SOME RESULTS ON FINITE GROUPS WHOSE ORDER CONTAINS A PRIME TO THE FIRST POWER

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In memory of TADASI NAKAYAMA

The author discussed questions of the type treated here at various times with his friend Tadasi Nakayama. There had been plans of a collaboration between Nakayama and him in an effort to broaden our knowledge of the part of the character theory on which this present work is based. Nakayama's untimely death destroyed the hope of such a collaboration. I wish to dedicate this paper to his memory.

§ 1. Introduction

In a previous investigation [1], the author has studied finite groups $\mathfrak{G}$ of an order $g=pg_0$ where p is a prime and g_0 an integer not divisible by p. This work has been continued by H. F. Tuan [5]. Let t denote the number of conjugate classes of $\mathfrak{G}$ which consist of element of order p. Tuan dealt with the groups $\mathfrak{G}$ for which $t\leqq 2$ and which have a faithful representation of degree less than $p-1$. We shall assume here that $t\geqq 3$.** We shall also suppose that $\mathfrak{G}$ does not have a normal subgroup of order p. We state here two results. We shall show (Corollary, Theorem 1) that if χ is a faithful irreducible character of $\mathfrak{G}$ of degree n which has $T>1$ conjugates over the field of the g_0-th roots of unity, then

$$n>\frac{1}{3}T^{-3/4}p^{5/4}.$$

In Theorem 2, we assume that $\mathfrak{G}$ has an irreducible faithful representation of degree $n<p-1$. It is then shown that

$$p\leqq t^3-t+1.$$

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** Some remarks communicated by H. F. Tuan to the author twenty years ago have been helpful.

Thus for a fixed t, only finitely many primes p are possible. In a later paper, S. Hayden and the author will study small values of t and give further extensions of the results.

§ 2. Preliminaries

Let $\mathfrak{G}$ be a finite group of order $g = pg_0$ where p is a fixed prime and where g_0 is an integer not divisible by p. Let $\mathfrak{P}$ be a p-Sylow subgroup of $\mathfrak{G}$. The centralizer $\mathfrak{C}(\mathfrak{P})$ and the normalizer $\mathfrak{N}(\mathfrak{P})$ of $\mathfrak{P}$ then have the form

$$(2.1) \qquad \mathfrak{C} = \mathfrak{C}(P) = \mathfrak{P} \times \mathfrak{V}, \ \mathfrak{N} = \mathfrak{N}(P) = \langle\ (\mathfrak{P}),\ M\rangle.$$

Here, $\mathfrak{V}$ is a group of order v prime to p and M is an element whose order m over $\mathfrak{C}(\mathfrak{P})$ divides $p-1$. If we set

$$(2.2) \qquad p - 1 = mt,$$

t is the number of conjugate classes of $\mathfrak{G}$ which contain elements of order p. Distribute the irreducible characters of $\mathfrak{V}$ into classes $F_0, F_1, \ldots, F_{l-1}$ of characters associated in $\mathfrak{N}$. It is shown in [1] that $\mathfrak{G}$ has l p-blocks $B_0, B_1, \ldots, B_{l-1}$ of full defect, and we have a one-to-one correspondence $F_\lambda \to B_\lambda$. Let θ_λ be a character belonging to F_λ and suppose that F_λ consists of τ_λ characters, i.e. that the inertial group of θ_λ has index τ_λ in $\mathfrak{N}$. Then $\tau_\lambda \mid m$; we set

$$(2.3) \qquad m = m_\lambda \tau_\lambda, \ t_\lambda = \tau_\lambda t$$

so that

$$(2.2^*) \qquad p - 1 = m_\lambda t_\lambda, \ (\lambda = 0, \ldots, l-1).$$

The degree of θ_λ will be denoted by $f_\lambda = \theta_\lambda(1)$.
As shown in [1], B_λ consists of m_λ "non-exceptional" characters $\zeta_0^{(\lambda)}, \zeta_1^{(\lambda)}, \ldots, \zeta_{m_\lambda-1}^{(\lambda)}$ and t_λ exceptional characters $\chi_1^{(\lambda)}, \chi_2^{(\lambda)}, \ldots \chi_{t_\lambda}^{(\lambda)}$. The values of these characters for p-singular elements of $\mathfrak{G}$ can be given explictly, only certain $\pm$ signs remain undetermined. Let ρ denote a primitive p-th root of unity and let c denote a primitive root (mod p). We form the Gauss periods of length m_λ

$$(2.4) \qquad \omega_k^{(\lambda)} = \sum_\nu{}' \rho^{c^\nu}$$

where ν ranges over the integers, $k, k + t_\lambda, k + 2t_\lambda, \ldots, k + (m_\lambda - 1)t_\lambda$. Clearly,

$\omega_k^{(\lambda)}$ remains unchanged, if k is changed (mod t_λ). If P is a fixed generator of $\mathfrak{P}$ and if $V \in \mathfrak{V}$, we have with $\theta^M(V) = \theta(M^{-1}VM)$

$$\chi_j^{(\lambda)}(PV) = \varepsilon^{(\lambda)} \sum_{\mu=0}^{\tau_\lambda - 1} \theta_\lambda^{M^\mu}(V) \omega_{j+\mu t}^{(\lambda)}, \tag{2.5}$$

$$\zeta_i^{(\lambda)}(PV) = \varepsilon_i^{(\lambda)} \sum_{\mu=0}^{\tau_\lambda - 1} \theta_\lambda^{M^\mu}(V) \tag{2.6}$$

with $\varepsilon^{(\lambda)}$, $\varepsilon_i^{(\lambda)} = \pm 1$; cf. [1] I, Theorem 4.

We take B_0 as the principal p-block of $\mathfrak{G}$. Then $\theta_0 = 1$, $\tau_0 = 1$, i.e. $t_0 = t$, $m_0 = m$. We shall choose here the notation such that $\zeta_0^{(0)} = 1$ and that $\varepsilon_i^{(0)} = 1$ for $i = 0, 1, \ldots, a-1$ and $\varepsilon_i^{(0)} = -1$ for $i = a, a+1, \ldots, m-1$.

Set $\omega_k^{(0)} = \eta_k$. The η_k are the Gauss periods of length m. It is seen easily that

$$\eta_i = \sum_{\nu=0}^{\tau_\lambda - 1} \omega_{i+\nu t}^{(\lambda)}. \tag{2.7}$$

As is well known, $\eta_1, \eta_2, \ldots, \eta_t$ form a $\mathbf{Z}$-basis for the ring of algebraic integers of the field $\mathbf{Q}(\eta_1) \subseteq \mathbf{Q}(\rho)$; *) we have

$$\sum_{i=1}^{t} \eta_i = -1. \tag{2.8}$$

With each η_i, the conjugate complex number $\bar{\eta}_i$ appears in $\{\eta_1, \eta_2, \ldots, \eta_t\}$. We shall use the notation

$$\bar{\eta}_i = \eta_{i'}. \tag{2.9}$$

We also give some formulas for the multiplication of the η_i. These can be proved easily directly, but we apply a group theoretical method.

