmod 9$ *were the signs are all independent of each other.*

Proof of Theorem 1. The set of conditions used in Theorem 1 arises from a theorem proved by Mordell [3] upon replacing his condition (3), in which he assumes that $U \not\equiv 0, \pm 1 \pmod 9$, by the condition (3) as shown in the statement of Theorem 1. Hence it suffices to prove that at that point of the argument where Mordell [3] obtains a contradiction by imposing the conditions $U \not\equiv 0, \pm 1 \pmod 9$ it is possible to obtain a contradiction by imposing instead the conditions $U \equiv 0 \pmod 3$ and $U \not\equiv 0 \pmod 9$ (i.e., $U \equiv 3$ or $6 \pmod 9$). Upon referring to the paper of Mordell [3] one sees that it is enough to show that the equation

(6) $$Y + \sqrt{k}Z^3 = (T \pm U\sqrt{k})(A + B\sqrt{k})^3$$

cannot be solved in rational integers Y, Z, A and B if $(Y, k) = 1$ and $U \equiv 3$ or $6 \pmod 9$.

Upon equating coefficients in (6) one obtains the two equations

(7) $$Z^3 = \pm AU(A^2 + 3kB^2) + TB(3A^2 + kB^2), \text{ and}$$

(8) $$Y = TA(A^2 + 3kB^2) \pm UkB(3A^2 + kB^2).$$

Upon taking residues modulo 3 in equation (7) one obtains $Z \equiv \pm UA \pmod 3$. Since it is being assumed that $U \equiv 0 \pmod 3$ it follows that $Z \equiv 0 \pmod 3$. Again, taking residues modulo 3 in equation (8) one obtains $Y \equiv TA \pmod 3$. Since $(Y, k) = 1$ it follows that $A \not\equiv 0 \pmod 3$ and $T \not\equiv 0 \pmod 3$. Hence $A^3 \equiv \pm 1 \pmod 9$. Next, taking residues modulo 9 in equation (7) one obtains

$$0 \equiv \pm U + 3TB\left(A^2 + \frac{k}{3}B^2\right) \pmod 9. \tag{9}$$

If $B \equiv 0 \pmod 3$ then $3TB\left(A^2 + \frac{k}{3}B^2\right) \equiv 0 \pmod 9$ which implies $U \equiv 0 \pmod 9$ contrary to the assumption on U. If $B \not\equiv 0 \pmod 3$ then $B^2 \equiv 1 \pmod 3$. Since $k \equiv -3 \pmod 9$ it follows that $\frac{k}{3} \equiv -1 \pmod 3$. Since $A \not\equiv 0 \pmod 3$ it follows that $A^2 \equiv 1 \pmod 3$. Hence upon assuming $B \not\equiv 0 \pmod 3$ one finds that $A^2 + \frac{k}{3}B^2 \equiv 0 \pmod 3$ so that once again $3TB\left(A^2 + \frac{k}{3}B^2\right) \equiv 0 \pmod 9$. Thus one obtains the contradiction $U \equiv 0 \pmod 9$ also in this case.

Proof of Theorem 2. The set of conditions used in Theorem 2 arises from a theorem proved by Chang [4] upon replacing his condition (3), in which he assumes that $U \not\equiv 0 \pmod 3$ and $U \not\equiv \pm 2 \pmod 9$ by the condition (3′) as shown in the statement of Theorem 2. The Pellian equation $p^2 - kq^2 = -4$ need not enter the discussion of the theorem proved by Chang [4] or Theorem 2 since this equation is insoluble whenever $k \equiv 0 \pmod 3$. It suffices to prove that at that point of the argument where Chang [4] obtains a contradiction by imposing the conditions $U \not\equiv 0 \pmod 3$ and $U \not\equiv \pm 2 \pmod 9$ it is possible to obtain a contradiction by imposing instead the conditions

$$U \equiv 0 \pmod 3 \text{ and } U \not\equiv 0 \pmod 9 \text{ (i.e., } U \equiv 3 \text{ or } 6 \pmod 9)\text{)}.$$

Upon referring to the paper of Chang [4] one sees that it is enough to show that the equation

$$Y + Z^3\sqrt{k} = \left(\frac{1}{2}T \pm \frac{1}{2}U\sqrt{k}\right)\left(\frac{1}{2}A + \frac{1}{2}B\sqrt{k}\right)^3 \tag{10}$$

cannot be solved in rational integers Y, Z, A and B if $(Y, k) = 1$ and $U \equiv 3$ or $6 \pmod 9$. Here (T, U) is the fundamental solution of the Pellian equation $p^2 - kq^2 = +4$.

