ON α-HARMONIC FUNCTIONS

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Chapter 1. Introduction and Preliminaries

M. Riesz [8] introduced the notion of α-superharmonic functions in $n(\geq 1)$-dimensional Euclidean space R^n in connection with the potential of order α. In this paper, we shall first define the α-superharmonic and α-harmonic functions in a domain D. In case $\alpha = 2$, they coincide with ones in the usual sense. Next we shall introduce generalized Laplacians $\underline{P}_f^\alpha(x)$ and $P_f^\alpha(x)$ of order α, which are, in the case $\alpha = 2$, equal to the well-known generalized Laplacians except for a universal constant. Then we shall prove the following equivalences.

1. *A Lebesgue measurable function $f(\not\equiv +\infty)$ in R^n is α-superharmonic in a domain D if and only if f is lower semicontinuous and $\underline{P}_f^\alpha(x) \leq 0$ in D.*

2. *A Lebesgue measurable function f in R^n is α-harmonic in a domain D if and only if f is finite continuous in D and $P_f^\alpha(x) = 0$ in D.*

Finally we shall prove Ninomiya's domination principle as an application of the above results.

In R^n, the potential of a given order α, $0 \leq \alpha < n$, of a measure μ in R^n is defined by

$$U_\alpha^\mu(x) = \int |x-y|^{\alpha-n} d\mu(y),$$

provided the integral on the right exists. We shall say that a measure μ in R^n is *α-finite* if the potential $U_\alpha^\mu(x)$ is finite *p.p.p.* in R^n. Here a property is said to hold *p.p.p.* on a subset X *in* R^n, when the property holds on X except for a set E which does not support any measure $\nu \neq 0$ with finite *α-energy* $\iint |x-y|^{\alpha-n} d\nu(y) d\nu(x)$. M. Riesz [8] proved that every α-finite measure can be balayaged to every closed set if $0 < \alpha \leq 2$, $0 < \alpha < 2$ *or* $0 < \alpha < 1$ according to $n \geq 3$, $n = 2$ *or* $n = 1$. This paper is based on this result. Let F be a closed set in R^n and x be a point in $\mathscr{C}F$. We shall denote the balayaged measure

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of a unit measure ε_x at x to F by $\mu_{x,F}^{(\alpha)}$. Let $B(x_0\,;\,r)$ be an open ball with center x_0 and radius r. If $\alpha \neq 2$, for any x in $B(x_0\,;\,r)$,

$$d\mu_{x,\mathscr{C}B(x_0;r)}^{(\alpha)}(y) = \lambda_{x_0,r}(x, y)dy$$

with

$$\lambda_{x_0,r}(x, y) = \begin{cases} a_\alpha(r^2 - |x - x_0|^2)^{\alpha/2}(|y - x_0|^2 - r^2)^{-\alpha/2}|y - x|^{-n} & \textit{in } \mathscr{C}B(x_0\,;\,r) \\ 0 \textit{ in } B(x_0\,;\,r), \end{cases}$$

where

$$a_\alpha = \pi^{-(\alpha/2+1)}\Gamma\left(\frac{n}{2}\right)\sin\frac{\alpha n}{2}.$$

It holds that

$$\int d\mu_{x,F}^{(\alpha)} \leq 1 \text{ and } \int \kappa_{x_0,r}(y)dy = 1,$$

where

$\kappa_{x_0,r}(y)$ stands for $\lambda_{x_0,r}(x_0, y)$. For a given real-valued function f Lebesgue measurable in R^n, we shall denote

$$\int f(y)\kappa_{x_0,r}(y)dy$$

by $\mathfrak{M}_\alpha(x_0\,;\,f,\,r)$. This is a generalization of Gauss' mean value.

Chapter 2. α-harmonic functions

Throughout this chapter, we assume that $0<\alpha<2$ or $0<\alpha<1$ according to $n\geq 2$ or $n=1$. A measure with density f, measurable in R^n, will be called the measure f. First we shall define α-superharmonic functions and α-harmonic functions.

§ 2.1. Definitions

DEFINITION 1.[1] Let D be a domain in R^n. We shall say that a function f defined in R^n is α-superharmonic in D if f satisfies the following three conditions:

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1) The notion of α-superharmonicity was first introduced by M. Riesz [8]. According to him, a function f is α-superharmonic in R^n if f satisfies the following conditions:

(1) $f(x)\geq 0$ and $f(x)\not\equiv +\infty$ in R^n,

(2) f is lower semicontinuous in R^n,

(3) for each x in R^n and each open ball $B(x\,;\,r)$, $f(x)\geq\mathfrak{M}_\alpha(x\,;\,f,\,r)$.

Another kind of α-superharmonicity was introduced by Frostman [4].

(S. 1) f is Lebesgue measurable in R^n,

(S. 2) f is lower semicontinuous in D,

(S. 3) for each x in D and each open ball $B(x\,;\,r)$ contained with its closure in D, $\mathfrak{M}_\alpha(x\,;\,f,\,r)$ exists and

$$f(x) \geq \mathfrak{M}_\alpha(x\,;\,f,\,r).$$

DEFINITION 2. Let D be a domain in R^n. We shall say that a function f defined in R^n is α-harmonic in D if f satisfies the following three conditions:

(H. 1) f is Lebesgue measurable in R^n,

(H. 2) f is finite continuous in D,

(H. 3) for each x in D and each open ball $B(x\,;\,r)$ contained with its closure in D, $\mathfrak{M}_\alpha(x\,;\,f,\,r)$ exists and

$$f(x) = \mathfrak{M}_\alpha(x\,;\,f,\,r).$$

It is easily seen that the potential $U_\alpha^\mu(x)$ of an α-finite positive measure μ is α-superharmonic in R^n and α-harmonic in $\mathscr{C}S_\mu$.[2]