Let $\mathfrak{M}_{p,t}$ denote the metacyclic group of order $p(p-1)/t$ defined as group with generators P, M with the relations

$$P^p = 1,\ M^m = 1,\ M^{-1}PM = P^{c^t}.$$

Then $\mathfrak{M}_{p,t}$ satisfies our conditions for $\mathfrak{G}$ and $t = (p-1)/m$ is the number of conjugate classes of elements of order p. Here, B_0 is the only p-block. The non-excepetional characters can be identified with those of $\mathfrak{M}_{p,t}/\langle P\rangle$, i.e. of a cyclic group of order m. For suitable choice of the notation, we have $\zeta_k^{(0)} = \zeta^k$, $(0 \leqq k < m)$, ζ a character of degree 1. The characters $\chi_i^{(0)}$ have degree

*) We use the notation $\mathbf{Q}$ for the field of rational numbers and the notation $\mathbf{Z}$ for the ring of integers.

m, $\varepsilon^{(0)}=1$. The product $\chi_i^{(0)}\bar{\chi}_j^{(0)}$ will contain ζ^k, if and only if $\chi_j^{(0)}\zeta^k=\chi_i^{(0)}$. Taking the element P, we see that this is so, if and only if $i=j$. Hence we may set

$$\chi_i^{(0)}\bar{\chi}_j^{(0)}=\sum_{r=1}^{t}c_{ijr}\chi_r^{(0)}+\delta_{ij}\sum_{k=0}^{m-1}\zeta^k$$

with integral $c_{ijr}\geqq 0$ and Kronecker δ_{ij}. For the element P, this yields

$$\eta_i\bar{\eta}_j=\sum_{r=1}^{t}c_{ijr}\eta_r+m\delta_{ij}, \tag{2.10}$$

while for the element 1, we find

$$\sum_{r=1}^{t}c_{ijr}=m-\delta_{ij}. \tag{2.11}$$

Since c_{ijr} is the multiplicity of the principal character in $\chi_i^{(0)}\bar{\chi}_j^{(0)}\bar{\chi}_r^{(0)}$, we see that

$$c_{ijr}=c_{irj}=c_{rij'}. \tag{2.12}$$

§ 3. Characters of $\mathfrak{N}$

The results stated in § 2 apply in particular to the group $\mathfrak{N}$ in (2.1). We shall write here b_λ instead of B_λ. Since $\mathfrak{P}\lhd\mathfrak{N}$, no p-block of defect 0 occurs for $\mathfrak{N}$.

The irreducible characters of $\mathfrak{B}$ have the form $\theta_\lambda^{M^\mu}$. They may be considered as characters of $\mathfrak{C}=\mathfrak{P}\times\mathfrak{B}$. Let (ρ) denote the linear character of $\mathfrak{C}$ defined by

$$(\rho)(P^iV)=\rho^i$$

for $V\in\mathfrak{B}$. If the element M is chosen suitably, we may assume that

$$(\rho)^M=(\rho)^{c^t}. \tag{3.1}$$

For any $j\in\mathbf{Z}$, the character $(\rho)^{c^j}\theta_\lambda$ of $\mathfrak{C}$ has m associates in $\mathfrak{N}$. It follows that this character induces an irreducible character

$$((\rho)^{c^j}\theta_\lambda)^*=\xi_j^{(\lambda)} \tag{3.2}$$

of $\mathfrak{N}$. Using (3.1) we see without difficulty that

$$\xi_j^{(\lambda)}|\mathfrak{C}=\sum_{\mu=0}^{\tau_\lambda-1}\theta_\lambda^{M^\mu}\sum_\nu(\rho)^{c^\nu} \tag{3.3}$$

where in the inner sum, ν ranges over all members of a residue system mod

$p-1$ for which $\nu \equiv j+\mu t \pmod{t_\lambda}$. Comparison of (3.3) with (2.5), (2.6) shows that $\xi_0^{(\lambda)}, \xi_1^{(\lambda)}, \ldots, \xi_{t_\lambda-1}^{(\lambda)}$ are the t_λ exceptional characters in b_λ. The signs $\varepsilon^{(\lambda)}$ here are $+1$. It is now also clear that $\xi_j^{(\lambda)}$ remains unchanged, if j is changed mod t_λ. The degree of $\xi_j^{(\lambda)}$ is $m\theta_\lambda(1)=mf_\lambda$.

We next form the irreducible character θ_λ^* of $\mathfrak{N}$ induced by the character θ_λ of $\mathfrak{V}$. If ψ is an irreducible constituent of θ_λ^*, necessarily

$$\psi \mid \mathfrak{C} = e \sum_{\nu=0}^{\tau_\lambda-1} \theta_\lambda^{M^\nu}$$

with integral $e>0$. Comparing this with (2.5), (2.6), we see that ψ is a non-exceptional character of b_λ and that $e=1$. Hence ψ has degree $\tau_\lambda f_\lambda$ and it appears only once in θ_λ^*. This shows that θ_λ^* splits into the m_λ non-exceptional characters of b_λ each appearing with multiplicity 1. In our present case, the signs $\varepsilon_i^{(\lambda)}$ are also $+1$.

The characters of the principal block b_0 have kernels including $\mathfrak{V}$. Since $\mathfrak{N}/\mathfrak{C}$ is cyclic of order m, we can find a linear character $\psi^{(0)} \in b_0$ such that $\psi^{(0)}(M)$ is a primitive m-th root of unity while $\psi^{(0)} \mid \mathfrak{C}=1$. Then the non-exceptional characters of b_0 are

$$\psi_i^{(0)}=(\psi^{(0)})^i, \quad (i=0,1,2,\ldots,m-1).$$

Let $\psi^{(\lambda)}$ now denote a non-exceptional character of b_λ. It is clear that, for any i,

$$\psi_i^{(\lambda)}=(\psi^{(0)})^i \psi^{(\lambda)} \tag{3.4}$$

is also an irreducible non-exceptional character of b_λ. Conversely, if ψ is an irreducible non-exceptional character of b_λ, then $\psi\overline{\psi}^{(\lambda)}$ must contain a constituent in b_0. This can be seen from (2.6). It also follows from the general theory. Since the kernel of $\psi\overline{\psi}^{(\lambda)}$ includes $\mathfrak{P}$, this constituent can only be a $\psi_i^{(0)}$. It then follows that ψ is the character $\psi_i^{(\lambda)}$. We see that the non-exceptional characters of b_λ are the $\psi_i^{(\lambda)}$ with $i=0,1,2,\ldots,m_\lambda-1$. Necessarily,

$$\psi_i^{(\lambda)}(\psi^{(0)})^{m_\lambda}=\psi_i^{(\lambda)}. \tag{3.5}$$

We now have

(3 A) *The block b_λ of $\mathfrak{N}$ consists of the t_λ exceptional characters $\xi_i^{(\lambda)}$ and the m_λ non-exceptional characters $\psi_i^{(\lambda)}$, $0 \leqq j < t_\lambda$; $0 \leqq i < m_\lambda$. The $\xi_j^{(\lambda)}$ have degree mf_λ while the $\psi_i^{(\lambda)}$ have degree $\tau_\lambda f_\lambda$. The kernel of each $\psi_i^{(\lambda)}$ includes $\mathfrak{P}$.*

Since the sign associated with each $\psi_i^{(\lambda)}$ is $+1$, the tree corresponding to b_λ is a "star" whose center corresponds to the set of exceptional characters while each free end point corresponds to a $\psi_i^{(\lambda)}$. This shows that the modular irreducible characters $\phi_i^{(\lambda)}$ of b_λ are simply the restrictions of the $\psi_i^{(\lambda)}$ to the set of p-regular elements; $i=0, 1, 2, \ldots, m_\lambda - 1$. The chief indecomposable character $\Phi_i^{(\lambda)}$ corresponding to $\phi_i^{(\lambda)}$ then is given by

$$\Phi_i^{(\lambda)} = \psi_i^{(\lambda)} + \sum_{j=1}^{t_\lambda} \xi_j^{(\lambda)}. \tag{3.6}$$

It also follows that, for p-regular elements $\sigma \in \mathfrak{N}$, we have

$$\xi_i^{(\lambda)}(\sigma) = \sum_{\nu=0}^{m_\lambda - 1} \phi_\nu^{(\lambda)}(\sigma). \tag{3.7}$$

Taking the element 1 in (3.6) and using (3 A) and (2.2), (2.3) we obtain

(3 B) *The character* $\Phi_i^{(\lambda)}$ *has the degree* $p\tau_\lambda f_\lambda$.

