SOME RESULTS IN THE FOURIER ANALYSIS

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To TADASI NAKAYAMA in memorial

There are many uses of Fourier analysis in the analytic number theory. In this paper we shall derive two fundamental theorems using Cramer's method (Mathematical methods of statistics, 1946). Let E, E^* be unit cubes in the whole n-dimensional Euclidean space X such that

$$E = \{(u_1 \cdots u_n):\ 0 \leqq u_1 \leqq 1, \ldots, 0 \leqq u_n \leqq 1\}$$

$$E^* = \left\{(u_1 \cdots u_n):\ x_1 - \frac{1}{2} \leqq u_1 \leqq x_1 + \frac{1}{2}, \ldots, x_n - \frac{1}{2} \leqq u_n \leqq x_n + \frac{1}{2}\right\}$$

We define $F(u)$ as follows

$$F(u) = 0\ (u < x-t),\ \frac{1}{2}\ (u = x-t),\ 1\ (x-t < u < x+t),$$
$$\frac{1}{2}\ (u = x+t),\ 0\ (x+t < u),$$

for fixed x and $t > 0$.

LEMMA 1. *For fixed x and $t\left(0 < t < \frac{1}{2}\right)$ the function*

$$\sum_{m=-k}^{k} \frac{\operatorname{sim} 2\pi m(x+t-u)}{2\pi m} - \sum_{m=-k}^{k} \frac{\sin 2\pi m(x-t-u)}{2\pi m} \tag{1}$$

is boundedly convergent to $F(u)$ as $k \to \infty$, where $x - \frac{1}{2} \leqq u \leqq x + \frac{1}{2}$.

Proof. Since (1) is equal to

$$2t + 2\int_0^{x+t-u} (\cos 2\pi z + \cdots + \cos 2\pi kz)\, dz - 2\int_0^{x-t-u} (\cos 2\pi z + \cdots + 2\pi kz)\, dz$$

$$= 2t + 2\int_{x-t-u}^{x+t-u} \frac{\sin\left(k+\frac{1}{2}\right)2\pi z - \sin\frac{1}{2}2\pi z}{2\sin\frac{1}{2}2\pi z}\, dz = \int_{x-t-u}^{x+t-u} \frac{\sin(2k+1)\pi z}{\sin \pi z}\, dz,$$

the lemma is obtained by proving that

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$$\int_0^a \frac{\sin(2k+1)\pi z}{\operatorname{sim} \pi z} dz \qquad \left(|a| \leqq t + \frac{1}{2}\right)$$

is boundedly convergent to $\frac{1}{2}(a>0)$, $0(a=0)$, $-\frac{1}{2}(a<0)$ as $k \to \infty$. If we put $2\pi z = v$, then

$$\int_0^a \frac{\sin(2k+1)\pi z}{\sin \pi z} dz = \frac{1}{z}\int_0^{2\pi a} \frac{\sin\left(k+\frac{1}{2}\right)v}{2\sin\frac{v}{2}} dv$$

$$= \frac{1}{\pi}\left\{\int_0^{(2k+1)\pi a} \frac{\sin v}{v} dv + \int_0^{2\pi a}\left(\frac{1}{2\sin\frac{v}{2}} - \frac{1}{v}\right)\sin\left(k+\frac{1}{2}\right)v\, dv\right\},$$

whence follows the result by the theorem of Dirichlet's integral.

LEMMA 2. *Let $f(u_1, \ldots, u_n)$ be a function with period* 1 *for each variable and be L-integrable over E. If we write*

$$a(m_1, \ldots, m_n) = \int_E \cdots \int e^{2\pi i(m_1u_1+\cdots+m_nu_n)} f(u_1\cdots u_n)\, du_1\cdots du_n,$$

then

$$\sum_{m_1=-k}^{k} \cdots \sum_{m_n=-k}^{k} \left(\frac{\sin 2\pi m_1 t_1}{2\pi m_1 t_1} \cdots \frac{\sin 2\pi m_n t_n}{2\pi m_n t_n}\right) e^{-2\pi i(m_1x_1+\cdots+m_nx_n)} a(m_1\cdots m_n) \tag{2}$$

is convergent to

$$\frac{1}{2t_1}\cdots\frac{1}{2t_n}\int_{x_1-t_1}^{x_1+t_1}\cdots\int_{x_n-t_n}^{x_n+t_n} f(n_1\cdots u_n)\, du_1\cdots du_n$$

as $k \to \infty$, provided that $0<t_1<\frac{1}{2}, \ldots, 0<t_n<\frac{1}{2}$.