§ 2.2. Elementary properties

PROPERTY 1. *Let f and f' be α-harmonic in a domain D. If $f(x) = f'(x)$ in D, then $f(x) = f'(x)$ almost everywhere in R^n.* In fact, for any open ball $B(x_0\,;\,r_0)$ contained with its closure in D and any x in $B(x_0\,;\,r_0)$, it holds that

$$\begin{aligned}&\int (f(y) - f'(y))\,\lambda_{x_0,r_0}(x,\,y)\,dy \\ &= a_\alpha (r_0^2 - |x - x_0|^2)^{\alpha/2} \int_{\mathscr{C}B(x_0;r_0)} (f(y) - f'(y))\,(|y - x_0|^2 - r_0^2)^{-\alpha/2} |y - x|^{-n} dy \\ &= f(x) - f'(x) = 0\end{aligned}$$

by Lemma 4 which we shall be given in § 2.3. Put

$$g(x) = \begin{cases} 0 \text{ in } B(x_0\,;\,r_0) \\ (f(x) - f'(x))\,(|x - x_0|^2 - r_0^2)^{-\alpha/2} \text{ on } CB(x_0\,;\,r_0). \end{cases}$$

Then the potential of order 0 of the measure g is equal to 0 in $B(x_0\,;\,r_0)$. By the unicity theorem of M. Riesz[3], $g(x) = 0$ almost everywhere in R^n. Hence $f(x) = f'(x)$ almost everywhere in $\mathscr{C}B(x_0\,;\,r_0)$. This completes the proof.

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2) Cf. [8], n°20.

3) Cf. [8], n°11.

PROPERTY 2. *If f is harmonic in the usual sense in R^n, it is α-harmonic there.* In fact, let x_0 be a point in R^n and r be a positive number. Using the polar coordinate (ρ, σ) with center at x_0, we have

$$\mathfrak{M}_\alpha(x_0 ; f, r) = a_\alpha r^\alpha \int_r^\infty (\rho^2 - r^2)^{-\alpha/2} \rho^{-1} \left(\int_{S(x_0; 1)} f_{\rho, \sigma} d\sigma \right) d\rho,$$

where $S(x_0 ; 1)$ is a unit sphere with center x_0. Since f is harmonic in the usual sense in R^n,

$$f(x_0) = \frac{1}{\omega_n} \int_{S(x_0; 1)} f_{\rho, \sigma} d\sigma,$$

where ω_n denotes the area of the unit sphere. Hence

$$f(x_0) = \mathfrak{M}_\alpha(x_0 ; f, r).$$

PROPERTY 3. *If f is α-harmonic and bounded from below in R^n, then it is constant.* In fact, without loss of generality we may assume that f is non-negative. By M. Riesz's decomposition theorem[4], there exist α-finite positive measure ν and a non-negative constant C such that

$$f(x) = U_\alpha^\nu(x) + C$$

in R^n. Suppose that f is non-constant. Then there exist a point x_0 in R^n and a positive number r_0 such that $\nu(B(x_0 ; r_0)) > 0$. Let ν' be the balayaged measure of ν to $\mathscr{C}B(x_0 ; r_0)$. For any x in $B(x_0 ; r_0)$,

$$U_\alpha^{\nu'}(x_0) = \int U_\alpha^\nu(y) \lambda_{x_0, r_0}(x, y) dy = \int |y - z|^{\alpha - n} \lambda_{x_0, r_0}(x, y)\, dy d\nu(z)$$
$$< \int U_\alpha^{\varepsilon_x}(y) d\nu(y) = U_\alpha^\nu(x).$$

In particular,

$$U_\alpha^\nu(x_0) > \int U_\alpha^\nu(y) \kappa_{x_0, r_0}(y) dy = \mathfrak{M}_\alpha(x_0 ; U_\alpha^\nu, r_0).$$

This contradicts our assumptions.

PROPERTY 4. *Let f be harmonic in the usual sense in R^n. If it is bounded from below, it is constant.* This follows from Properties 2 and 3.

PROPERTY 5. *Let f be α-harmonic in R^n. If there exist an α-finite positive*

[4] Cf. [8], n°31 and n°32.

measure ν *and a positive constant* C *such that*

$$|f(x)|\leq U_\alpha^\nu(x)+C$$

in R^n, *then* f *is constant.* In fact, for any x_0 in R^n and any positive number r,

$$|f(x_0)|=\left|\int_{\mathscr{C}B(x_0;\,r)} f(y)\kappa_{x_0,r}(y)\,dy\right|\leq\int_{\mathscr{C}B(x_0;\,r)}|f(y)|\,\kappa_{x_0,r}(y)\,dy$$
$$\leq\int_{\mathscr{C}B(x_0;\,r)}(U_\alpha^\nu(y)+C)\,\kappa_{x_0,r}(y)\,dy=\mathfrak{M}_\alpha(x_0\,;\,U_\alpha^\nu,\,r)+C.$$

Since $\lim_{r\to\infty}\mathfrak{M}_\alpha(x_0\,;\,U_\alpha^\nu,\,r)=0$[5], $|f(x_0)|\leq C$.
By Property 3, f is constant.

PROPERTY 6. *Let* f *be harmonic in the usual sense in* R^n. *If there exist an* α*-finite positive measure* ν *and a non-negative constant* C *such that*

$$|f(x)|\leq U_\alpha^\nu(x)+C$$

in R^n, *then* f *is constant.* This follows from Properties 2 and 5.