The next statement is also an immediate consequence of (3.6):

(3 C) *The* $\mathbf{Z}$*-module of functions spanned by the irreducible characters in* b_λ *has the basis* $\Phi_i^{(\lambda)}$, $\xi_j^{(\lambda)}$ *with* $i=0, \ldots, m_\lambda - 1$ *and* $j = 1, 2, \ldots, t_\lambda$.

We conclude § 3 with the proof of three lemmas concerning the products of the characters of $\mathfrak{N}$.

(3 D) *For* $i \not\equiv j \pmod t$, $\xi_i^{(\lambda)}\bar{\xi}_j^{(\lambda)}$ *is a sum of exceptional characters.*

Indeed, it can be seen from (3.3) that, for $i \not\equiv j \pmod t$, $\xi_i^{(\lambda)}\xi_j^{(\lambda)}|\mathfrak{C}$ is a sum of irreducible characters $\theta(\rho)^k$ where θ is a character whose kernel includes $\mathfrak{P}$ and where $(\rho)^k \neq 1$. It follows that each constituent of $\xi_i^{(\lambda)}\bar{\xi}_j^{(\lambda)}$ is non-trivial on $\mathfrak{P}$. By (3 A), all constituents are exceptional characters.

(3 E) *We have formulas*

$$\Phi_i^{(\alpha)}\bar{\xi}_j^{(\beta)} = \sum_{\Upsilon=0}^{l-1}\sum_{k=1}^{m_\Upsilon - 1} q_{ik}^{\alpha\beta\Upsilon}\Phi_k^{(\Upsilon)} \tag{3.8}$$

where $q_{ik}^{\alpha\beta\Upsilon} \in \mathbf{Z}$ *and where*

$$0 \leqq q_{ik}^{\alpha\beta\Upsilon} \leqq \tau_\alpha f_\alpha f_\beta / f_\Upsilon. \tag{3.9}$$

Proof. We can set

$$\psi_k^{(\gamma)}\psi_\nu^{(\beta)} = \sum_{\alpha=0}^{l-1}\sum_{i=0}^{m_\alpha-1} a_{i\nu k}^{\alpha\beta\gamma}\psi_i^{(\alpha)}$$

with $a_{ijk}^{\alpha\beta\gamma}\in \mathbf{Z}$, $a_{ijk}^{\alpha\beta\gamma}\geqq 0$. This formula remains valid, if each ψ is replaced by the corresponding ϕ. Now [4], (64) and (76) show that

$$\Phi_i^{(\alpha)}\overline{\phi}_\nu^{(\beta)} = \sum_{\gamma=1}^{l}\sum_{k=1}^{m_\gamma-1} a_{i\nu k}^{\alpha\beta\gamma}\Phi_k^{(\gamma)}. \tag{3.10}$$

We claim that (3.8) holds with

$$q_{ik}^{\alpha\beta\gamma} = \sum_{\nu=0}^{m_\beta-1} a_{i\nu k}^{\alpha\beta\gamma}. \tag{3.11}$$

Since the chief indecomposable characters Φ vanish for p-singular elements, it suffices to check (3.8) for p-regular elements σ. Here, (3.8) is obtained by summing (3.10) for $\nu = 0, 1, \ldots, m_\beta - 1$ and using (3.7) and (3.11). It is now evident that $q_{ik}^{\alpha\beta\gamma}\in \mathbf{Z}$, $q_{ik}^{\alpha\beta\gamma}\geqq 0$.

We note next that $a_{i\nu k}^{\alpha\beta\gamma}$ is the multiplicity of the principal character in

$$\psi_k^{(\gamma)}\psi_\nu^{(\beta)}\overline{\psi}_i^{(\alpha)} = \psi^{(\gamma)}\psi^{(\beta)}\overline{\psi}^{(\alpha)}(\psi^{(0)})^{k+\nu-i},$$

cf. (3.4). For fixed α, β, γ, then $a_{i\nu k}^{\alpha\beta k}$ depends only on $k+\nu-i$. Moreover, (3.5) shows that $a_{i\nu k}^{\alpha\beta\gamma}$ remains unchanged if one one of the indices i, ν, or k is changed modulo the greatest common divisor d of m_α, m_β, m_γ. Let dU denote the sum of the $a_{i\nu k}^{\alpha\beta\gamma}$ where one of the subscripts ranges over a residue system mod d. Then (3.11) reads

$$q_{ik}^{\alpha\beta\gamma} = m_\beta U = mU/\tau_\beta. \tag{3.12}$$

On the other hand, since $a_{ijk}^{\alpha\beta\gamma}$ is the multiplicity of $\psi_k^{(\gamma)}$ in $\psi_i^{(\alpha)}\overline{\psi}_j^{(\beta)}$,

$$\psi_i^{(\alpha)}\overline{\psi}_j^{(\beta)} = \sum_{\gamma=0}^{l-1}\sum_{k=0}^{m_\gamma-1} a_{ijk}^{\alpha\beta\gamma}\psi_k^{(\gamma)}. \tag{3.13}$$

On comparing degrees, we have

$$\tau_\alpha\tau_\beta f_\alpha f_\beta \geqq \sum_{k=1}^{m_\gamma} a_{ijk}^{\alpha\beta\gamma}\tau_\gamma f_\gamma = m_\gamma U\tau_\gamma f_\gamma = mUf_\gamma.$$

Now (3.12) shows that (3.9) holds.

(3 F) *We have formulas*

$$\Phi_i^{(\alpha)}\overline{\Phi}_\nu^{(\beta)} = \sum_{\gamma=0}^{l-1}\sum_{k=0}^{m_\gamma-1} A_{i\nu k}^{\alpha\beta\gamma}\Phi_k^{(\gamma)} \tag{3.14}$$

where $A_{i\nu k}^{\alpha\beta\gamma} \in \mathbf{Z}$ *and where*

$$0 \leqq A_{i\nu k}^{\alpha\beta\gamma} \leqq (t + \tau_\gamma^{-1}) \tau_\alpha \tau_\beta f_\alpha f_\beta / f_\gamma. \tag{3.15}$$

Proof. Again, it suffices to consider p-regular elements in (3.14). On adding (3.8) for $j = 1, 2, \ldots, t_\beta$ and (3.10), we see that (3.14) holds with

$$A_{i\nu k}^{\alpha\beta\gamma} = t_\beta q_{ik}^{\alpha\beta\gamma} + a_{i\nu k}^{\alpha\beta\gamma} = t\tau_\beta q_{ik}^{\alpha\beta\gamma} + a_{i\nu k}^{\alpha\beta\gamma}.$$

Now (3.15) is obtained from (3.9), since (3.13) implies $a_{i\nu k}^{\alpha\beta\gamma} \leqq \tau_\alpha \tau_\beta f_\alpha f_\beta / (\tau_\gamma f_\gamma)$.

