A NEW DEFINITION OF THE n-DIMENSIONAL QUASICONFORMAL MAPPINGS

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Introduction

In this note we shall extend, for arbitrary n, Pesin's [11] bidimensional definition for quasiconformal mappings and establish its equivalence with Gehring's [7] and Väisälä's [15] definitions.

The four Väisälä's [15] definitions are the following:

1° A homeomorphism $\bar{x}=f(x)$ of a domain $D\subset R^n$ is called *K-quasiconformal* $(1\leqq K<\infty)$, if $\delta(x)$ is uniformly bounded in D and $\delta(x)\leqq K$ a.e.[1] in D, where, for each r, $0<r<d\ (x, frD)$, we put (according to Väisälä [15]):

$$L(x, r)=\max_{|x'-x|=r}|f(x')-f(x)|, \qquad l(x, r)=\min_{|x'-x|=r}|f(x')-f(x)|,$$

$$T(x, r)=m\{f[B(x, r)]\},$$

$$\bar{\delta}(x)=\varlimsup_{r\to 0}\frac{A_nL(x, r)^n}{T(x, r)}, \qquad \underline{\delta}(x)=\varliminf_{r\to 0}\frac{T(x, r)}{A_nl(x, r)^n},$$

$$\delta(x)=\max[\bar{\delta}(x),\ \underline{\delta}(x)], \qquad \delta_L(x)=\varlimsup_{r\to 0}\frac{L(x, r)}{l(x, r)}$$

and A_nr^n is the volume of the n-dimensional ball $B(x, r)$ wtih the centre x and the radius r.

2° A homeomorphism $\bar{x}=f(x)$ *of a domain* $D\subset R^n$ is called *K-quasiconformal* $(1\leqq K<\infty)$ if

$$\frac{1}{K}M(\Gamma)\leqq M(\Gamma^*)\leqq KM(\Gamma)$$

for each curve family $\Gamma\subset D$, where $M(\Gamma)$ is the module of Γ and Γ^* is its image.

We recall (see Fuglede [5]) that

$$M(\Gamma)=\inf_{\rho\in F(\Gamma)}\int_{R^n}\rho^n d\tau,$$

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[1] a.e. = almost everywhere.

where $F(\Gamma)$ is the family of functions $\rho(x) \geqq 0$ Borel-measurable defined for all $x \in R^n$, so that $\int_\gamma \rho ds \geqq 1$ for each curve $\gamma \in \Gamma$ and $d\tau$ is the element of volume.

3° A homeomorphism $\bar{x} = f(x)$ of a domain $D \subset R^n$ is called *K-quasiconformal* $(1 \leqq K < \infty)$, if it is ACL (absolutely continuous on lines), a.e. differentiable and

$$\frac{1}{K} \max |f'(x)|^n \leqq |J(x)| \leqq K \min |f'(x)|^n$$

a.e. in D, where we put

$$\max |f'(x)| = \max_{|\Delta x| = 1} |f'(x) \Delta x|, \qquad \min |f'(x)| = \min_{|\Delta x| = 1} |f'(x) \Delta x|$$

and $f'(x)$ is the derivative operator of $f(x)$ i.e. the linear transformation of R^n so that

$$f(x + \Delta x) = f(x) + f'(x) \Delta x + O(|\Delta x|).$$

4° A homeomorphism $\bar{x} = f(x)$ of a domain $D \subset R^n$ is called *K-quasiconformal* $(1 \leqq K < \infty)$, if

$$\frac{1}{K} M(A) \leqq M(A^*) \leqq KM(A),$$

for all the rings A with closure $\bar{A} \subset D$, where $M(A)$ is the module of A and A^* is its image.

A homeomorphism $\bar{x} = f(x)$ is called *quasiconformal* according to one of the above definitions if it is K-quasiconformal for some K.

The equivalence of the preceding four definitions of the K-quasiconformal homeomorphisms has been established for $n = 3$ by Väisälä [15] and for arbitrary n by Chén Hàng-lén in [4] and by us in some lectures about the n-dimensional quasiconformal homeomorphisms delivered in Bucarest (1. I-31. III. 1964).