§ 2.3. Four Lemmas

Let D be a domain in R^n and a function f defined in R^n be $\mu_{x,\mathscr{C}D}^{(\alpha)}$-integrable for any x in D. We denote by $E_{f,D}(x)$ the following function

$$\begin{cases} f(x) \ \ in\ \mathscr{C}D \\ \int f(y)\,d\mu_{x,\mathscr{C}D}^{(\alpha)}(y)\ \ in\ \ D. \end{cases}$$

LEMMA 1. *Let* $B(x_0\,;\,r_0)$ *be an open ball and* f *be a Lebesgue measurable and bounded function in* R^n. *Then* $E_{f,B(x_0;\,r_0)}(x)$ *is* α*-harmonic in* $B(x_0\,;\,r_0)$.

Proof. Evidently $E_{f,B(x_0;\,r_0)}(x)$ is finite continuous in $B(x_0\,;\,r_0)$. Hence it is sufficient to prove the condition (H. 2). By Lusin's theorem, there exists a sequence (f_m) of functions of class C^2 with compact support such that $f_m(x)\to f(x)$ almost everywhere in R^n as $m\to\infty$, and

$$|f_m(x)|\leq M,\ |f(x)|\leq M\ \ in\ \ R^n,$$

where M is a positive constant. Since f_m is of class C^2 with compact support,

$$f_m(x)=\int|x-y|^{\alpha-n}k_m(y)\,dy$$

where

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[5] Cf. [8], n°31.

$$k_m(y) = \int |y-z|^{(2-\alpha)-n} \Delta f_m(z)\, dz.$$

Let μ_m be the balayaged measure of the measure k_m to $\mathscr{C}B(x_0\,;\,r_0)$. Then

$$U_\alpha^{\mu_m}(x) = \begin{cases} f_m(x) \ \ on\ \mathscr{C}B(x_0\,;\,r_0) \\ \int f_m(y)\lambda_{x_0,r_0}(x,\,y)\,dy \ \ in\ B(x_0\,;\,r_0). \end{cases}$$

By Lebesgue's bounded convergence theorem,

$$U_\alpha^{\mu_m}(x) \to E_{f,\,B(x_0;\,r_0)}(x)$$

almost everywhere in R^n as $m \to \infty$. On the other hand, being

$$\int \lambda_{x_0,r_0}(x,\,y)\,dy \leq 1,$$

it holds that

$$|U_\alpha^{\mu_m}(x)| \leq M \ in\ R^n.$$

Hence by Lebesgue's bounded convergence theorem,

$$\int U_\alpha^{\mu_m}(y)\,\kappa_{x_1,r}(y)\,dy \to \int E_{f,\,B(x_0;\,r_0)}(y)\,\kappa_{x_1,r}(y)\,dy$$

as $m \to \infty$ for any open ball $B(x_1\,;\,r)$ contained with its closure in $B(x_0\,;\,r_0)$. Since $S_{\mu_m} \subset \mathscr{C}B(x_0\,;\,r_0) \subset \mathscr{C}B(x_1\,;\,r)$,

$$U_\alpha^{\mu_m}(x_1) = \int U_\alpha^{\mu_m}(y)\,\kappa_{x_1,r}(y)\,dy.$$

Consequently

$$E_{f,\,B(x_0;\,r_0)}(x_1) = \mathfrak{M}_\alpha(x_1,\,E_{f,\,B(x_0;\,r_0)},\,x).$$

This completes the proof.

LEMMA 2. *Let $B(x_0\,;\,r_0)$ be an open ball and a function f be Lebesgue measurable in R^n. If f is κ_{r_0,r_0}-integrable, for any fixed x in $B(x_0\,;\,r_0)$ f is $\lambda_{x_0,x_0}(x,\,y)$-integrable and $E_{f,\,B(x_0;\,r_0)}(x)$ is α-harmonic in $B(x_0\,;\,r_0)$.*

Proof. First we shall show that in $B(x_0\,;\,r_0)$

$$\int |f(y)|\,\lambda_{r_0,r_0}(x,\,y)\,dy < +\infty.$$

In fact, for any fixed x in $B(x_0\,;\,r_0)$, there exists a positive constant M such that

$$|y-x|^{-n} \leq M|y-x_0|^{-n}$$

for any y in $\mathscr{C}B(x_0\,;\,r_0)$. Now

$$\begin{aligned}
&\int |f(y)|\,\lambda_{x_0,r_0}(x,y)\,dy \\
&= a_\alpha (r_0^2-|x-x_0|^2)^{\alpha/2} \int_{\mathscr{C}B(x_0\,;\,r_0)} |f(y)|(|y-x_0|^2-r_0^2)^{-\alpha/2}|y-x|^{-n}dy \\
&\leq M(r_0^2-|x-x_0|^2)^{\alpha/2} r_0^{-\alpha} \int |f(y)|\,\kappa_{x_0,r_0}(y)\,dy < +\infty.
\end{aligned}$$

Similarly as Lemma 1, $E_{f,B(x_0\,;\,r_0)}(x)$ is finite continuous in $B(x_0\,;\,r_0)$. Put

$$f_m^+(x) = \inf\,(f^+(x),\ m),\ f_m^-(x) = \inf\,(f^-(x),\ m),$$

where

$$f^+(x) = \sup\,(f(x),\ 0),\ f^-(x) = -\inf\,(f(x),\ 0).$$

By Lemma 1, $E_{f_m^+,B(x_0\,;\,r_0)}(x)$ and $E_{f_m^-,B(x_0\,;\,r_0)}(x)$ are α-harmonic in $B(x_0\,;\,r_0)$. Hence

$$E_{f_m^+,B(x_0\,;\,r_0)}(x) = \mathfrak{M}_\alpha(x\,;\,E_{f_m^+,B(x_0\,;\,r_0)},\ r),$$

and

$$E_{f_m^-,B(x_0\,;\,r_0)}(x) = \mathfrak{M}_\alpha(x\,;\,E_{f_m^-,B(x_0\,;\,r_0)},\ r),$$