§ 4. Some results concerning the characters of $\mathfrak{G}$

It follows from (2.5) and the results of § 3 that the difference $\chi_j^{(\lambda)} | \mathfrak{N} - \varepsilon^{(\lambda)} \xi_j^{(\lambda)}$ is a generalized character of $\mathfrak{N}$ which vanishes for all p-singular elements; $(\lambda = 0, \ldots, l-1;\ j = 1, 2, \ldots, t_\lambda)$. Consequently, the difference is a linear combination of the chief indecomposable characters $\Phi_k^{(\alpha)}$ of $\mathfrak{N}$ with coefficients u in $\mathbf{Z}$, (cf. for instance [2], Theorem 17). Moreover, since the values of the $\Phi_k^{(\alpha)}$ lie in the field Ω of the g_0-th roots of unity over $\mathbf{Q}$, application of an element of the Galois group of $\Omega(\rho)$ over Ω shows that the coefficients u do not depend on j. Hence we may set

$$\chi_j^{(\lambda)} | \mathfrak{N} = \varepsilon^{(\lambda)} \xi_j^{(\lambda)} + \sum_{\alpha, r} u_{\alpha, r}^{(\lambda)} \Phi_r^{(\alpha)} \tag{4.1}$$

where in the sum on the right, α ranges over $0, \ldots, l-1$ and, for given α, r ranges over $0, 1, \ldots, m_\alpha - 1$.

(4 *A*). *In* (4.1), *the integers* $u_{\alpha, r}^{(\lambda)}$ *are non-negative. If* $\varepsilon^{(\lambda)} = -1$, *there exists an* r *for which* $u_{\lambda, r}^{(\lambda)}$ *is positive.*

This becomes evident, if we use the formulas (3.6) to express (4.1) by means of the irreducible charcters of $\mathfrak{N}$. Since $\chi_j^{(\lambda)} | \mathfrak{N}$ is a character of $\mathfrak{N}$, the coefficients of each $\varphi_k^{(\alpha)}$ and each $\xi_i^{(\lambda)}$ must be non-negative.

(4 *B*) *Choose a fixed value of* λ. *There exist coefficients* $h_{\alpha\beta r} \in \mathbf{Z}$ *such that*

$$\chi_i^{(\lambda)} \bar{\chi}_i^{(\lambda)} = \sum_{\mu=0}^{a-1} \zeta_\mu^{(0)} + \sum_{r=1}^{t} h_{iir} \chi_r^{(0)} + \Gamma + \Delta_{ii}\ ; \tag{4.2 a}$$

$$\chi_i^{(\lambda)} \bar{\chi}_j^{(\lambda)} = \sum_{\nu=a}^{m-1} \zeta_\nu^{(0)} + \sum_{r=1}^{t} h_{ijr} \chi_r^{(0)} + \Gamma + \Delta_{ij} \tag{4.2 b}$$

for $i \not\equiv j \bmod t_\lambda$. *Here* Γ *is a character whose irreducible constituents are non-exceptional characters* $\zeta_k^{(0)} \in B_0$ *and which does not depend on* i, j, *and* $\Delta_{\alpha\beta}$ *is a* *character whose irreducible constituents do not lie in B_0. (The integers $h_{\alpha\beta r}$ and the characters Γ and $\Delta_{\alpha\beta}$ will depend on λ).*

Proof. Choose $i \not\equiv j \pmod{t_\lambda}$ and form the generalized character

$$\Xi = \chi_i^{(\lambda)}\bar{\chi}_j^{(\lambda)} - \chi_k^{(\lambda)}\bar{\chi}_k^{(\lambda)}.$$

Then Ξ vanishes for all p-regular elements of $\mathfrak{G}$. Express Ξ by the irreducible characters of $\mathfrak{G}$ and let Ξ_0 denote the sum of the terms which lie in B_0, say

$$\Xi_0 = \sum_{\mu=0}^{a-1} x_\mu \zeta_\mu^{(0)} + \sum_{\nu=a}^{m-1} y_\lambda \zeta_\nu^{(0)} + \sum_{r=1}^{t} z_r \chi_r^{(0)}.$$

Then Ξ_0 also vanishes for p-regular elements. On the other hand, for every value of r,

$$\sum_{\mu=0}^{a-1} \zeta_\mu^{(0)} - \sum_{\nu=a}^{m-1} \zeta_\nu^{(0)} - \varepsilon^{(0)}\chi_r^{(0)}$$

vanishes for p-regular elements, cf. [1] I, Theorem 6. If we set $z = \sum_r z_r$, it follows that

$$\sum_{\mu=0}^{a-1} (x_\mu + \varepsilon^{(0)}z)\zeta_\mu^{(0)} + \sum_{\nu=a}^{m-1} (y_\nu - \varepsilon^{(0)}z)\zeta_\nu^{(0)}$$

vanishes for p-regular elements. Since the $\zeta_i^{(0)}$ are still linearly independent on the set of p-regular elements, we find $x_\mu = -\varepsilon^{(0)}z$, $y_\nu = \varepsilon^{(0)}z$. Now, the prinicipal character $\zeta_0^{(0)}$ appears with the multiplicity -1 in Ξ; we have $x_0 = -1$. Hence

$$\varepsilon^{(0)} = z = \sum_r z_r, \ x_\mu = -1, \ y_\nu = 1. \tag{4.3}$$

Thus,

$$\chi_i^{(\lambda)}\bar{\chi}_j^{(\lambda)} = \chi_k^{(\lambda)}\bar{\chi}_k^{(\lambda)} - \sum_{\mu=0}^{a-1} \zeta_\mu^{(0)} + \sum_{\nu=a}^{m-1} \zeta_\nu^{(0)} + \sum_{r=1}^{t} z_r \chi_r^{(0)} + \Xi_{ijk}^* \tag{4.4}$$

where Ξ_{ijk}^* is a character of $\mathfrak{G}$ whose irreducible constituents lie in blocks other than B_0. In particular, this shows that $\zeta_0^{(0)}, \zeta_1^{(0)}, \ldots, \zeta_{a-1}^{(0)}$ are constituents of $\chi_k^{(\lambda)}\bar{\chi}_k^{(\lambda)}$ and that $\zeta_a^{(0)}, \ldots, \zeta_{m-1}^{(0)}$ are constituents of $\chi_i^{(\lambda)}\bar{\chi}_j^{(\lambda)}$.

We may choose our notations such that (4.2 a) holds for $i=1$ and that h_{11r}, Γ, Δ_{11} have the significance stated in (4 B). Then (4.4) with $k=1$ shows that (4.2 b) holds, if we set $h_{ijr} = h_{11r} + z_r$. It follows from (4.3) that

$$\sum_{r=1}^{t} h_{ijr} = \sum_{r=1}^{t} h_{11r} + \varepsilon^{(0)}. \tag{4.5}$$

If we now apply (4.4) with any k, we see that (4.2 a) holds in general. In addition, we have

$$\sum_{r=1}^{t} h_{kkr} = \sum_{r=1}^{t} h_{ijr} - \varepsilon^{(0)}. \tag{4.6}$$

This completes the proof of (4 B).

(4 C) *In (4 B), the h_{ijr} are non-negative. There exists a number H such that*

$$\sum_{r=1}^{t} h_{ijr} = H \textit{ for } i \not\equiv j \pmod{t_\lambda}, \quad \sum_{r=1}^{t} h_{iir} = H - \varepsilon^{(0)}. \tag{4.7}$$

The first statement is obvious since $h_{\alpha\beta r}$ is the multiplicity of $\chi_r^{(0)}$ in $\chi_\alpha^{(\lambda)}\overline{\chi}_\beta^{(\lambda)}$. The second statement is a consequence of (4.5) and (4.6).