Upon equating coefficients in (10) one obtains the two equations

$$16Z^3 = \pm AU(A^2 + 3kB^2) + TB(3A^2 + kB^2), \text{ and} \tag{11}$$

$$16Y = TA(A^2 + 3kB^2) \pm UkB(3A^2 + kB^2). \tag{12}$$

Upon taking residues modulo 3 in equation (11) one obtains $Z \equiv \pm UA \pmod 3$. Since it is being assumed that $U \equiv 0 \pmod 3$ it follows that $Z \equiv 0 \pmod 3$.

Again, taking residues modulo 3 in equation (12) one obtains $Y \equiv TA \pmod 3$. Since $(Y, k) = 1$ it follows that $A \not\equiv 0 \pmod 3$ and $T \not\equiv 0 \pmod 3$. Hence $A^3 \equiv \pm 1 \pmod 9$. Next, taking residues modulo 9 in equation (11) one obtains a contradiction in the form $U \equiv 0 \pmod 9$, just as in the proof of Theorem 1.

It seems natural to ask whether it is possible to make any progress when one assumes $k \equiv 1 \pmod 8$ and simultaneously $k \equiv -3 \pmod 9$ i.e., $k \equiv 33 \pmod{72}$. If one parallels the work of Chang [4] it is found that the equation

$$Y^2 - kZ^6 = X^3 \tag{13}$$

can be obtained. The symbols X, Y and Z have the meanings ascribed to them by Chang [4] and the conditions $(Y, Z) = (X, Z) = 1$ obtain. Upon assuming k to be square free one also obtains $(Y, k) = 1$. Since $k \equiv 1 \pmod 8$ both odd and even values for X are conceivable. If $X \equiv 1 \pmod 2$ then the argument proceeds exactly as in Chang [4], provided (2) through (5) of Chang [4] (or (2′) through (5′) of Theorem 2) are assumed. Hence in these two cases one can conclude that there are no solutions of equation (13) with $X \equiv 1 \pmod 2$. It may therefore now be assumed that $X \equiv 0 \pmod 2$. Upon factorizing the lefthand side of equation (13) one obtains the ideal equation

$$[Y + Z^3\sqrt{k}][Y - Z^3\sqrt{k}] = [X]^3. \tag{14}$$

Let A be the greatest common divisor of the two ideals $[Y + Z^3\sqrt{k}]$ and $[Y - Z^3\sqrt{k}]$. Then it can be shown that $A|[2]$. To prove this fact it is enough to show that $2 \in A$, since $A|[2]$ can equivalently be expressed by saying that A includes (as a set of algebraic integers from the field $R(\sqrt{k})$) [2]. By the definition of A one has

$$\begin{aligned} A &= ([Y + Z^3\sqrt{k}],\ [Y - Z^3\sqrt{k}]) \\ &= [Y + Z^3\sqrt{k}k,\ Y - Z^3\sqrt{k}]. \end{aligned} \tag{15}$$

It will suffice to prove the existence of rational integers a, b, c and d having the properties

$$2 = \left(\frac{a + b\sqrt{k}}{2}\right)\left(\frac{Y + Z^3\sqrt{k}}{2}\right) + \left(\frac{c + d\sqrt{k}}{2}\right)\left(\frac{Y - Z^3\sqrt{k}}{2}\right), \tag{16}$$

$$a \equiv b \pmod 2, \quad c \equiv d \pmod 2. \tag{17}$$

The form for the general integer of $R(\sqrt{k})$ follows from the assumption $k \equiv 1 \pmod 4$. Upon equating coefficients on both sides of equation (16) and simplifying, one obtains

(18) $$(a+c)Y+(b-d)kZ^3=4, \text{ and}$$

(19) $$(b+d)Y+(a-c)Z^3=0.$$

Equation (19) can be satisfied by putting $a=c$ and $b=-d$. Then equation (18) becomes