Proof. By making use of Lebesgue's dominated convergence theorem, it follows from Lemma 1 that

$$\int_E \cdots \int \prod_{h=1}^{n}\left\{\sum_{m_h=-k}^{k} \frac{e^{2\pi i m_h(u_h-x_h)} \sin 2\pi m_h t_h}{\pi m_h}\right\} f(u_1\cdots u_n)\, du_1\cdots du_n \tag{3}$$

$$= \int_{E^*} \cdots \int \prod_{h=1}^{n}\left\{\sum_{m_h=-k}^{k} \frac{\sin 2\pi m_h(x_h+t_h-u_h)}{2\pi m_h} - \sum_{m_h=-k}^{k} \frac{\sin 2\pi m_h(x_h-t_h-u_h)}{2\pi m_h}\right\}$$

$$f(u_1\cdots u_n)\, du_1\cdots du_n \qquad \text{(by translation)}$$

$$\to \int_{x_1-t_1}^{x_1+t_1}\cdots\int_{x_n-t_n}^{x_n+t_n} f(u_1\cdots u_n)\, du_1\cdots du_n \qquad (\text{as } k\to\infty).$$

On the other hand, it is easy to verify that $(2) = \frac{1}{2t_1}\cdots\frac{1}{2t_n}(3)$. Hence we get the lemma.

THEOREM 1. *Let $f(u_1\cdots u_n)$ and $a(m_1\cdots m_n)$ be the same as in Lemma 2. If*

$$\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}|a(m_1\cdots m_n)| \tag{4}$$

is convergent, then there exists

$$\lim_{t_1\to 0,\dots,t_n\to 0}\frac{1}{2t_1}\cdots\frac{1}{2t_n}\int_{x_1-t_1}^{x_1+t_1}\cdots\int_{x_n-t_n}^{x_n+t_n}f(u_1\cdots u_n)du_1\cdots du_n \tag{5}$$

and equals

$$\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}e^{-2\pi i(m_1x_1+\cdots+m_nx_n)}a(m_1\cdots m_n).$$

Proof. Since

$$g(t_1\cdots t_n)$$
$$=\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}\left(\frac{\sin 2\pi m_1t_1}{2\pi m_1t_1}\cdots\frac{\sin 2\pi m_nt_n}{2\pi m_nt_n}\right)e^{-2\pi i(m_1x_1+\cdots+m_nx_n)}a(m_1\cdots m_n)$$

is convergent absolutely and uniformly in the neighbourhood of $(t_1\cdots t_n)=(0\cdots 0)$, we have

$$\lim_{t_1\to 0,\dots,t_n\to 0}g(t_1\cdots t_n)=\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}e^{-2\pi i(m_1x_1+\cdots+m_nx_n)}a(m_1\cdots m_n),$$

whence follows the result by Lemma 2.

Firstly we note that if $f(u_1\cdots u_n)$ is continuous at $(x_1\cdots x_n)$, then (5) is equal to $f(x_1\cdots x_n)$. Secondly we note that if

$$\frac{\partial^{p_1+\cdots+p_n}}{\partial^{p_1}x_1\cdots\partial^{p_n}x_n}f(x_1\cdots x_n)$$

are continuous in X, where $p_i(1\leqq i\leqq n)$ is 0 or 1, then (4) of Theorem 1 is convergent. This can be obtained by the device of using Bessel's inequality.

THEOREM 2. *Let $f(u_1\cdots u_n)$ be L-integrable over X. We write*

$$g(v_1\cdots v_n)=\int_X\cdots\int e^{2\pi i(v_1u_1+\cdots v_nu_n)}f(u_1\cdots u_n)\,du_1\cdots du_n,$$

and assume that $g(v_1\cdots v_n)$ is L-integrable over X. Then, there exists

$$\lim_{t_1\to 0,\dots,t_n\to 0}\frac{1}{2t_1}\cdots\frac{1}{2t_n}\int_{x_1-t_1}^{x_1+t_1}\cdots\int_{x_n-t_n}^{x_n+t_n}f(u_1\cdots u_n)\,du_1\cdots du_n \tag{6}$$