Gehring [7] gives the two following definitions:

The metric definition. A homeomorphism $\bar{x} = f(x)$ of a domain $D \subset R^n$ is said to be K-quasiconformal if $\delta_L(x)$ is uniformly bounded in D and

$$\delta_L(x) \leqq K$$

a.e. in D.

The geometric definition. The terms of this definition are the same as those of the fourth Väisälä's definition, but the meaning of the module of a ring is somewhat different. So, the module of Väisälä's definition is the quantity (see p. 7 of [15]):

$$M(A) = \frac{\omega_n}{M(\Gamma_A)},$$

where ω_n is the $(n-1)$-dimensional Lebesgue measure of the sphere $|x|=1$ and $M(\Gamma_A)$ is the module of the family of arcs which join the boundary components of the ring A in A; the module of Gehring's definition is the quantity

$$\operatorname{mod} A = \left[\frac{\omega_n}{C(A)}\right]^{1/(n-1)},$$

where $C(A)$ is the conformal capacity of the ring A (see Loewner [9]). From

$$M(\Gamma_A) = C(A)$$

(see Krivov in [8] and Šabat in a unpublished Note), which also follows immediately from Gehring's theorem 1 of [6] (its tridimensional proof remaining the same for arbitrary n), we obtain

$$M(A) = \operatorname{mod}^{n-1} A,$$

which implies the equivalence Gehring's geometric definition of K-quasiconformal homeomorphisms with Väisälä's definitions of K^{n-1}-quasiconformal homeomorphisms.

Väisälä's inequalities (5.2) of [15]:

$$\delta(x) \leqq \delta_L^{n-1}(x), \qquad \delta_L(x) \leqq \delta^{2/n}(x),$$

which hold a.e. in D, imply that

The class of K-quasiconformal homeomorphisms according to Gehring's metric definition is contained in the Väisälä's class of K^{n-1}-quasiconformal homeomorphisms and the Väisälä's class of K-quasiconformal homeomorphisms is contained in the class of $K^{2/n}$-quasiconformal homeomorphisms according to Gehring's metric definition. The bound $K^{2/n}$ is best possible.

Indeed, the affine mapping

$$y^i = K^{2(i-1)/n(n-1)} x^i \qquad (i = 1, \ldots, n)$$

is K-quasiconformal according to Väisälä's third (analytic) definition, because it is ACL and satisfies the corresponding inequalities, the latter being implied by some of Väisälä's relations (5.1) of [15]:

$$\frac{\max|y'(x)|^n}{|J(x)|}=\frac{\lambda_1^{n-1}}{\lambda_2\cdots\lambda_n}, \qquad \frac{|J(x)|}{\min|y'(x)|^n}=\frac{\lambda_1\cdots\lambda_{n-1}}{\lambda_n^{n-1}},$$

which hold in every point of differentiability with $J \neq 0$ and where $y'(x)$ maps the unit ball onto an ellipsoid with semi-axes $\lambda_1 \geqq \cdots \geqq \lambda_n > 0$, which in our case are of the form $\lambda_i = K^{2(i-1)/n(n-1)}$. But

$$\delta_L(x)=\frac{\lambda_1}{\lambda_n}$$

(see also (5.1) of [15]) implies

$$\delta_L(x)=\frac{\lambda_1}{\lambda_n}=K^{2/n}$$

and the bound $K^{2/n}$ cannot be improved.

The equivalence of Väisälä's definitions of K^{n-1}-quasiconformal homeomorphisms and Gehring's geometric definition of K-quasiconformal homeomorphisms, combined with the preceding relationship between Väisälä's and Gehring's metric definitions, imply that

Gehring's class of K-quasiconformal homeomorphisms according to the metric definition is contained in his class of K-quasiconformal homeomorphisms according to the geometric definition, which in its turn is contained in his class of $K^{2(n-1)/n}$-quasiconformal homeomorphisms according to the metric definition. The bound $K^{2(n-1)/n}$ is best possible.

