

HEREDITARY LOCAL RINGS

P. M. COHN

To the memory of TADASI NAKAYAMA

1. Introduction

Many questions about free ideal rings (= firs, cf. [5] and §2 below) which at present seem difficult become much easier when one restricts attention to local rings. One is then dealing with hereditary local rings, and any such ring is in fact a fir (§2). Our object thus is to describe hereditary local rings. The results on firs in [5] show that such a ring must be a unique factorization domain; in §3 we prove that it must also be rigid (cf. the definition in [3] and §3 below). More precisely, for a semifir¹⁾ R with prime factorization rigidity is necessary and sufficient for R to be a local ring.

§4 gives an example of a right fir (in fact a principal right ideal domain) with prime factorization, which is not left hereditary and hence is not a left fir. Since the example is of a local ring, this provides an example of a rigid unique factorization domain which is a semifir but not a fir.

The final section concerns the centre of a hereditary local ring. If this is not a field, then both the ring and its centre are discrete valuation rings. This improves a result of Northcott [8] who showed that the centre, if not a field, must be a 1-dimensional regular local ring. The actual result proved in §5 is rather more general (apart from the stronger conclusion) in that the hypothesis is weaker: we do not require the existence of a central non-unit ($\neq 0$) but merely a 'large' non-unit, and in an integral domain every central non-unit $\neq 0$ is large.

2. Hereditary and semihereditary local rings

Throughout, all rings are associative with 1, and all modules are unital. We recall that a ring R is said to be p -trivial (= projective-trivial) if there is

Received May 27, 1965.

¹⁾ Semifirs, called 'local firs' in [5] have been renamed here (by analogy with 'semihereditary') to avoid confusion with local rings.

a projective R -module P such that each finitely generated projective R -module has the form P^n for a unique integer n . In particular, such a ring has invariant basis number (i.e. any two bases of a free R -module have the same number of elements). An integral domain (not necessarily commutative) with invariant basis number, in which all (finitely generated) right ideals are free, is called a *right (semi) fir*; ¹⁾ *left (semi) firs* are defined similarly. In [5] it was shown that every right semifir is a left semifir, so we may simply speak of semifirs. By a *local ring* we understand as usual a ring in which the sum of two non-units is a non-unit.

Since a local ring has a skew field as homomorphic image, it certainly has invariant basis number. Further, all projectives are free²⁾ and this shows that a local ring must be \mathfrak{p} -trivial. Now it was proved in [6] (Theorem 2) that any right (or left) semihereditary \mathfrak{p} -trivial ring R is a total matrix ring over a semifir, $R \cong T_n$, say. If $n > 1$, the equation $1 = e_{11} + (1 - e_{11})$ in R contradicts the fact that R is local; hence $n = 1$ and $R \cong T$ is itself a semifir. Similarly, if R is right hereditary (and local) it is a right fir ([6], Theorem 1) so we obtain

THEOREM 1. *A right or left semihereditary local ring is a semifir; a right hereditary local ring is a right fir.*

In particular this shows that a right hereditary local ring is an integral domain. We recall from [5] that a right fir satisfies the ascending chain condition for principal right ideals and hence that every non-unit has a left factor which is a prime ([5], proof of Theorem 2.8). Moreover, by Theorem 2.8 of [5], every left and right hereditary local ring is a unique factorization domain, (UFD) in the sense of [4]. We return to this result in the next section.

3. Rigid unique factorization domains

We begin with an obvious general remark.

LEMMA 1. *Let R be a semihereditary local ring. Given $a, b \in R$ such that $aR \cap bR \neq 0$, then $bR \subseteq aR$ or $aR \subseteq bR$.*

²⁾ This is only needed here for finitely generated projectives. The general result (projectives over a local ring are free) was first explicitly proved by Kaplansky [7], but it may be worth noting that this is an immediate consequence of Azumaya's form of the Krull-Schmidt theorem [1].