(20) $$aY+bkZ^3=2.$$

Now since $X\equiv 0 \pmod 2$ by assumption, it is necessary to have $Y\equiv Z\equiv 1 \pmod 2$. Then it follows that $Y\equiv kZ^3\equiv 1 \pmod 2$ from which it follows that if (a, b) is to be a solution of equation (20) then $a\equiv b \pmod 2$ is necessary. This last condition is in accord with equation (17). Equation (20) is a linear diophantine equation in the two quantities a and b and has solutions in a and b since $(Y, kZ^3)=1|2$. Finally, since $a\equiv b \pmod 2$ is required by equation (20) the previously imposed conditions $a=c$ and $b=-d$ imply that $b\equiv d \pmod 2$. Hence it follows that it is possible to find rational integers a, b, c and d satisfying equations (16) and (17) and so $A|[2]$ as stipulated.

It will be of use in the sequel to know the canonical decomposition of the ideal [2] in the field $R(\sqrt{k})$. Since it is being assumed that $k\equiv 1 \pmod 4$ it follows (Theorem 872, page 172, Landau [5]) that the discriminant Δ of $R(\sqrt{k})$ is given by $\Delta=k\equiv 1 \pmod 8$. Hence Δ is a quadratic residue modulo 8. From Theorem 879, page 178, Landau [6] with $p=2$ it follows that $[2]=PQ$ where $P=[2, R+\omega]$ and $Q=[2, R+\omega']$ for a suitable rational integer R. Here $\omega=\frac{1+\sqrt{k}}{2}$ and $\omega'=\frac{1-\sqrt{k}}{2}$. Also since $2\nmid\Delta$ it follows from Theorem 880, page 180, Landau [7] that $P\neq Q$. P and Q are prime ideals.

It can be shown that one can choose the prime ideal factors of [2] as $P=[2, \omega]$ and $Q=[2, \omega']$. Upon writing $PQ=[2, \omega][2, \omega']=[4, 2\omega, 2\omega', \omega\omega']$ one sees that 4, 2ω, $2\omega'$ and $\omega\omega'$ are integral (algebraic) multiples of 2 and so $[2]|PQ$. The element $\omega\omega'$ has the value $\frac{1-k}{4}$ and since $k\equiv 1 \pmod 8$ it follows that $\omega\omega'$ is an even rational integer. Also $2=2\omega+2\omega'$ so that $PQ|[2]$. Hence $PQ=[2]$.

The next step is to determine under what conditions P and Q are principal ideals. In order that P and Q be principal ideals it is necessary and sufficient that the number 2 have a non-trivial representation of the form

$$2 = \left(\frac{a + b\sqrt{k}}{2}\right)\left(\frac{u + v\sqrt{k}}{2}\right) \tag{21}$$

where a, b, u and v are rational integers satisfying the conditions $a \equiv b \pmod 2$, $u \equiv v \pmod 2$. The term non-trivial refers to the requirement that

$$\frac{a + b\sqrt{k}}{2} \text{ and } \frac{u + v\sqrt{k}}{2} \text{ not be units of } R(\sqrt{k}).$$

From the ideal equation corresponding to equation (21) it follows that one can identify P with $\left[\frac{a + b\sqrt{k}}{2}\right]$ and Q with $\left[\frac{u + v\sqrt{k}}{2}\right]$. Now it is known that $N(P) = N(Q) = 2$, and so, using the fact that $N([\beta]) = |N(\beta)|$ where β is any integer of $R(\sqrt{k})$, one sees that the two equations

$$|a^2 - kb^2| = 8 \tag{22}$$

$$|u^2 - kv^2| = 8 \tag{23}$$

must be satisfied. Since $x^2 - ky^2 = +8$ is insoluble whenever $k \equiv 0 \pmod 3$, equations (22) and (23) become

$$a^2 - kb^2 = -8, \tag{24}$$

$$u^2 - kv^2 = -8. \tag{25}$$

Upon equating coefficients on both sides of equation (21) one obtains the two equations

$$au + bvk = 8 \tag{26}$$

$$av + bu = 0. \tag{27}$$

If one multiplies equation (26) by v and substitutes for av from equation (27) it is found, using equation (25), that $b = v$. Hence also $u = -a$ and thus equation (21) becomes

$$2 = \left(\frac{a + b\sqrt{k}}{2}\right)\left(\frac{-a + b\sqrt{k}}{2}\right). \tag{28}$$

It is seen that, since $k \equiv 1 \pmod 4$, the parity restrictions on a, b, u and v must be met if equations (24) and (25) are to be satisfied.