Thus all the six definitions of quasiconformal homeomorphisms are equivalent.

1. A new class of n-dimensional quasiconformal mappings

We begin with some preliminary definitions.

A family of surfaces $\{\Sigma_\alpha\}$ $(0<\alpha<1)$ is called *regular of parameter k $(1 \leqq k < \infty)$ relatively to the point x_0*, if the surfaces Σ_α are the images of the spheres $|t|=\alpha$ by the homeomorphism $x=\varphi(t)$ of the ball $|t|<1$ on a neighbourhood of the point $x_0=\varphi(0)$ and

$$\varlimsup_{\alpha\to 0}\frac{\max\limits_{x\in\Sigma_\alpha}|x-x_0|}{\min\limits_{x\in\Sigma_\alpha}|x-x_0|}=k.$$

A homeomorphism $\bar{x}=f(x)$ is called *regular in* x_0 if it maps a regular family of parameter k relatively to x_0 in a regular family of parameter k' $(1\leqq k'<\infty)$ relatively to $f(x_0)$. The quantity

$$q(x_0)=\inf kk', \tag{1}$$

where the infimum is taken over all regular families relatively to x_0, is called the *characteristic of the mapping* $x=f(x)$ in x_0.

A homeomorphism $x=f(x)$ is called *regular in a domain* $D\subset R^n$, if it is regular in each of its points.

A homeomorphism $x=f(x)$ of a domain $D\subset R^n$ is called *Q-quasiconformal in Pesin's sense* in D, if it is regular in D, the characteristic $q(x)$ is uniformly bounded in D and

$$q(x)\leqq Q$$

a.e. in D. A *quasiconformal homeomorphism in Pesin's sense* is a Q-quasiconformal one for some Q.

Remark 1. This definition of Q-quasiconformal mappings is a generalisation of Pesin's [11] corresponding to bidimensional definition of the "general Q-quasiconformal mappings"; in Pesin's definition the preceding inequality must hold everywhere in D. Our general definition than Pesin's original one has the advantage to be equivalent with Gehring's metric definition (as we shall prove at the end of this note).

Remark 2. Markushevitch [10] considered the class of continuous mappings $\bar{x}=f(x)$ in a domain $D\subset R^n$, so that for every $\bar{x}\in Z\subset D$, where $mZ=0$,

(a) $f(x)$ is one-to-one in a neighbourhood $U(x)$ of x and

(b) in $U(x)$ there is a sequence $\{\Gamma_i(x)\}$ of surfaces that are homeomorphic to spheres and

$$(A)\quad \varliminf_{i\to\infty}\frac{r_{i+1}(x)}{r_i(x)}>0,\quad \varliminf_{i\to\infty}\frac{r_i(x)}{R_i(x)}=\frac{1}{k}>0,\quad \varliminf_{i\to\infty}\frac{r'_i(\bar{x})}{R'_i(\bar{x})}=\frac{1}{k'}>0,\quad \lim_{i\to\infty}R_i(x)=0,$$

where $r_i(x)$, $R_i(x)$ denote the minimum, respectively the maximum, of the distances from x to $\Gamma_i(x)$, $\Gamma'_i(x)$ the image of $\Gamma_i(x)$ and $r'_i(\bar{x})$, $R'_i(\bar{x})$ the corresponding distances from $\bar{x}$ to $\Gamma'_i(x)$. He observed that one can substitute each sequence $\{\Gamma_i(x)\}$ by the family $\{\Gamma_\lambda(x)\}$, which fills a neighbourhood of x and then the first condition of (A) became unnecessary.

We remark that if we impose to this class of mappings the additional condition to be one-to-one everywhere in D (i.e. $Z=\phi$) and that $\inf k(x)k'(x)$, where the infimum is taken over all the families $\{\Gamma_\lambda(x)\}$ of surfaces that are homeomorphic to spheres, must be uniformly bounded and $\inf k(x)k'(x) \leqq K$ a.e. in D, then the class of Markushevitch's continuous locally one-to-one mappings reduce to the class of K-quasiconformal mappings in Pesin's sense. The main result Markushevitch proved about his class from above is its differentiability a.e. in D.