for any open ball $B(x\,;\,r)$ contained with its closure in $B(x\,;\,r)$. Since $(E_{f_m^+,B(x_0\,;\,r_0)})$ tends increasingly to $E_{f^+,B(x_0\,;\,r_0)}$,

$$\mathfrak{M}_\alpha(x\,;\,E_{f_m^+,B(x_0\,;\,r_0)},\ r) \to \mathfrak{M}_\alpha(x\,;\,E_{f^+,B(x_0\,;\,r_0)},\ r)$$

as $m\to\infty$. Consequently

$$E_{f^+,B(x_0\,;\,r_0)}(x) = \mathfrak{M}_\alpha(x\,;\,E_{f^+,B(x_0\,;\,r_0)},\ r)$$

for any x in $B(x_0\,;\,r_0)$ and any open ball $B(x\,;\,r)$ contained with its closure in $B(x_0\,;\,r_0)$. Similarly we obtain that

$$E_{f^-,B(x_0\,;\,r_0)}(x) = \mathfrak{M}_\alpha(x\,;\,E_{f^-,B(x_0\,;\,r_0)},\ r).$$

Therefore

$$\begin{aligned}
E_{f,B(x_0\,;\,r_0)}(x) &= E_{f^+,B(x_0\,;\,r_0)}(x) - E_{f^-,B(x_0\,;\,r_0)}(x) \\
&= \mathfrak{M}_\alpha(x\,;\,E_{f^+,B(x_0\,;\,r_0)},\ r) - \mathfrak{M}_\alpha(x\,;\,E_{f^-,B(x_0\,;\,r_0)},\ r) \\
&= \mathfrak{M}_\alpha(x\,;\,E_{f,B(x_0;r_0)},\ r).
\end{aligned}$$

This completes the proof.

For a general domain D, we get in the same way the following

LEMMA 2'. *Let D be a domain in R^n and a function f be Borel measurable in R^n. If is $\mu^{(\alpha)}_{x, \mathscr{C}D}$-integrable for any x in D, $E_{f,D}(x)$ is α-harmonic in D.*

LEMMA 3. *Let a function f be α-harmonic in a bounded domain D. If f is finite continuouus on $\overline{D}$ and $f(x)=0$ almost everywhere in $\mathscr{C}D$, then $f(x)=0$ in D.*

Proof. Let x_0 be a point in $\overline{D}$ such that

$$f(x_0) = \max\{f(x)\ ;\ x \in \overline{D}\}.$$

Suppose that $f(x_0) > 0$. Then x_0 is not on the boundary of D. Let $B(x_0\ ;\ r)$ be an open ball contained with its closure in D. Then

$$\begin{aligned} \mathfrak{M}_\alpha(x_0\ ;\ f, r) &= \int f(y)\,\kappa_{x_0,r}(y)\,dy \\ &= \int_{\mathscr{C}B(x_0\ ;\ r)\cap D} f(y)\,\kappa_{x_0,r}(y)\,dy \leq \int_{\mathscr{C}B(x_0\ ;\ r)\cap D} f(x_0)\,\kappa_{x_0,r}(y)\,dy \\ &< \int f(x_0)\,\kappa_{x_0,r}(y)\,dy = f(x_0). \end{aligned}$$

This contradicts the α-harmonicity of f. Therefore $f(x) \leq 0$ in D. Similarly we obtain $f(x) \geq 0$ in D, and hence $f(x)=0$ in D.

LEMMA 4. *Let f be α-harmonic in a domain D. For each open ball contained with its closure in D,*

$$f(x) = \int f(y)\,\lambda_{x_0,r}(x, y)\,dy$$

in $B(x_0\ ;\ r)$ and f is analytic in D.

Proof. Similarly as Lemma 2, for any x in $B(x_0\ ;\ r)$,

$$\int |f(y)|\,\lambda_{x_0,r}(x, y)\,dy < +\infty.$$

By Lemma 2, $E_{f, r(x_0\ ;\ r)}(x)$ is α-harmonic in $B(x_0\ ;\ r)$. Put

$$g(x) = f(x) - E_{f, B(x_0\ ;\ r)}(x).$$

Then $g(x)=0$ in $B(x_0\ ;\ r)$. Consequently in $B(x_0\ ;\ r)$,

$$f(x) = \int f(y)\,\lambda_{x_0\ r}(x, y)\,dy.$$

Hence by M. Riesz's theorem[6], f is analytic in $B(x_0 ; r)$. $B(x_0 ; r)$ being arbitrary, f is analytic in D. This completes the proof.

§ 2.4. Extension of generalized Laplacian

Now we shall introduce another mean value of a function. Let f be a Lebesgue measurable function in R^n. If

$$\gamma\int_1^{\infty}\rho^{-\gamma-1}(\rho^2-1)^{\gamma/2-1}\mathfrak{M}_\alpha(x ; f, r\rho)\,d\rho$$

exists for a positive nember γ, we denote it by $\mathscr{A}_{\alpha,\gamma}(x ; f, r)$. Since

$$\gamma\int_1^{\infty}\rho^{-\gamma-1}(\rho^2-1)^{\gamma/2-1}d\rho=1,$$

$\mathscr{A}_{\alpha,\gamma}(x ; f, r)$ is considered as a kind of mean values of f. By M. Riesz's formula,

$$\begin{aligned}&\mathscr{A}_{\alpha,\gamma}(x ; f, r)\\&=C_{\alpha,\gamma,n}r^{\alpha}\int_{\mathscr{C}B(x;r)}(|x-y|^2-r^2)^{\gamma/2-\alpha/2}|x-y|^{-\gamma-n}f(y)\,dy,\end{aligned}$$

where

$$C_{\alpha,\gamma,n}=\frac{\pi^{-n/2}\Gamma\left(\frac{n}{2}\right)\Gamma\left(1+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}\right)}.$$