Since $k \equiv 1 \pmod 4$ and since $Y \equiv Z^3 \equiv 1 \pmod 2$ it follows that $\frac{Y + Z^3\sqrt{k}}{2}$ and $\frac{Y - Z^3\sqrt{k}}{2}$ are integers of $R(\sqrt{k})$. In other words $[2] \mid [Y + Z^3\sqrt{k}]$ and $[2] \mid [Y - Z^3\sqrt{k}]$. Putting this fact together with the previous result that $A \mid [2]$ shows that $A = [2]$. From equation (14), using the fact that $X \equiv 0 \pmod 2$

one obtains the equation

$$\left[\frac{Y+Z^3\sqrt{k}}{2}\right]\left[\frac{Y-Z^3\sqrt{k}}{2}\right]=[2]\left[\frac{X}{2}\right]^3 \tag{29}$$

where the two ideals on the left-hand side of equation (29) are relatively prime. Upon using the unique factorization of ideals in an algebraic number field, one obtains the two equations

$$\left[\frac{Y+Z^3\sqrt{k}}{2}\right]=I_1D_1^3, \tag{30}$$

$$\left[\frac{Y-Z^3\sqrt{k}}{2}\right]=I_2D_2^3, \tag{31}$$

Where I_1, I_2, D_1 and D_2 are ideals in $R(\sqrt{k})$ which satisfy the conditions $(I_1, I_2)=[1]$,

$$I_1I_2=[2],\ (D_1, D_2)=[1] \text{ and } D_1D_2=\left[\frac{X}{2}\right].$$

If it is now assumed that the Pellian equation $a^2-kb^2=-8$ can be solved, it follows that the ideals I_1 and I_2 are principal ideals in every case, according to remarks made previously. Then from equations (30) and (31) it follows that D_1^3 and D_2^3 are also principal ideals. Finally, the assumption $3\nmid H$ leads one to conclude that D_1 and D_2 are principal ideals. Thus, in particular, one can write $I_1=\left[\frac{a+b\sqrt{k}}{2}\right]$ and $D_1=\left[\frac{c+d\sqrt{k}}{2}\right]$. From equation (30) one obtains the equation

$$\left[\frac{Y+Z^3\sqrt{k}}{2}\right]=\left[\frac{a+b\sqrt{k}}{2}\right]\left[\frac{c+d\sqrt{k}}{2}\right]^3. \tag{32}$$

From equation (32) one obtains the equation

$$\frac{Y+Z^3\sqrt{k}}{2}=\varepsilon\left(\frac{a+b\sqrt{k}}{2}\right)\left(\frac{c+d\sqrt{k}}{2}\right)^3 \tag{33}$$

where ε is a unit of the field $R(\sqrt{k})$. It follows that one can write $\frac{Y-Z^3\sqrt{k}}{2}$ in the form

$$\frac{Y-Z^3\sqrt{k}}{2}=\varepsilon\left(\frac{a-b\sqrt{k}}{2}\right)\left(\frac{c-d\sqrt{k}}{2}\right)^3 \tag{34}$$

and a corresponding equation in ideals would be

$$\left[\frac{Y-Z^3\sqrt{k}}{2}\right]=\left[\frac{a-b\sqrt{k}}{2}\right]\left[\frac{c-d\sqrt{k}}{2}\right]^3. \tag{35}$$

From equations (31) and (35) one obtains the equation

$$I_2 D_2^3 = \left[\frac{a-b\sqrt{k}}{2}\right]\left[\frac{c-d\sqrt{k}}{2}\right]^3. \tag{36}$$

From equation (36) one has $\left[\frac{c-d\sqrt{k}}{2}\right] \mid D_2$ for if there were a prime ideal R with the properties $R \mid \left[\frac{c-d\sqrt{k}}{2}\right]$ and $R \nmid D_2$ then one would necessarily have $R^3 \mid I_2$, which is impossible since $I_2 \mid [2]$. In the same way, one finds that $D_2 \mid \left[\frac{c-d\sqrt{k}}{2}\right]$ since the conditions on $\left[\frac{a-b\sqrt{k}}{2}\right]$ make it impossible to have the cube of a prime ideal dividing $\left[\frac{a-b\sqrt{k}}{2}\right]$. Hence $D_2 = \left[\frac{c-d\sqrt{k}}{2}\right]$ and $I_2 = \left[\frac{a-b\sqrt{k}}{2}\right]$. Since one now has $I_1 I_2 = \left[\frac{a+b\sqrt{k}}{2}\right]\left[\frac{a-b\sqrt{k}}{2}\right] = [2]$, the two possibilities $I_1 = [1]$ and $I_1 = [2]$ cannot arise.