We give now a slight generalisation of our former definitions of Q-quasiconformal homeomorphisms with one and two sets of characteristics [2, 3], which represent particular cases of the definition of the Q-quasiconformal mappings in Pesin's sense.

We recall first that the characteristics of an ellipsoid E are the quantities

$$(C) \qquad \gamma_k^i, \quad p_m = \frac{a_m}{a_n} \quad (i, k = 1, \ldots, n;\ m = 1, \ldots, n-1),$$

where γ_k^i are the directing cosines of the axes of E and a_m, a_n ($a_1 \geqq \cdots \geqq a_n > 0$) are its semi-axes. The quantity p_1 is called the principal characteristic.

We say that a mapping $\bar{x}=f(x)$ maps an infinitesimal ellipsoid $E[(C), x]$ into an infinitesimal ellipsoid $E[(C'), f(x)]$, if it is one-to-one and continuous in a neighbourhood of x and maps every ellipsoid $E_h[(C), x]$ with the centre x, the characteristics (C) and the minimum semi-axe $a_n = h$ sufficiently small on a Jordan surface $f(E_h)$ comprised between two homothetical ellipsoids $E_{h_1'}[(C'), f(x)]$ and $E_{h_2'}[(C'), f(x)]$ with the centre $f(x)$, the characteristics (C') and the minimam semi-axes h_1', h_2' so that

$$\lim_{h\to 0} \frac{h_1'}{h_2'} = 1,$$

as E_h shrinks itself homotheticaly to the point x.

A mapping $\bar{x}=f(x)$ of a domain $D \subset R^n$ is called Q *quasiconformal with two sets of characteristics* (C), (C'), if it is one-to-one and maps every infinitesimal ellipsoid $E[(C), x]$ into an infinitesimal ellipsoid $E[(C'), f(x)]$, where the principal characteristics $p_1(x)$, $p_1'(x)$ are uniformly bounded in D and

$$p_1(x)p_1'(x) \leqq Q$$

a.e. in D.

Remark. The continuity of $f(x)$ is implied by the continuity of a mapping which maps infinitesimal ellipsoids into infinitesimal ellipsoids.

The class of *Q-quasiconformal homeomorphisms with a set of characteristics* in D is obtained from the preceding one by putting $p_1'(x)\equiv 1$ in D. The *conformal* mappings are obtained for $Q=1$.

2. Some properties of the homeomorphisms Q-quasiconformal in Pesin's sense

In this chapter as we deal only with Q-quasiconformal homeomorphisms in Pesin's sense we shall call them simply Q-quasiconformal homeomorphisms.

Obviously, the inverse of a Q-quasiconformal homeomorphism is also a Q-quasiconformal homeomorphism. Then, the composite of a Q-quasiconformal homeomorphism and a Q'-quasiconformal homeomorphism is a QQ'-quasiconformal homeomorphism.

We recall that the module of dilatation [2] is the quantity

$$\left|\frac{\partial f(x_0)}{\partial s}\right| = \lim_{|\Delta x|_s\to 0}\frac{|\Delta f(x_0)|}{|\Delta x|_s},$$

where $\Delta x = x - x_0$, $\Delta f(x_0) = f(x) - f(x_0)$ and $|\Delta x|_s\to 0$ means that $x\to x_0$ in the direction s.

THEOREM 1. *Let $\bar{x}=f(x)$ be a quasiconformal homeomorphism differentiable in x_0 and let $\left|\frac{\partial f(x_0)}{\partial s_0}\right|$ be the module of dilatation in the direction s_0 in x_0. Then for the module of dilatation $\left|\frac{\partial f(x_0)}{\partial s}\right|$, in every direction s where it exists, holds*

$$\frac{1}{q(x_0)}\left|\frac{\partial f(x_0)}{\partial s_0}\right| \leqq \left|\frac{\partial f(x_0)}{\partial s}\right| \leqq q(x_0)\left|\frac{\partial f(x_0)}{\partial s_0}\right|, \tag{2}$$