We denote the mean value corresponding to $\gamma=\alpha$ by $\mathscr{A}_\alpha(x ; f, r)$. Thus

$$\mathscr{A}_\alpha(x ; f, r)=\frac{\alpha r^{\alpha}}{\omega_n}\int_{\mathscr{C}B(x;r)}|x-y|^{-\alpha-n}f(y)\,dy$$

We denote

$$\varliminf_{\varepsilon\to 0}\frac{\omega_n}{\alpha\varepsilon^{\alpha}}(\mathscr{A}_\alpha(x ; f, \varepsilon)-f(x))$$

by $\underline{P}^{\alpha}_{f}(x)$. In particular, when

$$\lim_{\varepsilon\to 0}\frac{\omega_n}{\alpha\varepsilon^{\alpha}}(\mathscr{A}_\alpha(x ; f, \varepsilon)-f(x))$$

exists, we denote it by $P^{\alpha}_{f}(x)$. For $\alpha=2$, $P^{\alpha}_{f}(x)$ coincides with the generalized Laplacian except for a universal constant[7].

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[6] Cf. [8], n°26.

[7] Cf. [1], pp. 17-18,

§ 2.5. Inverse distribution of $r^{\alpha-n}$

We consider the distribution D_α such that

$$D_\alpha * r^{\alpha-n} = -\delta,$$

where δ is Dirac's distribution. By Deny's theorem[8],

$$D_\alpha = C_{\alpha,n}\,\mathrm{pf.}\, r^{-\alpha-n}$$

where

$$C_{\alpha,n} = \pi^{-n}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}$$

and the distribution pf. $r^{-\alpha-n}$ is defined as follows:

$$\mathrm{pf.}\, r^{-\alpha-n}(\varphi) = \mathrm{pf.}\int |x|^{-\alpha-n}\varphi(x)\,dx$$ [9]

for a function φ of class C^∞ with compact support.

LEMMA 5. *Let f be a measurable function defined in R^n, and x_0 be a point in R^n. If f is a function of class C^2 in a neighborhood of x_0 and*

$$\int_{\complement B(x_0;\,\varepsilon_0)} |y|^{-\alpha-n} f(x_0-y)\,dy < +\infty$$

for a positive number ε, then $P_f^\alpha(x_0)$ exists and

$$P_f^\alpha(x_0) = \mathrm{pf.}\int |y|^{-\alpha-n} f(x_0-y)\,dy.$$

Proof. Without loss of generality we may assume that $x_0 = 0$. By our assumptions, for any y in some neighborhood of 0,

$$f(y) = f(0) + \sum_{i=1}^n y_i \frac{\partial f}{\partial y_i}(0) + \frac{1}{2}\sum_{i,j=1}^n y_i y_j \frac{\partial^2 f}{\partial y_i \partial y_j}(0) + \psi(y),$$

where $\psi(y) = o(|y^2|)$ and $y = (y_1, y_2, \ldots, y_n)$. Hence

$$\int_{B(0;\,\varepsilon)} \psi(y)\,|y|^{-\alpha-n}\,dy < +\infty$$

for any sufficiently small positive number ε. Hence

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[8] Cf. [2], p. 153.

[9] Cf. [9], p. 42.

$$\text{pf.}\int f(-y)\,|y|^{-\alpha-n}dy$$

exists, and

$$\text{pf.}\int f(-y)|y|^{-\alpha-n}dy = \text{pf.}\int f(y)|y|^{-\alpha-n}dy$$
$$= \lim_{\varepsilon\to 0}\Big(\int_{\mathscr{C}B(O\,;\,\varepsilon)}|y|^{-\alpha-n}f(y)\,dy + f(0)I^{(1)}(\varepsilon) + \sum_{i=1}^{n}\frac{\partial f}{\partial y_i}(0)I_i^{(2)}(\varepsilon)$$
$$+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial y_i\partial y_j}(0)\,I_{ij}^{(3)}(\varepsilon)\Big).$$

where $I^{(1)}$, $I_i^{(2)}$ and $I_{ij}^{(3)}$ are functions in $r(r=|y|)$ satisfying the following conditions:

(1) $$\frac{dI^{(1)}}{dr}(r) = \omega_n r^{-\alpha-1},$$

(2) $$\frac{dI_i^{(2)}}{dr}(r) = r^{-\alpha-1}\int_{S_1} y_i\,ds,$$

(3) $$\frac{dI_{ij}^{(3)}}{dr}(r) = r^{-\alpha-1}\int_{S_1} y_i y_j\,ds,$$

(4) *their integral constants are* 0,

where S_1 is the unit sphere with center 0 and ds is the area-element on S_1. Since y_i and $y_iy_j(i\neq j)$ are harmonic in the usual sense in R^n,

$$\int_{S_1} y_i\,ds = 0 \ \textit{and}\ \int_{S_1} y_iy_j\,ds = 0\,(i\neq j).$$

On the other hand

$$\int_{S_1} y_i^2\,ds = \frac{1}{n}\int_{S_1} y^2\,ds = \frac{\omega_n}{n}|y|^2.$$

Therefore

$$\text{pf.}\int |y|^{-\alpha-n}f(y)\,dy$$
$$= \lim_{\varepsilon\to 0}\Big(\int_{\mathscr{C}B(O\,;\,\varepsilon)}|y|^{-\alpha-n}f(y)\,dy - \frac{\omega_n}{\alpha\varepsilon^\alpha}f(0) + \frac{\omega_n}{n(2-\alpha)}\varepsilon^{2-\alpha}\Delta f(0)\Big)$$
$$= \lim_{\varepsilon\to 0}\Big(\int_{\mathscr{C}B(O\,;\,\varepsilon)}|y|^{-\alpha-n}f(y)\,dy - \frac{\omega_n}{\alpha\varepsilon^\alpha}f(0)\Big)$$
$$= \lim_{\varepsilon\to 0}\frac{\omega_n}{\alpha\varepsilon^\alpha}(\mathscr{A}_\alpha(0\,;\,f,\varepsilon) - f(0)).$$

Consequently

$$P_f^\alpha(0) = \text{pf.}\int |y|^{-\alpha-n}f(-y)\,dy.$$

This completes the proof.