Our next aim is to give an estimate for the number H. We shall now make the following assumptions:

(I). $t \geqq 3$

(II). *There does not exist a normal subgroup $\mathfrak{K}$ of $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{K}$ is isomorphic with the metacyclic group $\mathfrak{M}_{p,t}$ of order $p(p-1)/t$* *.

Suppose that in (4.1) for $\lambda = 0$, all $u_{\alpha,k}^{(0)}$ vanish. Necessarily $\varepsilon^{(0)} = 1$ and $\chi_j^{(0)}(1) = m < (p-1)/2$. We take $\mathfrak{K}$ as the kernel of $\chi_1^{(0)}$. By (2.5), $P \notin \mathfrak{K}$ and hence $\mathfrak{K}$ has an order prime to p. Then the principal blocks of $\mathfrak{G}$ and of $\overline{\mathfrak{G}} = \mathfrak{G}/\mathfrak{K}$ can be identified, see e.g. [3], Theorem 1. In particular, we see that the number m remains the same if $\mathfrak{G}$ is replaced by $\overline{\mathfrak{G}}$. Now the results of [1] II show that $\overline{\mathfrak{G}} \simeq \mathfrak{M}_{p,t}$, a contradiction.

We may therefore assume that (4.1) for $\lambda = 0$ reads

$$\chi_q^{(0)} | \mathfrak{N} = \varepsilon^{(0)}\xi_q^0 + u_{\gamma,k}^{(0)}\Phi_k^{(\gamma)} + \cdots ; \quad u_{\gamma,k}^{(0)} \geqq 1. \tag{4.8}$$

Since $t \geqq 3$, we can choose $i \not\equiv j \pmod t$ in (4.2 b). Restrict the arguments to $\mathfrak{N}$ and express all characters of $\mathfrak{N}$ by means of the $\xi_i^{(\alpha)}$, $\Phi_j^{(\alpha)}$, cf. (3 C), (3.6). Now (4.8) shows that $\Phi_k^{(\gamma)}$ will appear at least with the coefficient H in (4.2 b).

On the other hand, on account of (4.1), we have

$$\chi_i^{(\lambda)}\overline{\chi}_j^{(\lambda)} | \mathfrak{N} = \xi_i^{(\lambda)}\overline{\xi}_j^{(\lambda)} + \sum_{\alpha, r}\sum_{\beta, s} u_{\alpha r}^{(\lambda)} u_{\beta s}^{(\lambda)} \Phi_r^{(\alpha)}\overline{\Phi}_s^{(\beta)}$$
$$+ \varepsilon^{(\lambda)} \sum_{\alpha, r} u_{\alpha,r}^{(\lambda)} [\xi_i^{(\lambda)}\overline{\Phi}_r^{(\alpha)} + \overline{\xi}_j^{(\lambda)}\Phi_r^{(\alpha)}]$$

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* The assumption (II) is only used to eliminate the case that $\chi_j^{(0)}(1) = (p-1)/t \leqq (p-1)/3$.

where (α, r) and (β, s) range over the pairs (q, μ) with $q=0, \ldots, l-1$ and $\mu=0, 1, \ldots, m_q-1$. Again express all the characters on the right in accordance with (3 C). It follows from (3 D), (3 E), (3 F) that the coefficient of $\Phi_k^{(\gamma)}$ is equal to

$$\sum_{\alpha, r}\sum_{\beta, s} u_{\alpha r}^{(\lambda)} u_{\beta s}^{(\lambda)} A_{rsk}^{\alpha\beta\gamma} + \varepsilon^{(\lambda)} \sum_{\alpha, r} u_{\alpha, r}^{(\lambda)} [q_{rk}^{\alpha\lambda\gamma} + q_{rk^*}^{\alpha\lambda\gamma^*}]$$

where we set $\overline{\Phi}_k^{(\gamma)} = \Phi_{k^*}^{(\gamma^*)}$. Now (3.9) and (3.15) show that in the case $\varepsilon^{(\lambda)}=1$, this quantity and hence H is at most equal to

$$(t+\tau_\gamma^{-1}) f_\gamma^{-1} [\sum_{\alpha, r} u_{\alpha r}^{(\lambda)} \tau_\alpha f_\alpha]^2 + 2 f_\lambda f_\gamma^{-1} \sum_{\alpha, r} u_{\alpha r}^{(\lambda)} \tau_\alpha f_\alpha.$$

In the case $\varepsilon^{(\lambda)} = -1$, the second summand can be deleted. In either case we set

(4.9) $$R = \sum_{\alpha, r} u_{\alpha r}^{(\lambda)} \tau_\alpha f_\alpha.$$

Then our result shows that

(4.10 a) $H \leqq (t+1)R^2 + 2 f_\lambda R$ for $\varepsilon^{(\lambda)} = 1$;

(4.10 b) $H \leqq (t+1)R^2$ for $\varepsilon^{(\lambda)} = -1$.

Set $n = \chi_j^{(\lambda)}(1)$. Then (4.1) in conjunction with (3 A) and (3 B) yields

(4.11) $$n = \varepsilon^{(\lambda)} m f_\lambda + p \sum_{\alpha, r} u_{\alpha r}^{(\lambda)} \tau_\alpha f_\alpha = \varepsilon^{(\lambda)} m f_\lambda + pR.$$

Suppose first that $\varepsilon^{(\lambda)} = 1$. If we put $n/p = K$, we have

$$f_\lambda = p/(K-R)/m \leqq (t+1)(K-R)$$

and (4.10) becomes

$$H \leqq (t+1)(R^2 + 2(K-R)R) < (t+1)K^2.$$

Suppose then that $\varepsilon^{(\lambda)} = -1$. As remarked in (4 A), some $u_{\lambda r}^{(\lambda)}$ is positive and hence

$$R \geqq \tau_\lambda f_\lambda,$$
$$n \geqq -m f_\lambda + p\tau_\lambda f_\lambda > -m f_\lambda + m t \tau_\lambda f_\lambda.$$

Thus, $m f_\lambda < n(t_\lambda - 1)^{-1}$. For $n/p = K$, by (4.11)

$$R < K(1 + (t_\lambda - 1)^{-1}) = K t_\lambda / (t_\lambda - 1) \leqq \frac{3}{2} K.$$

This can be substituted in (4.10 b). We have now shown.

(4 D) *If $\mathfrak{G}$ satisfies the assumptions* (I) *and* (II), *then*

$$H<(t+1)n^2/p^2 \qquad (\text{for } \varepsilon^{(\lambda)}=1).$$
$$H<(t+1)(t_\lambda^2/(t_\lambda-1)^2)(n^2/p^2)\leqq\frac{9}{4}(t+1)n^2/p^2 \qquad (\text{for } \varepsilon^{(\lambda)}=-1).$$

Here, n is the degree of the exceptional characters $\chi_j^{(\lambda)}$.

§ 5. Proof of Theorem 1

THEOREM 1. *Let $\mathfrak{G}$ be a group of order $g=pg_0$, where p is a prime and where g_0 is an integer not divisible by p. Assume that the p-Sylow group $\mathfrak{P}$ of $\mathfrak{G}$ is not normal in $\mathfrak{G}$ and that $\mathfrak{G}$ contains $t\geqq 3$ conjugate classes of elements of order p. If $\chi^{(\lambda)}$ of degree n is a faithful exceptional irreducible character of $\mathfrak{G}$ (for the prime p), then*

$$p-1<w(w-2\,\varepsilon^{(0)})/(t-2) \tag{5.1}$$

where $w=(t+1)t_\lambda n^2/p^2$ in the case $\varepsilon^{(\lambda)}=1$ and $w=(t+1)\cdot(t_\lambda^3/(t_\lambda-1)^2)\cdot(n^2/p^2)$ in the case $\varepsilon^{(\lambda)}=-1$. Here $\varepsilon^{(\lambda)}$ and $\varepsilon^{(0)}$ are the signs belonging to the exceptional characters of the p-blocks B_λ and B_0.