where $q(x)$ is the characteristic (1) *of $\bar{x}=f(x)$.*

Proof. Let $\{\Sigma_\alpha\}$ be a regular family of surfaces of parameter k relatively to x_0. According to the hypotheses of the theorem, $\bar{x}=f(x)$ maps $\{\Sigma_\alpha\}$ into a regular family of surfaces of parameter k'. Hence

$$\overline{\lim_{\alpha\to 0}}\frac{|f(x_\alpha)-f(x_0)|_s}{|f(x_\alpha)-f(x_0)|_{s_0}}\leqq k', \qquad \overline{\lim_{\alpha\to 0}}\frac{|x_\alpha - x_0|_{s_0}}{|x_\alpha - x_0|_s}\leqq k,$$

where $x_\alpha\in\Sigma_\alpha$ and $|f(x_\alpha)-f(x_0)|_s$, $|x_\alpha - x_0|_s$ are norms of vectors with $x_\alpha - x_0$ of direction s. Then

$$\left|\frac{\partial f(x_0)}{\partial s}\right| = \overline{\lim_{\alpha\to 0}}\left[\frac{|f(x_\alpha)-f(x_0)|_s}{|f(x_\alpha)-f(x_0)|_{s_0}}\cdot\frac{|f(x_\alpha)-f(x_0)|_{s_0}}{|x_\alpha - x_0|_{s_0}}\cdot\frac{|x_\alpha - x_0|_{s_0}}{|x_\alpha - x_0|_s}\right] \leqq kk'\left|\frac{\partial f(x_0)}{\partial s_0}\right|.$$

But this inequality holds for every regular family relatively to x_0, so that

$$\left|\frac{\partial f(x_0)}{\partial s}\right| \leqq q(x_0)\left|\frac{\partial f(x_0)}{\partial s_0}\right|.$$

Now, changing between them s and s_0, we obtain also the first part of the inequality (2).

THEOREM 2. *The Jacobian J of a homeomorphism $\bar{x} = f(x)$, quasiconformal in $D \subset R^n$ is zero in a point x_0 of differentiability if and only if all its partial derivatives of the first order are zero.*

Proof. Let $J(x_0) = 0$ and suppose, by absurde, that at least one partial derivative of the first order, say $\bar{x}^p_q(x_0) \neq 0$. Then $\left|\frac{\partial f(x_0)}{\partial x^q}\right| \neq 0$ and the preceding theorem yields $\left|\frac{\partial f(x_0)}{\partial s}\right| \neq 0$ in all the directions s. Hence, the theorem 6 of [2] implies $J(x_0) \neq 0$. The absurdity obtained establishes the theorem.

THEOREM 3. *Let the quasiconformal homeomorphism $\bar{x} = f(x)$ be differentiable at x_0. Then*

$$\begin{aligned} \Lambda_f(x_0) &= \overline{\lim_{\Delta x\to 0}}\frac{|\Delta f(x_0)|}{|\Delta x|} = \sup_s\left|\frac{\partial f(x_0)}{\partial s}\right| = \sqrt[n]{p_1\cdots p_{n-1}|J(x_0)|}, \\ \lambda_f(x_0) &= \underline{\lim_{\Delta x\to 0}}\frac{|\Delta f(x_0)|}{|\Delta x|} = \inf_s\left|\frac{\partial f(x_0)}{\partial s}\right| = \sqrt[n]{\frac{p_2\cdots p_{n-1}|J(x_0)|}{p_1^{n-2}}}, \end{aligned} \tag{3}$$

where $\Lambda_f(x_0)$, $\lambda_f(x_0)$ are respectively the maximal and the minimal dilatation of $f(x)$ in x_0 and p_m $(m = 1, \ldots, n-1)$ are the characteristic parameters of the affine transformation

$$f_1(x, x_0) = f(x_0) + \frac{\partial f(x_0)}{\partial x^i}\Delta x^i. \tag{4}$$

We can speak of the characteristic parameters of an affine transformation, because it can be considered as a quasiconformal homeomorphism with a set of characteristics.