Proof. By Theorem 1, R is a semifir and hence $aR + bR = dR$, for some $d \in R$. This means that $a = da_0$, $b = db_0$ and $a_0u - b_0v = 1$. Since R is a local ring, a_0 or b_0 must be a unit, and accordingly, $bR \subseteq aR$ or $aR \subseteq bR$.

If R is a left and right fir, then R is a UFD, by Theorem 2.8 of [5]. More generally, this holds for a semifir in which each non-zero element not a unit can be written as a product of primes. The next theorem gives a criterion for such a ring to be local. We recall the following definition:

A (not necessarily commutative) UFD R is said to be *rigid* if, for any two prime factorizations of an element a of R :

$$(1) \quad a = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_r$$

there exist units u_0, u_1, \dots, u_r ($u_0 = u_r = 1$) such that

$$q_i = u_{i-1}^{-1} p_i u_i \quad (i = 1, 2, \dots, r).$$

THEOREM 2. *Let R be a semifir with prime factorization (and hence a UFD). Then R is rigid if and only if it is a local ring.*

By the preceding remarks the theorem applies in particular to rings which are left and right firs.

Proof. Let R be a local ring and consider any two prime factorizations (1) of an element a . Then $p_1 R \cap q_1 R \neq 0$; since p_1 is a non-unit and q_1 a prime, we deduce from Lemma 1 that $p_1 R \subseteq q_1 R$, and by symmetry, $q_1 R \subseteq p_1 R$. Hence $p_1 R = q_1 R$, i.e. $q_1 = p_1 u_1$ for some unit $u_1 \in R$. Cancelling p_1 in (1) we obtain

$$p_2 \cdots p_r = u_1 q_2 \cdots q_r$$

and induction on r shows R to be rigid.

Conversely, let R be rigid; we shall show that R is a local ring by showing that the sum of a non-unit is a unit. Let a be a non-unit and u a unit; since R is a UFD we can write $a = pa_1$ where p is a prime. Now we have the identity

$$(1 + pa_1)p = p(1 + a_1p).$$

If $1 + pa_1$ were a non-unit, then by rigidity $1 + pa_1 = pb$ for some $b \in R$, whence $1 = p(b - a_1)$, which contradicts the fact that p is a non-unit. Hence $1 + a$ is a unit, as we wished to show.

Combining this result with Theorem 1, we obtain the

COROLLARY. *A (left or right) semihereditary local ring with prime factorization (and in particular, a left and right hereditary local ring) is a rigid UFD.*

**4. A principal right ideal domain with prime factorization
which is not left hereditary**

Let R be a UFD with invariant basis number, which is not a field, and consider R^N , the direct product of a countable number of copies of R . We propose to show that R^N , as right R -module, cannot be projective. Put $A^{(i)} \cong R$, $A_n = \prod_{i=1}^n A^{(i)}$, then $A_n \cong R^n$ for $n = 1, 2, \dots$. If R^N were projective, we would have an equation

$$(2) \quad A_1 \oplus B = \sum_I \oplus C_i \quad (C_i \cong R)$$

where i runs over some index set I . Denote by f_i the projection onto C_i and take $a \in R$, where a is not zero or a unit. Then $\bigcap_n Ra^n = 0$ and hence $\bigcap_n C_i a^n = 0$ for each $i \in I$. It follows by a theorem of Chase [2] that there exist integers m, n such that

$$f_i(A_n a^m) = 0 \text{ for all but a finite number of } i \in I.$$

Since $f_i(A_n a^m) = f_i(A_n) a^m$ and a^m annihilates no element of C_i , we have

$$(3) \quad f_i(A_n) = 0$$

for all but a finite number of $i \in I$. Suppose that (3) holds except when $i = i_1, \dots, i_k$ and write $C' = \sum_1^k C_{i_j}$, $C'' = \sum_{j \neq i_j} C_j$ then (2) may be rewritten as