§ 2.6. Main theorems

THEOREM 1. *Let f be a Lebesgue measurable function defined in R^n and D be a domain in R^n. Assume that*

(1) f is lower semicontinuous and $f(x) > -\infty$ in D,

(2) f is $\kappa_{x,r}$-integrable for any x in D and any open ball $B(x\,;\,r)$ contained with its closure in D. Then f is α-superharmonic in D if and only if $\underline{P}^{\alpha}_{f}(x) \leq 0$ in D.

Proof. First suppose that f is α-superharmonic in D. For any x in D and any open ball $B(x\,;\,r)$ contained with its closure in D,

$$\int_{\mathscr{C}B(x\,;\,r)} |x-y|^{-\alpha-n}|f(y)|\,dy < +\infty.$$

In fact,

$$\begin{aligned}&\int_{\mathscr{C}B(x\,;\,r)} |f(y)|\,\kappa_{x,r}(y)\,dy\\ &= a_\alpha r^\alpha \int_{\mathscr{C}B(x\,;\,r)} |f(y)|(|y-x|^2-r^2)^{-\alpha/2}|x-y|^{-n}dy\\ &\geq a_\alpha r^\alpha \int_{\mathscr{C}B(x\,;\,r)} |f(y)|\,|y-x|^{-\alpha-n}dy.\end{aligned}$$

Hence

$$\int_{\mathscr{C}B(x\,;\,r)} |x-y|^{-\alpha-n}|f(y)|\,dy < +\infty.$$

f being α-superharmonic in D, there exists a positive number r_x such that

$$f(x) \geq \mathfrak{M}_\alpha(x\,;\,f,\,r)$$

for any $0<r\leq r_x$. We take an arbitrary positive number ε such that $\varepsilon<r_x$. Then

$$\begin{aligned}&\mathscr{A}_\alpha(x\,;\,f,\,\varepsilon)-f(x)\\ &= \alpha\int_1^\infty \rho^{-\alpha-1}(\rho^2-1)^{\alpha/2-1}(\mathfrak{M}_\alpha(x\,;\,f,\,\varepsilon\rho)-f(x))\,d\rho\\ &\leq \alpha\int_{r_x/\varepsilon}^\infty \rho^{-\alpha-1}(\rho^2-1)^{\alpha/2-1}(\mathfrak{M}_\alpha(x\,;\,f,\,\varepsilon\rho)-f(x))\,d\rho.\end{aligned}$$

Now

$$\alpha\left|\int_{r_x/\varepsilon}^\infty \rho^{-\alpha-1}(\rho^2-1)^{\alpha/2-1}(\mathfrak{M}_\alpha(x\,;\,f,\,\varepsilon\rho)-f(x))\,d\rho\right|$$

$$\leq \alpha\int_{r_x/\varepsilon}^{\infty}\rho^{-\alpha-1}(\rho^2-1)^{\alpha/2-1}|\mathfrak{M}_\alpha(x\,;f,\,\varepsilon\rho)-f(x)|\,d\rho$$
$$\leq \alpha\int_{r_x/\varepsilon}^{\infty}\rho^{-\alpha-1}\Big(\rho^2-\Big(\frac{r_x}{\varepsilon}\Big)^2\Big)^{\alpha/2-1}|\mathfrak{M}_\alpha(x\,;f,\,\varepsilon\rho)-f(x)|d\rho.$$

Putting $r=\frac{\varepsilon}{r_x}\rho$, we obtain

$$\alpha\int_{r_x/\varepsilon}^{\infty}\rho^{-\alpha-1}\Big(\rho^2-\Big(\frac{r_x}{\varepsilon}\Big)^2\Big)^{\alpha/2-1}|\mathfrak{M}_\alpha(x\,;f,\,\varepsilon\rho)-f(x)|\,d\rho$$
$$=\alpha\Big(\frac{\varepsilon}{r_x}\Big)^2\int_1^{\infty}r^{-\alpha-1}(r^2-1)^{\alpha/2-1}|\mathfrak{M}_\alpha(x\,;f,\,rr_x)-f(x)|\,dr$$
$$\leq\alpha\Big(\frac{\varepsilon}{r_x}\Big)^2\int_1^{\infty}r^{-\alpha-1}(r^2-1)^{\alpha/2-1}(\mathfrak{M}_\alpha(x\,;|f|,\,rr_x)+|f(x)|)\,dr$$
$$\leq\Big(\frac{\varepsilon}{r_x}\Big)^2(\mathfrak{M}_\alpha(x\,;|f|,\,r_x)+|f(x)|).$$

Since we may assume that $f(x)$ is finite, $\mathfrak{M}_\alpha(x\,;|f|,r_x)+|f(x)|$ is finite. Hence

$$\underline{P}_f^\alpha(x)\leq\lim_{\varepsilon\to0}\frac{\omega_n\varepsilon^{2-\alpha}}{\alpha r_x^2}\cdot(\mathfrak{M}_\alpha(x\,;|f|,\,r_x)+|f(x)|=0.$$