Proof. If $J(x_0) \neq 0$, (3) holds from the Theorem 14 of [3]. If $J(x_0) = 0$, then Theorem 6 of [2], Theorem 14 of [3] and Theorem 1 imply $\Lambda_f(x_0) = \lambda_f(x_0) = 0$, that is, (3) holds again.

LEMMA 1. *Let $\bar{x}=A(x)$ be an affine transformation with the principal characteristic p_1. Then $p_1=q$, where q is the caracteristic in* (1).

Proof. Obviously, $p_1 \geqq q$. We shall prove that $p_1=q$.

If we consider a regular family of parameter $k \geqq p_1$ in a point x_0, obviously $kk' \geqq p_1$.

Let $E_h=E_h[(C), x_0]$ be the ellipsoid mapped by $\bar{x}=A(x)$ in a sphere. E_h is comprised between two concentric spheres S_1, S_2, with the radius r_1, r_2 so that $\frac{r_1}{r_2}=p_1$, i.e. E_h lies in S_1 and is tangent to it and lies outside S_2 and is tangent to it. For the family of ellipsoids $\{E_h\}$ we have $kk'=p_1$, where $k'=1$ because $A(E_h)$ is a sphere.

Let us consider now a regular family $\{\Sigma\}$ of parameter $k<p_1$, and let $\Sigma_0 \in \{\Sigma\}$ and comprised between two spheres S_1^*, S_2^*, with the radius r_1^*, r_2^*, so that $\frac{r_1^*}{r_2^*}=k_0<p_1$, Let E_h^* be an ellipsoid comprised between S_1^*, S_2^*, with the same directing cosines γ_k^i $(i, k=1, \ldots, n)$ and the same distribution of the semi-axes as E_h. Obviously, r_2^*, $r_1^*=k_0 r_2^*$ are respectively the minimum and the maximum semi-axes of E_h^*. But $\bar{x}=A(x)$ maps E_h in a sphere by stretching the minimal semi-axe $p_1\lambda$ times and the maximal semi-axe only λ times. Hence, the direction of the semi-axes of E_h^* being the same as those of E_h, we conclude that $\bar{x}=A(x)$ maps E_h^* into another ellipsoid $A(E_h^*)$ by stretching the minimal semi-axe $p_1\lambda$ times and the maximal λ times. Thus, the maximal semi-axe of $A(E_h^*)$ is $p_1\lambda r_2^*$ and the minimal $k_0\lambda r_2^*$. Hence, the principal characteristic of $A(E_h^*)$ is $p_1^*=\frac{p_1}{k_0}$ and $A(E_h^*)$ is comprised between the spheres with radii $R_2^*=k_0\lambda r_2^*$ and $R_1^*=p_1\lambda r_2^*$. Thus $\frac{r_1^*}{r_2^*}\cdot\frac{R_1^*}{R_2^*}=p_1$. If instead E_h^* we consider Σ_0, then obviously $\frac{r_1^*}{r_2^*}=k_0$ is unchanged, but $\frac{R_1^*}{R_2^*}$ does not decrease, so that, in this case $\frac{r_1^* R_1^*}{r_2^* R_2^*} \geqq p_1$. But this inequality holds for all Σ_0 with the corresponding $k_0<p_1$. Then, as for all Σ with a diameter sufficiently small $kk_0<p_1$, we conclude that $kk' \geqq p_1$, where k' is the parameter of the family $\{A(\Sigma)\}$.

Thus in both cases ($k \geqq p_1$ and $k<p_1$) we have $kk' \geqq p_1$. But $kk'=p_1$ for the family of ellipsoids $\{E_h\}$ and we conclude that $q(x_0)=\inf kk'=p_1(x_0)$ where x_0 is an arbitrary point.

THEOREM 4. *Let the quasiconformal homeomorphism $\bar{x}=f(x)$ be differentiable at x_0. Then*

(5) $$|J(x_0)| \geqq \frac{\Lambda_f(x_0)}{q^{n-1}(x_0)}.$$

Proof. This theorem is a consequence of the preceding theorem and lemma and of Theorem 15 of [3].