Remark. The assumption that $\chi^{(\lambda)}$ is faithful is not needed in the following two cases: 1) if $\mathfrak{M}_{p,t}$ is not a homomorphic image of $\mathfrak{G}$. 2) if $\chi^{(\lambda)}(1)>(p-1)/3$.

Proof. We shall first make the additional assumption that $\mathfrak{M}_{p,t}$ is not a homomorpic image of $\mathfrak{G}$. Then (4 D) applies. We give a lower estimate for H. Form

$$S_{\alpha\beta}^{(\lambda)}=v^{-1}\sum_{V\in\mathfrak{V}}\chi_\alpha^{(\lambda)}(PV)\bar{\chi}_\beta^{(\lambda)}(PV).$$

It follows at once from (2.5) that

$$S_{\alpha\beta}^{(\lambda)}=\sum_{\nu=0}^{\tau_\lambda-1}\omega_{\alpha+\nu t}^{(\alpha)}\bar{\omega}_{\beta+\nu t}^{(\alpha)}.$$

On the other hand, we can use (4.2) to find $S_{\alpha\beta}^{(\lambda)}$. The equation (2.6) implies that

$$\sum_{V\in\mathfrak{V}}\zeta_i^{(0)}(PV)=\varepsilon_i^{(0)}v.$$

Similarly, by (2.5)

$$\sum_{V\in\mathfrak{V}}\chi_j^{(0)}(PV)=\varepsilon^{(0)}v\omega_j^{(0)}=\varepsilon^{(0)}v\eta_j.$$

Since $\Delta_{\alpha\beta}$ consists of constituents not in B_0, a similar argument shows that

$$\sum_{V\in\mathfrak{V}} \Delta_{\alpha\beta}(PV) = 0.$$

Since $\varepsilon_i^{(0)} = 1$ for $0 \leqq i < a$ and $\varepsilon_i^{(0)} = -1$ for $a \leqq i < m$, (4.2) implies

$$S_{\alpha\beta}^{(\lambda)} = \varepsilon^{(0)} \sum_{r=1}^{t} h_{\alpha\beta r}\eta_r + x + m\delta_{\alpha\beta}^{(\lambda)}$$

where $x \in \mathbf{Z}$ does not depend on α, β and where $\delta_{\alpha\beta}^{(\lambda)} = 1$ or 0 according as to whether or not $\alpha \equiv \beta (\mathrm{mod}\ t_\lambda)$.

Hence

$$\sum_{\nu=1}^{\tau_\lambda} \omega_{\alpha+\nu t}^{(\lambda)} \bar{\omega}_{\beta+\nu t}^{(\lambda)} = \varepsilon^{(0)} \sum_{r=1}^{t} h_{\alpha\beta r}\eta_r + x + m\delta_{\alpha\beta}^{(\lambda)}. \tag{5.2}$$

It follows from (2.7) that

$$\eta_i\bar{\eta}_j = \sum_{\nu=0}^{\tau_\lambda-1} \sum_{\mu=0}^{\tau_\lambda-1} \omega_{i+\nu t}^{(\lambda)} \bar{\omega}_{j+\mu t}^{(\lambda)} = \sum_{\nu=0}^{\tau_\lambda-1} \sum_{\mu=0}^{\tau_\lambda-1} \omega_{i+\nu t}^{(\lambda)} \bar{\omega}_{j+\mu t+\nu t}^{(\lambda)}. \tag{5.3}$$

If we set

$$C_{ijr} = \sum_{\mu=0}^{\tau_\lambda-1} h_{i,j+\mu t,r}, \tag{5.4}$$

substitution of (5.2) into (5.3) yields

$$\eta_i\bar{\eta}_j = \varepsilon^{(0)} \sum_{r=1}^{t} C_{ijr}\eta_r + \tau_\lambda x + m\delta_{ij}^{(0)} \tag{5.5}$$

where $\delta_{ij}^{(0)} = 1$ or 0 according as to whether or not $i \equiv j (\mathrm{mod}\ t)$. On account of (4.7), we find

$$\sum_{r=1}^{t} C_{ijr} = \tau_\lambda H - \delta_{ij}^{(0)} \varepsilon^{(0)}. \tag{5.6}$$

We may now compare (5.5) and (2.10). Since $\eta_1, \eta_2, \ldots, \eta_r$ are linearly independent over $\mathbf{Z}$ and since their sum is -1, this yields

$$\varepsilon^{(0)} C_{ijr} - \tau_\lambda x = c_{ijr}. \tag{5.7}$$

In particular, (2.12) shows that

$$C_{\alpha\beta\gamma} = C_{\alpha\gamma\beta} = C_{\gamma\alpha\beta'}. \tag{5.8}$$

It also follows from (5.7), (5.6) and (2.11) that

$$\varepsilon^{(0)} \tau_\lambda H - t_\lambda x = m, \tag{5.9}$$

Since $t \geqq 3$, we can choose $i \not\equiv j \pmod{t}$. We use (5.5) to express

$$(\eta_i \bar{\eta}_i)\bar{\eta}_j = (\eta_i \bar{\eta}_j)\bar{\eta}_i$$

in two ways as linear combination of $\eta_1, \eta_2, \ldots, \eta_m$. Comparing the coefficient of $\bar{\eta}_j = \eta_{j'}$, we find

$$\sum_{r=1}^{t} C_{iir} C_{rjj'} - \varepsilon^{(0)} \sum_{r=1}^{t} C_{iir}\tau_\lambda x - \varepsilon^{(0)} m C_{iij} + \tau_\lambda x + m$$
$$= \sum_{r=1}^{t} C_{ijr} C_{rij'} - \varepsilon^{(0)} \sum_{r=1}^{t} C_{ijr}\tau_\lambda x - \varepsilon^{(0)} C_{iji} m.$$

On account of (5.8) and (5.6), this becomes

$$\sum_{r=1}^{t} C_{iir} C_{jjr} + m + 2\,\tau_\lambda x = \sum_{r=1}^{t} C_{ijr}^2.$$

Since all $C_{\alpha\beta\gamma}$ are non negative,

$$m + 2\,\tau_\lambda x \leqq \sum_{r=1}^{t} C_{ijr}^2 \leqq \Big(\sum_{r=1}^{t} C_{ijr}\Big)^2 = \tau_\lambda^2 H^2.$$

On account of (5.9), this becomes

$$mt^2 + 2(\varepsilon^{(0)}\tau_\lambda H - m)t \leqq t_\lambda^2 H^2$$

which can rewritten in the form

(5.10) $$(p-1)(t-2) \leqq t_\lambda H (t_\lambda H - 2\,\varepsilon^{(0)}).$$

Now, (5.1) is a consequence of (4 D).