$$A^{(1)} \oplus \dots \oplus A^{(n-1)} \oplus A_n \oplus B = C' \oplus C'';$$

now (3) shows that $A_n \subseteq C'$ and it follows that A_n is complemented in C' , i.e. there exists D such that

$$(4) \quad A_n \oplus D = C'.$$

Since $A_n \cong R^n$, we have $R \oplus A_n \cong A_n$. Adding R to (4) and remembering that $C' \cong R^k$, we obtain

$$R^{k+1} \cong R^k,$$

which is impossible in a ring with invariant basis number. Thus we have proved

THEOREM 3. *If R is any UFD with invariant basis number which is not a field, then R^N is not projective.*

COROLLARY. *If R is a semifir with prime factorization then R^N is not projective.*

Let k be any field with an endomorphism σ which is not surjective³⁾, and consider the ring $R = k[[x; \sigma]]$ of all skew power series, consisting of all infinite series

$$\sum x^n \alpha_n \quad (\alpha_n \in k),$$

with multiplication according to the rule

$$\alpha x = x \alpha^\sigma.$$

The mapping σ can be extended to an endomorphism of R by defining $x^\sigma = x$.

It is clear that R is a local ring with maximal ideal xR and is a principal right ideal domain (in fact R is a discrete valuation ring in the sense defined below, §5); moreover, R has prime factorization and so is a rigid UFD. All these facts are easily verified; we shall now show that R is not a left fir, thus in R we have an example of a one-sided fir.

To show that R is not a left fir we must find a left ideal which is not free; now by Theorem 3, Cor., R^N is not projective, so it is enough to find a left ideal isomorphic to R^N . To obtain such an ideal we need only take a sequence of elements (u_n) say, tending to zero, which is left R -free. For then $\sum a_n u_n \in R$ for all $(a_n) \in R^N$ and by freeness, the mapping

$$(a_n) \rightarrow \sum a_n u_n$$

is an isomorphism of left R -modules. Let ρ belong to k but not to k^σ . Then we assert that the sequence $(x \rho x^n)$ has the required properties. Clearly it tends to zero, and if

$$\sum c_i x \rho x^i = 0,$$

then by cancelling a power of x , if possible, we may assume that $c_0 \neq 0$. Then

$$\sum x^{i+1} c_i^{\sigma^{i+1}} \rho^{\sigma^i} = 0.$$

Hence $x c_0^{\sigma^2} \rho \in R^\sigma$, whereas not all coefficients of $x c_i^{\sigma^2} \rho$ lie in k^σ . This is a con-

³⁾ It must be injective, since $1^\sigma = 1$ by definition.

tradition, and it shows that the elements $x\rho x^n$ are left R -free, as asserted.

5. The centre of a hereditary local ring

In [4] it was shown that a rigid UFD which is commutative is necessarily a discrete valuation ring. In general we define a *discrete valuation ring* (DVR) as an integral domain R (not necessarily commutative) with a prime p such that every non-zero element of R is of the form

$$p^r u \quad (r \geq 0, u \text{ a unit}).$$

When R is commutative, this reduces to the usual definition. A general DVR is clearly a rigid UFD and it may be verified that a rigid UFD is a DVR if and only if any two primes are right associated.

An element a of an arbitrary ring R is said to be *large* if aR is a large right ideal in R , i.e. if

$$aR \cap c \neq 0 \text{ for all non-zero right ideals } c.$$

It is easily verified that in any integral domain the set S of large elements is closed under multiplication and left division. Moreover, in a semifir (or more generally in any weak Bezout ring) the set S of large elements satisfies the generalized Ore condition, hence in this case the ring R admits a ring of right fractions with respect to S , RS^{-1} say, and this ring is again a semifir. We shall omit the (easy) proofs as we do not need these results. In fact, to adjoin inverses of large elements would be a retrograde step, as the next theorem shows.