THEOREM 5. *Let $\bar{x} = f(x)$ be a homeomorphism quasiconformal in D and $x = f^{-1}(\bar{x})$ its inverse. Then $\Lambda_f(x) < \infty$ a.e. in D and $\Lambda_{f^{-1}}(\bar{x}) < \infty$ a.e. in $f(D)$.*

The theorem is established arguing exactly as in the proof of lemma 1 in [2], which asserts that the conclusions of our theorem hold for quasiconformal homeomorphisms with two sets of characteristics. We have only to change the set of families of ellipsoids of characteristics (C) by a set of regular families of surfaces of a parameter $k(x)$ sufficiently small for kk' be uniformly bounded.

We shall give now an elegant proof of the Rademacher-Stepanov theorem. We precede it by some definitions.

We recall (see Bouligand p. 66 of [1]) that a half-line OT from an accumulating point 0 of a set E is called semi-tangent at 0 to the set E if every right circular cone, with the vertex in 0, the axis OT the opening and the altitude sufficiently small, contains at least a point of E different from 0. The set of all semi-tangents is called the contingent of the set E at 0. The contingent of an isolated point is considered the empty set. The bilateral contingent (see Rogerin [13]) is called the set of all straight-lines with the property that the pairs of half-lines which composed them belong to the contingent.

We recall also the following

ROGER'S THEOREM. In every cartesian q-dimensional set, with the eventual exception of a set of p-dimensional Carathéodory measure zero, the subset where the bilateral contingent does not contain a $(q-p)$-dimensional linear manifold coincides with that in which the bilateral contingent reduces to a p-dimensional linear manifold and the whole contingent to a system of $(p+1)$-dimensional linear semi-manifolds admitting the preceding one as a base.

RADEMACHER-STEPANOV THEOREM. *Let $u(x)$ be a real continuous function in a domain $D \subset R^n$. Then $u(x)$ is differentiable a.e. in a measurable set $E \subset D$, if and only if*

(6) $$\Lambda_u(x) < \infty$$

a.e. in E.

Proof. The necessity is proved as in Theorem 25 of [2].

For the sufficiency let G be the n-dimensional surface $u=u(x)$ of R^{n+1}, which has as orthogonal projection on the n-dimensional plane $u=0$ just D and let G_0 be the subset of G, which has as orthogonal projection the subset $E_0 \subset E$ where (6) holds. This implies that in every point $P \in G_0$ the contingent to G does not contain the semi-tangent Ou and therefore can be neither the whole $(n+1)$-dimensional space, nor the semi-space. Hence, by the preceding Roger's theorem, applied in the particular case $q=n+1$, $p=n$, we obtain that, with an eventual exception of a set of n-dimensional Carathéodory measure zero, at the points of G_0 the contingent is reduced to an n-dimensional plane, namely the tangent plane to the n-dimensional surface G at that point. But every set of n-dimensional Carathéodory measure zero is projected on any of the coordinate planes in a set of n-dimensional Lebesgue measure zero. Besides, the existence of a tangent plane not parallel with Ou in a point of G_0 implies the differentiability of $u=u(x)$ in the corresponding point $x \in E_0$. Thus we have proved the differentiability of $u=u(x)$ a.e. in E_0 and by the hypotheses of the theorem, a.e. in E. This completes our proof of Rademacher-Stepanov theorem.

THEOREM 6. *Let $\bar{x}=f(x)$ be a quasiconformal homeomorphism in $D \subset R^n$. Then $f(x)$ is differentiable a.e. in D.*

Proof. Theorem 5 implies that for the n functions $\bar{x}^i(x)$ $(i=1, \ldots, n)$ of the mapping $\bar{x}=f(x)$ the hypotheses of Rademacher-Stepanov theorem hold. Hence, every $x^i(x)$ is differentiable a.e. in D and then all the n functions are differentiable simultaneously a.e. in D, which is the same for the differentiability of $f(x)$ a.e. in D.