Proof. Instead of (1), we use the formula

$$\lim_{z\to\infty}\frac{1}{\pi}\int_{-z}^{z}\frac{\sin 2\pi tv}{v}e^{2\pi i(u-x)v}dv$$

$$=\lim_{z\to\infty}\left\{\int_{-z}^{z}\frac{\sin 2\pi v(x+t-u)}{2\pi v}dv-\int_{-z}^{z}\frac{\sin 2\pi v(x-t-u)}{2\pi v}dv\right\}=F(u) \qquad (7)$$

(boundedly convergent)

By the assumption, the integral

$$\int_{-z}^{z}\cdots\int_{-z}^{z}\prod_{h=1}^{n}\left\{\frac{\sin 2\pi t_h v_h}{2\pi t_h v_h}e^{-2\pi i x_h v_h}\right\}g(v_1\cdots v_n)dv_1\cdots dv_n.$$

exists and equals

$$\frac{1}{2t_1}\cdots\frac{1}{2t_n}\int_X\cdots\int\prod_{h=1}^{n}\left\{\frac{1}{\pi}\int_{-z}^{z}\frac{\sin 2\pi t_h v_h}{v_h}e^{2\pi i(v_h-x_h)v_h}dv_h\right\}f(u_1\cdots u_n)du_1\cdots du_n$$

by Fubini's theorem. Letting $z\to\infty$ and using the Lebesque dominated convergence theorem, we obtain

$$\frac{1}{2t_1}\cdots\frac{1}{2t_n}\int_{x_1-t_1}^{x_1+t_1}\cdots\int_{x_n-t_n}^{x_n+t_n}f(u_1\cdots u_n)du_1\cdots du_n$$

$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{h=1}^{n}\left\{\frac{\sin 2\pi t_h v_h}{2\pi t_h v_h}\right\}e^{-2\pi i(x_1v_1+\cdots+x_nv_n)}g(v_1\cdots v_n)dv_1\cdots dv_n$$

by (7). By the same manner as in Theorem 1, we can prove the theorem, and if $f(u_1\cdots u_n)$ is continuous at $(x_1\cdots x_n)$, then (6) is equal to $f(x_1\cdots x_n)$. It should be noted that $g(v_1\cdots v_n)$ is always continuous in X by the Lebesque domominated convergence theorem.

Finally we add the Poisson summation formula as an application of Theorem 1. Let $f(x_1\cdots x_n)$ be continuous and L-integrable over X and

$$\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty}f(x_1+k_1\cdots x_n+k_n)$$

be uniformly convergent in E. We put

$$a(m_1\cdots m_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{2\pi i(m_1t_1+\cdots+m_nt_n)}f(t_1\cdots t_n)\,dt_1\cdots dt_n$$

and assume that $\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}|a(m_1\cdots m_n)|$

is convergent. Then

$$\sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} f(m_1 \cdots m_n) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} a(m_1 \cdots m_n)$$

To prove this, let

$$F(x_1 \cdots x_n) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} f(x_1 + k_1 \cdots x_n + k_n).$$

Since

$$a(m_1 \cdots m_n)$$

$$= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \int_{k_1}^{k_1+1} \cdots \int_{k_n}^{k_n+1} e^{2\pi i(m_1(t_1-k_1)+\cdots+m_n(t_n-k_n))} f(t_1 \cdots t_n) dt_1 \cdots dt_n$$

$$= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \int_E \cdots \int e^{2\pi i(m_1x_1+\cdots+m_nx_n)} f(x_1 + k_1 \cdots x_n + k_n) dx_1 \cdots dx_n$$

$$= \int_E \cdots \int e^{2\pi i(m_1x_1+\cdots+m_nx_n)} F(x_1 \cdots x_n) dx_1 \cdots dx_n,$$

it follows from the assumption and Theorem 1 that

$$F(x_1 \cdots x_n) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} a(m_1 \cdots m_n) e^{-2\pi i(m_1x_1+\cdots+m_nx_n)}$$

$$= \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{2\pi i(m_1(t_1-x_1)+\cdots+m_n(t_n-x_n))} f(t_1 \cdots t_n) dt_1 \cdots dt_n$$

$$= \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{2\pi i(m_1u_1+\cdots+m_nu_n)} f(u_1 + x_1 \cdots u_n + x_n) du_1 \cdots du_n,$$

whence follows the result by putting $x_1 = \cdots = x_n = 0$.

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