It remains to deal with the case that $\mathfrak{G}$ contains a normal subgroup $\mathfrak{K}$ such that $\bar{\mathfrak{G}} = \mathfrak{G}/\mathfrak{K} \simeq \mathfrak{M}_{p,t}$. Clearly, $\mathfrak{K}$ then is the maximal normal subgroup of $\mathfrak{G}$ of an order prime to p. This implies that the principal p-blocks of $\mathfrak{G}$ and of $\bar{\mathfrak{G}}$ coincide, cf. [3], Theorem 1. Since the p-Sylow subgroup of $\bar{\mathfrak{G}}$ is self-centralizing, the natural homorphism of $\mathfrak{G}$ onto $\bar{\mathfrak{G}}$ maps $\mathfrak{V}$ into $\mathfrak{K}/\mathfrak{K}$ and we have $\mathfrak{V} \subseteq \mathfrak{K}$. It is now clear that

(5.11) $$\mathfrak{G} = \langle \mathfrak{K}\mathfrak{P},\ M\rangle$$

and that $\mathfrak{K}\mathfrak{P}$ is a normal subgroup of index m in $\mathfrak{G}$.

We assume now that χ is a faithful character of $\mathfrak{G}$.

We now use induction with regard to m to prove that

(5.12) $$(p-1)(t-2) \leqq w_0(w_0 - 2)$$

with $w_0 = (t+1)t_\lambda n^2/p^2$. This will imply (5.1). Since $t \geqq 3$, we have $p \geqq 7$. As

$\mathfrak{P}$ is not normal in $\mathfrak{G}$, it cannot be normal in $\mathfrak{K}\mathfrak{P}$.

If $m=1$, then $t=t_\lambda=p-1$. It will suffice to show $n\geqq p-1$ since then $w_0>(p-1)(p-2)$ and hence (5.12) will hold. If $n<p-1$, it follows from [1] II, Theorem 1 and Corollary 2 that $n=f_\lambda$ and that for the single non-exceptional character $\zeta_0^{(\lambda)}$ of B_λ, we have $\zeta_0^{(\lambda)}|\mathfrak{C}=\theta_\lambda$, while $\chi_j^{(\lambda)}$ is obtained from $\zeta_0^{(\lambda)}$ by multiplication with a linear character of $\mathfrak{P}\simeq\mathfrak{G}/\mathfrak{K}$. Thus, $\mathfrak{P}$ belongs to the kernel $\mathfrak{H}$ of $\zeta_0^{(\lambda)}$ and $\mathfrak{H}$ cannot contain p-regular elements $G\neq 1$ of $\mathfrak{G}$. Hence $\mathfrak{H}=\mathfrak{P}$ and we have $\mathfrak{P}\lhd\mathfrak{G}$, a contradiction.

Suppose then that $m>1$. Let s be a prime dividing m and set $M^*=M^s$,

$$\mathfrak{G}^*=\langle\mathfrak{K}\mathfrak{P},\ M^*\rangle.$$

Then $\mathfrak{G}^*$ is a normal subgroup of $\mathfrak{G}$ of index s. Moreover, $\mathfrak{G}^*$ is of the same structure as $\mathfrak{G}$ with m replaced by $m^*=m/s$, i.e. with t replaced by $t^*=ts$. In (4.2 a), the $\zeta_0^{(0)},\ldots,\zeta_{a-1}^{(0)}$ are the m linear characters of the cyclic group $\mathfrak{G}/\mathfrak{K}\mathfrak{P}$ of order m; $a=m$. Since s of them have a kernel including $\mathfrak{G}^*$, it follows from (4.2 a) that $\chi_j^{(\lambda)}|\mathfrak{G}^*$ is reducible. Consequently, $\chi_j^{(\lambda)}|\mathfrak{G}^*$ splits into s irreducible characters of degree $n^*=n/s$. The formulas (2.5) show that some of these constituents belong to the block B_λ^* of $\mathfrak{G}^*$ associated with θ_λ. If τ_λ^*, t_λ^* have the same significance for B_λ^* as τ_λ, t_λ have for B_λ, clearly, $\tau_\lambda^*\leqq\tau_\lambda$ and $t_\lambda^*\leqq st_\lambda$. It we set $w_0^*=(t^*+1)t_\lambda^*n^{*2}/p^2$, we have $w_0^*\leqq w_0$. It is now clear that the analogue of (5.12) for $\mathfrak{G}^*$ implies (5.12) and the proof of Theorem 1 is complete.

COROLLARY. *In Theorem 1, we have*

$$n>\frac{1}{3}t_\lambda^{-3/4}p^{5/4}.$$

Proof. If we have $w-2\varepsilon^{(0)}>2w$, then $\varepsilon^{(0)}=-1$, $w<2$ and (5.1) reads $p-1\leqq 8/(t-2)$. This is only possible for $p=7$, $t=3$. Since $w<2$ we have $12n^2<98$ and hence $n\leqq 2$. Then $n=2$. In this case, the result holds.

Assume then that $w-2\varepsilon^{(0)}\leqq 2w$. By (5.1),

$$p-1<2w^2/(t-2)\leqq 2(t+1)^2(t-2)^{-1}t_\lambda^6(t_\lambda-1)^{-4}n^4p^{-4}.$$

Here,

$$6p/7\leqq p-1,\ 3(t+1)^2\leqq 16(t-2)t,\ 2t_\lambda\leqq 3(t_\lambda-1)$$

and we obtain

$$p^5 < 63\, t t_\lambda^2 n^4 < 3^4 t_\lambda^3 n^4$$

and this yields the desired result.

Remark. If an irreducible character χ of $\mathfrak{G}$ has defect 0, it vanishes for p-singular elements. Thus, all values of χ lie in the field of the g_0-th roots of unity. This shows that the Corollary can be stated in the form given in the Introduction.

§6. Proof of Theorem 2

THEOREM 2. *Let $\mathfrak{G}$ be a group of order $g = pg_0$ where p is a prime and g_0 an integer not divisible by p. Assume that (1) the p-Sylow group $\mathfrak{P}$ of $\mathfrak{G}$ is not normal in $\mathfrak{G}$ and (2) that the number t of conjugate classes of elements of order p is at least 3. If $\mathfrak{G}$ has a faithful irreducible character χ of degree $n < p-1$ then $n = p - (p-1)/t$ and*

$$p \leqq t^3 - t + 1. \tag{6.1}$$

Proof. Since χ has degree $n < p-1$, it follows from [1] II, Corollary 2 that χ is an exceptional character $\chi_j^{(\lambda)}$ of a p-block of defect 1. It is also clear that we have one of the cases

Case 1. $\quad \varepsilon^{(\lambda)} = 1$

$$\chi_j^{(\lambda)} | \mathfrak{N} = \xi_j^{(\lambda)}. \tag{6.2 a}$$

Case 2. $\quad \varepsilon^{(\lambda)} = -1$

$$\chi_j^{(\lambda)} | \mathfrak{N} = -\xi_j^{(\lambda)} + \Phi_\mu^{(\lambda)} \tag{6.2 b}$$

where μ is one of the values $0, \ldots, m_\lambda - 1$.