If one parallels the treatment of Mordell [3] the following equations result in those cases where the unit cannot be totally absorbed

$$\frac{Y+Z^3\sqrt{k}}{2} = \left(\frac{T \pm U\sqrt{k}}{2}\right)\left(\frac{a+b\sqrt{k}}{2}\right)\left(\frac{C+D\sqrt{k}}{2}\right)^3, \tag{37}$$

$$C^2 - kD^2 = -2X. \tag{38}$$

In those situations where total absorption of the unit factor is possible, equation (38) still applies but equation (37) is replaced by the equation

$$\frac{X+Z^3\sqrt{k}}{2} = \left(\frac{a+b\sqrt{k}}{2}\right)\left(\frac{C+D\sqrt{k}}{2}\right)^3. \tag{39}$$

From equation (39) one obtains, upon equating coefficients, the equation

$$8Y = aC(C^2 + 3kD^2) + bkD(3C^2 + kD^2). \tag{40}$$

Upon taking residues modulo 9 in equation (40) it is found, using the fact that $C \not\equiv 0 \pmod 3$, that $Y \equiv \pm a \pmod 9$. Now if it is assumed that $b \equiv 0 \pmod 3$ then the equation $a^2 - kb^2 = -8$ forces the condition $a^2 \equiv 1 \pmod 9$. Thus $Y^2 \equiv 1 \pmod 9$ and upon referring back to equation (13) it can be seen that $Z \equiv 0 \pmod 3$ is necessary. Upon equating coefficients of $\sqrt{k}$ in equation (39) one obtains the equation

$$8Z^3 = aD(3C^2 + kD^2) + bC(C^2 + 3kD^2). \tag{41}$$

Upon taking residues modulo 9 in equation (41) it is found that $b \equiv 0 \pmod 9$ is required. Thus one cannot find rational integers Y, Z, C and D which satisfy equation (39) if it is assumed that $b\equiv 0 \pmod 3$ and simultaneously $b\not\equiv 0 \pmod 9$.

From equation (37) one obtains, upon equating coefficients of k, the equation

$$16Z^3=(Ta\pm Ubk)(3C^2+kD^2)D + (Tb\pm Ua)(C^2+3kD^2)C. \tag{42}$$

In equation (42) it is enough to consider the positive sign, upon replacing b by $-b$, D by $-D$ and leaving a and C unchanged. This replacement has the effect of changing Y to $-Y$. Hence one can replace equation (42) by the equation

$$16Z^3=(Ta+Ubk)(3C^2+kD^2)D + (Tb+Ua)(C^2+3kD^2)C. \tag{43}$$

Upon taking residues modulo 9 in equation (43) one obtains the relation

$$-2Z^3\equiv \pm(Tb+Ua) \pmod 9. \tag{44}$$

With the assumptions on U and b it follows that $Z\equiv 0 \pmod 3$ so that one would require $Tb+Ua\equiv 0 \pmod 9$.

The following result has been established:

THEOREM 3. *The equation $y^2=x^3+k$ has no rational solutions if k is a square free positive integer and if the following conditions obtain:*

(a) $k\equiv 1 \pmod 8$ *and* $k\equiv -3 \pmod 9$,

i.e., $k\equiv 33 \pmod{72}$,

(b) *the conditions* (2′) *through* (5′) *of Theorem* 2,

(c) *the Pellian equation* $X^2-kY^2=-8$ *is soluble and possesses a solution* (a,b) *for which* $b\equiv 0 \pmod 3$ *and* $b\not\equiv 0 \pmod 9$,

i.e., $b\equiv 3$ *or* $6 \pmod 9$,

(d) $Tb+Ua\not\equiv 0 \pmod 9$.

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