In order to prove the converse, suppose that $\underline{P}_f^\alpha(x)\leq0$ in D, and let $B(x_0\,;r_0)$ be an open ball contained with its closure in D. Then it is sufficient to prove the following inequality:

$$f(x)\geq\int_{\mathscr{C}B(x_0\,;\,r_0)}f(y)\,\lambda_{x_0,r_0}(x,\,y)\,dy$$

in $B(x_0\,;r_0)$. By the condition (2),

$$\int|f(y)|\,\lambda_{x_0,r_0}(x,\,y)\,dy<+\infty.$$

We take an open ball $B(x_0\,;r_1)$ such that $\overline{B(x_0\,;r_0)}\subset B(x_0\,;r_1)\subset\overline{B(x_0\,;r_1)}\subset D$. Since f is lower semicontinuous and $f(x)>-\infty$ in D, there exists a sequence (φ_m) of continuous functions with compact support in R^n which tends increasing to f on $\overline{B(x_0\,;r_1)}$. Put

$$f_m(x)=\begin{cases}\varphi_m(x)\ \ in\ \ B(x_0\,;r_1)\\ f(x)\ \ on\ \ \mathscr{C}B(x_0\,;r_1).\end{cases}$$

Then $(E_{f_m,B(x_0;r_0)})$ tends increasingly to $E_{f,B(x_0;r_0)}$ as $m\to\infty$. Hence it is sufficient to prove that $f(x)\geq E_{f_m,B(x_0;r_0)}(x)$ in $B(x_0\,;r_0)$ for any m. Now let φ be a function of class C^∞ with compact support in R^n such that $\varphi(x)\geq0$ in R^n and $\varphi(x)=1$ in $B(x_0\,;r_0)$. And let μ_φ be the balayaged measure of the measure φ to $\mathscr{C}B(x_0\ ;\ r_0)$. Put

$$g(x)=\int|x-y|^{\alpha-n}\varphi(y)\,dy-\int|x-y|^{\alpha-n}d\mu_\varphi(y).$$

Then $g(x)$ is finite continuous in R^n and $g(x)=0$ on $\mathscr{C}B(x_0\ ;\ r_0)$. Moreover for any x in $B(x_0\ ;\ r_0)$, $P_g^\alpha(x)$ exists and

$$P_g^\alpha(x)=D_\alpha*(r^{\alpha-n}*\varphi)(x)-P_{U_\alpha^{\mu_\varphi}}^\alpha(x).$$

Since S_{μ_φ} is contained in $\mathscr{C}B(x_0\ ;\ r_0)$. $P_{U_\alpha^{\mu_\varphi}}^\alpha(x)=0$ in $B(x_0\ ;\ r_0)$. Hence

$$D_\alpha*g(x)=-\varphi(x)$$

in $B(x_0\ ;\ r_0)$. Now for any positive number ε, we denote $E_{f_m, B(x_0\,;\,r_0)}-f-\varepsilon g$ by h. The function h is upper semicontinuous and $h(x)<+\infty$ in $B(x_0\ ;\ r_1)$, and it is equal to 0 on $\mathscr{C}B(x_0\ ;\ r_1)$. By Lemma 2, $E_{f_m, B(x_0\,;\,r_0)}$ is α-harmonic in $B(x_0\ ;\ r_0)$. Suppose that there exists a point x_1 in $B(x_0\ ;\ r_0)$ such that $h(x_1)>0$ and

$$h(x_1)=\sup\{h(x)\ ;\ x\varepsilon B(x_0\ ;\ r_0)\}.$$

Then for any open ball $B(x_1\ ;\ r)$ contained with its closure in $B(x_0\ ;\ r_0)$,

$$\begin{aligned}\mathscr{A}_\alpha(x_1\ ;\ h,r)&=\frac{\alpha r^\alpha}{\omega_n}\int_{\mathscr{C}B(x_1\,;\,r_0)}|x_1-y|^{-\alpha-n}h(y)\,dy\\ &\le\frac{\alpha r^\alpha}{\omega_n}\int_{\mathscr{C}B(x_1\,;\,r)\cap B(x_0\,;\,r_0)}|x_1-y|^{-\alpha-n}h(y)\,dy\\ &\le\frac{\alpha r^\alpha}{\omega_n}\int_{\mathscr{C}B(x_1\,;\,r)\cap B(x_0\,;\,r_0)}|x_1-y|^{-\alpha-n}h(x_1)\,dy\\ &<\frac{\alpha r^\alpha}{\omega_n}\int_{\mathscr{C}B(x_1\,;\,r)}|x_1-y|^{-\alpha-n}h(x_1)\,dy=h(x_1).\end{aligned}$$

Hence

$$\overline{\lim_{\varepsilon\to0}}\frac{\omega_n}{\alpha\varepsilon^\alpha}(\mathscr{A}_\alpha(x_1\ ;\ h,\varepsilon)-h(x_1))\le0.$$

On the other hand

$$\underline{P}_{-h}^\alpha(x)\le-\varepsilon\varphi(x)=-\varepsilon$$

in $B(x_0\ ;\ r_0)$. This is a contradiction. Consequently $h(x)\le0$ in $B(x_0\ ;\ r_0)$, i.e.,

$$E_{f_m, B(x_0\,;\,r_0)}(x)\le f(x)$$

in $B(x_0\ ;\ r_0)$. Therefore

$$f(x)\ge E_{f, B(x_0\,;\,r_0)}(x)$$

in $B(x_0\ ;\ r_0)$. In particular

$$f(x) \geq \mathfrak{M}_\alpha(x_0\ ;\ f, r),$$

i.e., f is α-superharmonic in D. This completes the proof.

THEOREM 2. *Let D be a domain in R^n and a function f defined in R^n be finite continuous in D. Then f is α-harmonic in D if and only if $P_f^\alpha(x)$ exists in D and $P_f^\alpha(x)=0$ in D.*

Proof. Suppose that $P_f^\alpha(x)=0$ in D. Since

$$\int_{\complement B(x\,;\,r)} |x-y|^{-\alpha-n} |f(y)|\, dy < +\infty$$

for any in D and any positive number r, it holds that

$$\int_{\complement B(x\,;\,r)} |f(y)|\, \kappa_{x,r}(y)\, dy < +\infty$$

for any x in D and any open ball $B(x_0\ ;\ r)$ contained with its closure in D. Consequently, by Theorem 1, f is α-harmonic in D. The converse is evident by Theorem 1.