Suppose that $\mathfrak{G}$ has a normal subgroup $\mathfrak{K}$ for which $\mathfrak{G}/\mathfrak{K} \simeq \mathfrak{M}_{p,t}$. As in §5 we see that $\mathfrak{K}\mathfrak{P}$ is a normal subgroup of index m in $\mathfrak{G}$ and that $\mathfrak{V} \subseteq \mathfrak{K}$. It is also clear that the irreducible constituents of $\chi | \mathfrak{P}\mathfrak{K}$ lie in the τ_λ blocks B_μ^* of $\mathfrak{K}\mathfrak{P}$ determined by the τ_λ associates of $\theta_\lambda^{M^\nu}$. All irreducible characters of B_μ^* have the same degree n^* and there is exactly one non-exceptional character $\zeta^{\mu *}$ in B_μ^*. Since $n^* < p-1$, Corollary 2 of [1] II shows that $\mathfrak{P}$ belongs to the kernel of $\zeta^{(\mu)*}$. Again, the exceptional characters of B_μ^* are obtained from the $\zeta^{(\mu)*}$ by multiplication with the linear characters of $\mathfrak{P}\mathfrak{K}/\mathfrak{K} \simeq \mathfrak{P}$. If we form the character

$$Z=\sum_{\mu}\zeta^{(\mu)*}$$

of $\mathfrak{K}\mathfrak{P}$, we see that an element $K\in\mathfrak{K}$ belongs to the kernel $\mathfrak{H}$ of Z only if it belongs to the kernel of $\chi_j^{(\lambda)}$, i.e. if $K=1$. Since $\mathfrak{P}\subseteq\mathfrak{H}$, we have $\mathfrak{P}\lhd\mathfrak{P}\mathfrak{K}$ and then $\mathfrak{P}\lhd\mathfrak{G}$, a contradiction. Hence $\mathfrak{G}$ satisfies the hypothesis (I), (II) in § 4, and all results of § 4 can be used.

We show next that the number H in (4.7) cannot vanish. Indeed, if $H=0$, (5.6) shows that, for $i, j=1, 2, \ldots, t$ with $i\neq j$, we have

$$C_{ijr}=0, \qquad (i, j=1, 2, \ldots, t).$$

By (5.9), $x=-m/t_\lambda$. Then (5.7) yields $c_{ijr}=m/t$ for the same i, j, r. It now follows from (2.10) that

$$\eta_i\bar{\eta}_j=-m/t \qquad (\text{for } i\neq j).$$

This is clearly impossible, since $\eta_1, \eta_2, \ldots, \eta_t$ are distinct and $t\geqq 3$. Hence $H\geqq 1$.

It is now easy to see that the first case is impossible. Indeed, in this case, $R=0$ by (4.9), while (4.10 a) shows that $R\neq 0$.

Suppose then that we have Case 2. By (6.2 b)

$$n=\chi_j^{(\lambda)}(1)=-mf_\lambda+pf_\lambda\tau_\lambda=\tau_\lambda f_\lambda(p-m_\lambda).$$

Since $m_\lambda\leqq m\leqq(p-1)/3$, the assumption $n<p-1$ implies that $\tau_\lambda=f_\lambda=1$. Thus, $n=p-(p-1)/t$.

It now follows from the results of § 3 that

$$\xi_j^{(\lambda)}|\mathfrak{V}=m\theta_\lambda, \qquad \psi_i^{(\lambda)}|\mathfrak{V}=\theta_\lambda$$

for all i and j. Then (3.6) shows that $\Phi_\mu^{(\lambda)}|\mathfrak{V}=p\theta_\lambda$. By (6.2 b),

$$\chi_j^{(\lambda)}|\mathfrak{V}=n\theta_\lambda.$$

Since $\chi_j^{(\lambda)}$ was faithful, this implies that $\mathfrak{V}$ is cyclic and that it belongs to the center $\mathfrak{Z}(\mathfrak{G})$ of $\mathfrak{G}$. Moreover, $\chi_i^{(\lambda)}\bar{\chi}_j^{(\lambda)}$ is trivial on $\mathfrak{V}$. Then the formulas (2.5) and (2.6) show that the irreducible constituents of $\chi_i^{(\lambda)}\bar{\chi}_j^{(\lambda)}|\mathfrak{N}$ belong to b_0. We can therefore express $\chi_i^{(\lambda)}\bar{\chi}_j^{(\lambda)}|\mathfrak{N}$ as a linear combination of the $\psi_\mu^{(0)}$ and the exceptional characters in b_0. Now $\xi_\alpha^{(\lambda)}\bar{\xi}_\beta^{(\lambda)}$ contains the linear character $\psi_\mu^{(0)}$, if and only if $\xi_\alpha^{(\lambda)}=\psi_\mu^{(0)}\xi_\beta^{(\lambda)}$. Since $\psi_\mu^{(0)}$ is trivial on $\mathfrak{P}\times\mathfrak{V}$, this is so, if and only if $\alpha=\beta$. Moreover, for $\alpha=\beta$, the character $\psi_\mu^{(0)}$ appears with multiplicity 1 in $\xi_\alpha^{(\lambda)}\bar{\xi}_\alpha^{(\lambda)}$. A similar argument shows that $\psi_\mu^{(0)}$ cannot occur in $\xi_i^{(\lambda)}\bar{\psi}_\mu^{(\lambda)}$. Finally,

$\psi_\mu^{(\lambda)}\overline{\psi}_\mu^{(\lambda)} = 1$. It now follows from (6.2 b) and (3.6) that

$$\chi_i^{(\lambda)}\overline{\chi}_j^{(\lambda)} | \mathfrak{N} = (t - 1 + \delta_{ij}^{(0)})\psi_0^{(0)} + \sum_{\mu=1}^{t-1}(t - 2 + \delta_{ij}^{(0)})\psi_\mu^{(0)} + \cdots \tag{6.3}$$

where the dots on the right stand for exceptional characters in b_0. On account of (3.6), this can be written in the form

$$\chi_i^{(\lambda)}\overline{\chi}_j^{(\lambda)} | \mathfrak{N} = (t - 1 + \delta_{ij}^{(0)})\Phi_0^{(0)} + \sum_{\mu=1}^{t-1}(t - 2 + \delta_{ij}^{(0)})\Phi_\mu^{(0)} + \cdots$$

where the characters not written are again exceptional characters in b_0.

We can apply the same method as in § 4 and compare our formula with (4.2) (restricted to $\mathfrak{N}$). Since $H \neq 0$, it follows at once that in (4.8), only terms $\Phi_k^{(\gamma)}$ with $\gamma = 0$ can appear with coefficients $u_k^{(\gamma)} \neq 0$. Moreover, $H \leqq t - 1$.

Actually, our method shows that we can have $H = t - 1$ only if (4.8) has the form

$$\chi_q^{(0)} | \mathfrak{N} = \varepsilon^{(0)}\xi_q^{(0)} + \Phi_0^{(0)}. \tag{6.4}$$

We can then also compare the multiplicity of $\psi_0^{(0)}$ in (6.3) for $i = j$ and in (4.2 a), restricted to $\mathfrak{N}$. On account of (6.4) and (4.7) this yields

$$t \geqq 1 + H - \varepsilon^{(0)}.$$

Thus, if $H = t - 1$, we must have $\varepsilon^{(0)} = 1$. Then $\chi_q^{(0)}(1) = m + p$. Since $\varepsilon^{(0)} \neq \varepsilon^{(\lambda)}$, we have $\lambda \neq 0$. It follows that $\mathfrak{V} \neq 1$ and hence that $\mathfrak{Z}(\mathfrak{G}) \neq 1$. Since $\varepsilon^{(0)} = 1$, (6.1) is an immediate consequence of (5.10).

On the other hand, if $H \leqq t - 2$, (5.10) yields

$$p - 1 \leqq t^2(t - 2) + 2t = t^3 - 2t^2 + 2t < t^3 - t.$$

This completes the proof of Theorem 2. We also have

THEOREM 2*. *If $\mathfrak{G}$ has center* 1, *the inequality* (6.1) *in Theorem* 2 *can be replaced by*

$$p \leqq t^3 - 2t^2 + 2t + 1.$$

The same is true, if the degrees of the exceptional characters in the principal p-block of $\mathfrak{G}$ are different from $p + (p - 1)/t$.

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