THEOREM 4. *Let R be a (left or right) semihereditary local ring with prime factorization. Then R is a discrete valuation ring if and only if it contains a large non-unit.*

Proof. Let R be a left or right semihereditary local ring with prime factorization. Then by Theorem 2, R is a rigid UFD. Now assume that R contains a large non-unit c , say. Given any prime $p \in R$, we have $cR \cap pR \neq 0$ and hence $c = pc'$. Write

$$(5) \quad c = p^k u,$$

where $k \geq 1$ is chosen maximal (clearly k is bounded by the number of factors in a prime decomposition of c). Then u must be a unit, for by (5), p is large

and so $pR \cap uR \neq 0$; if u were not a unit, this would mean (by Lemma 1) that $u = pu'$, which, inserted in (5), contradicts the maximality of k . The argument applies to all primes of R ; it shows that all primes are large and hence are right associated. Therefore R is a discrete valuation ring. Conversely, in a DVR, the unique prime is a large non-unit.

The result applies in particular to non-zero centre elements (which in an integral domain are always large), to show that a left and right hereditary local ring with centre not a field is a DVR. But now it is no longer necessary to assume that the ring is a UFD. The precise result is

THEOREM 5. *Let R be a right hereditary local ring whose centre is not a field. Then R is a discrete valuation ring, and so is its centre.*

Proof. Let c be a central element which is not zero or a unit. As before, we can show that for any prime p , we have $c = pc'$. Suppose that for every n there exists $u_n \in R$ such that

$$(6) \quad c = p^n u_n.$$

Since c is central, $p^n u_n = u_n p^n$ and we have the strictly ascending sequence of right ideals

$$cR = u_0 R \subset u_1 R \subset \dots$$

But by Theorem 1, R is a right fir, and this satisfies the ascending chain condition for principal right ideals. Hence (6) cannot hold for all n , and this means that we can again choose k in (5) to be maximal. Then u must be a unit, and as before it follows that all primes are right associated. To complete the proof we need only show that every non-unit $\neq 0$ has a prime factorization. Let $a \in R$ and suppose that for each n there exists a_n such that

$$a = p^n a_n.$$

Taking $n = kr$ where k is as in (5) and r is arbitrary, we have

$$a = p^{kr} a_{kr} = (cu^{-1})^r a_{kr} = c^r u^{-r} a_{kr} = u^{-r} a_{kr} c^r,$$

hence

$$aR \subset u^{-1} a_k R \subset u^{-2} a_{2k} R \subset \dots$$

which again gives a contradiction. This shows R to be a DVR. If we now choose c , in the centre of R , so that k in (5) takes its least positive value, then

it is easily seen that every centre element has the form $c^n v$, where v is a unit, again in the centre. Hence the centre of R is also a DVR, and the proof is complete.

REFERENCES

- [1] G. Azumaya, Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem, *Nagoya Math. J.* **1** (1950), 117-124.
- [2] S. U. Chase, On direct sums and products of modules, *Pacific J. Math.* **12** (1962), 847-854.
- [3] P. M. Cohn, Factorization in non-commutative power series rings, *Proc. Cambridge Phil. Soc.* **58** (1962), 452-464.
- [4] P. M. Cohn, Noncommutative unique factorization domains, *Trans. Amer. Math. Soc.* **109** (1963), 313-331.
- [5] P. M. Cohn, Free ideal rings, *J. Algebra* **1** (1964), 47-69.
- [6] P. M. Cohn, A remark on total matrix rings over firs, *Proc. Cambridge Phil. Soc.* **62** (1966) 1-4.
- [7] I. Kaplansky, Projective modules, *Ann. of Math.* **68** (1958), 372-377.
- [8] D. G. Northcott, The centre of a hereditary local ring, *Proc. Glasgow Math. Association* **5** (1962), 101-102.

*Queen Mary College,
University of London.*