Chapter 3. Ninomiya's dominarion principle

In this chapter, we assume that $0<\alpha\leq 2$, $0<\alpha<2$ or $0<\alpha<1$ according to $n\geq 3$, $n=2$ or $n=1$.

THEOREM 3.[10] *Let μ be a positive measure with compact support such that*

$$\iint |x-y|^{\alpha-n} d\mu(y)\, d\mu(x) < +\infty,$$

and let ν be a positive measure. If

$$U_\alpha^\mu(x) \leq U_\alpha^\nu(x)$$

on S_μ, then

$$U_\beta^\mu(x) \leq U_\beta^\nu(x)$$

in R^n for any β such that $\alpha\leq\beta<n$.

Proof. By Ninomiya's theorem[11], it is sufficient to prove the following

[10] N. Ninomiya [7] proved this when $n\geq 3$. An alternate proof of this theorem was given in [5].

[11] Cf. [6], p. 142.

assertion. Let α and β be the same as Theorem 3, let λ be a positive measure with compact support, and let p be a point in $\mathscr{C}S_\lambda$. If

$$U_\alpha^\lambda(x) \leq |x-p|^{\beta-n}$$

in S_λ, then

$$U_\alpha^\lambda(x) \leq |x-p|^{\beta-n}$$

in R^n. To exclude the trivial case, we may assume that $\alpha < \beta$. First we shall show that $|x-p|^{\beta-n}$ is α-superharmonic in R^n. In fact, by M. Riesz's formula[12],

$$|x-p|^{\beta-n} = \frac{1}{K_{\alpha,\beta-\alpha}} \int |x-y|^{\alpha-n} |y-p|^{(\beta-\alpha)-n} dy,$$

where

$$K_{\alpha,\beta-\alpha} = \pi^{n/2} \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\beta-\alpha}{2}\right) \Gamma\left(\frac{n-\beta}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right) \Gamma\left(\frac{n-\beta+\alpha}{2}\right) \Gamma\left(\frac{\beta}{2}\right)}.$$

Since the measure $\frac{1}{K_{\alpha,\beta-\alpha}} |y-p|^{(\beta-\alpha)-n}$ is an α-finite positive measure, $|x-p|^{\beta-n}$ is α-superharmonic in R^n. On the other hand, $U_\alpha^\lambda(x)$ is α-harmonic in $\mathscr{C}S_\lambda$. Put

$$f(x) = |x-p|^{\beta-n} - U_\alpha^\lambda(x).$$

Then f is α-superharmonic in $\mathscr{C}S_\lambda$. Next we shall show that f is non-negative at infinity. In fact, let ε be a positive number. Then S_λ being compact, there exists a positive number ρ such that

$$|x-y|^{\alpha-n} \leq (1+\varepsilon)|x-p|^{\alpha-n}$$

for any x in $\mathscr{C}B(O\,;\rho)$ and any y in S_λ. Hence for any x in $\mathscr{C}B(O\,;\rho)$,

$$U_\alpha^\lambda(x) \leq (1+\varepsilon)\lambda(R^n)|x-p|^{\alpha-n}.$$

Since $\beta > \alpha$, there exists a positive number R_0 such that $R_0 \geq \rho$, S_λ is contained in $B(O\,;R_0)$ and

$$|x-p|^{\beta-n} \geq (1+\varepsilon)\lambda(R^n)|x-p|^{\alpha-n}$$

for any in $\mathscr{C}B(O\,;R_0)$. Finally put

[12] Cf. [2], p. 151.

$$\bar{f}(x) = \begin{cases} f(x) \text{ in } \mathscr{D}S_\lambda, \\ \varliminf\limits_{\substack{y \to x \\ y \in \mathscr{C}S_\lambda}} f(y) \text{ on the boundary of } \mathscr{C}S_\lambda. \end{cases}$$

Then $\bar{f}$ is lower semicontinuous on $\overline{\mathscr{C}S_\lambda}$ and $\bar{f}$ is non-negative at infinity. By Frostman's theorem[13],

$$\bar{f}(x) \geq 0$$

on $\partial\mathscr{C}S_\lambda$. Hence there exists x_1 in $\overline{\mathscr{C}S_\lambda} \cap \overline{B(O\ ;\ R_0)}$ such that $\bar{f}(x_1)$ attains the minimum of $\bar{f}(x)$ on $\overline{\mathscr{C}S_\lambda} \cap \overline{B(O\ ;\ R_0)}$. Assume that $\bar{f}(x_1)$ is negative. Then x_1 is contained in $\mathscr{C}S_\lambda$. For any ball $B(x_1\ ;\ r)$ contained with its closure in $\mathscr{C}S_\lambda$,

$$\begin{aligned} \mathfrak{M}_\alpha(x_1\ ;\ f,\ r) &= \int f(y)\,\kappa_{x_1,r}(y)\,dy \\ &\geq \int_{\mathscr{C}S_\lambda \cap B(O\ ;\ R_0)} f(y)\kappa_{x_1,r}(y)\,dy \geq \int_{\mathscr{C}S_\lambda \cap B(O\ ;\ R_0)} f(x_1)\,\kappa_{x_1,r}(y)\,dy \\ &> \int f(x_1)\,\kappa_{x_1,r}(y)\,dy = f(x_1). \end{aligned}$$

This contradicts the α-superharmonicity of f. Consequently

$$U_\alpha^\lambda(x) \leq |x-p|^{\beta-n}$$

in R^n. This completes the proof.

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13) Cf. [3], p. 69.