THEOREM 7. *Let $\bar{x}=f(x)$ be a Q-quasiconformal homeomorphism of $|x|<1$ in $|\bar{x}| \leqq 1$, so that $f(0)=0$. Let $\Sigma \subset \{|x| \leqq 1\}$ be a Jordan surface, $B=[2^{(n-1)(n-4)/2} 3^{n-2}\pi A_n Q^{n-1}]^{1/n}$, $R_0=\min_{x\in\Sigma}|f(x)|$, $r_0=d(0, \Sigma)$ i.e. the distance from 0 to Σ and A_n the volume of $|x| \leqq 1$. Then*

$$R_0 e^{-(B/|x|)^n} < |f(x)| < \frac{B}{\left(\log \frac{r_0}{|x|}\right)^{1/n}} \tag{7}$$

for every $x \in \{[f^{-1}(|\bar{x}|<R_0)] \cap (|x|<r_0)\}$.

Proof. Let $x_1 \in \{|x| \leqq r_0\}$, $x_1 \neq 0$ and $|x_1|=r_1$. Let us estimate the volume of $f(r_1 \leqq |x| \leqq r_0)$. The preceding theorem implies that $f(x)$ is differentiable a.e. in $|x| \leqq 1$. Let E be the set of points of differentiability of $f(x)$. We assume, without loss of generality, that $q(x) \leqq Q$ in E. De la Vallée-Poussin decomposition theorem (see Saks p. 151 of [14]) and

$$\lim_{p\to\infty} \frac{mf[C(\alpha_p)]}{mC(\alpha_p)} = |J(x)|,$$

which holds in any point of differentiability and where $C(\alpha_p)$ is a cube with the centre in x and the side length α_p, with $\lim_{p\to\infty} \alpha_p = 0$ (see Rado and Reichelderfer chap. V, § 2 of [12]) imply

$$A_n > mf(r_1 \leqq |x| \leqq r_0) \geqq mf[r_1 \leqq |x| \leqq r_0) \cap E] = \int_{(r_1 \leqq |x| \leqq r_0)\cap E} |J|\, d\tau.$$

Hence, by (3) and $mE = m(|x| \leqq 1)$

$$(8) \qquad A_n > \frac{1}{Q^{n-1}} \int_{(r_1<|x|<r_0)\cap E} \Lambda_f^n d\tau = \frac{1}{Q^{n-1}} \int_{r_1<|x|<r_0} \Lambda_f^n d\tau = \frac{1}{Q^{n-1}} \int_{r_1}^{r_0} r^{n-1} dr \int_S \Lambda_f^n d\sigma =$$

$$= \frac{1}{Q^{n-1}} \int_{r_1}^{r_0} r^{n-1} dr \int_0^{\pi} \cdots \int_0^{\pi}\int_0^{2\pi} \Lambda_f^n \sin^{n-2}\vartheta_1 \cdots \sin\vartheta_{n-2}\, d\vartheta_1 \cdots d\vartheta_{n-1} =$$

$$= \frac{1}{Q^{n-1}} \int_{r_1}^{r_0} r^{n-2} dr \int_0^{\pi} \cdots \int_0^{\pi} \sin^{n-2}\vartheta_1 \cdots \sin\vartheta_{n-2}\, d\vartheta_1 \cdots d\vartheta_{n-2} \int_0^{2\pi} \Lambda_f^n r\, d\vartheta_{n-1},$$

where S is the sphere $|x| = 1$. Hölder's inequality yields

$$(9) \qquad \left(\int_0^{2\pi} \Lambda_f r\, d\vartheta_{n-1}\right)^n \leqq \int_0^{2\pi} \Lambda_f^n r\, d\vartheta_{n-1} \left(\int_0^{2\pi} r\, d\vartheta_{n-1}\right)^{n-1} = (2\pi r)^{n-1} \int_0^{2\pi} \Lambda_f^n r\, d\vartheta_{n-1}.$$

But, by (3)

$$(10) \qquad \int_0^{2\pi} \Lambda_f r\, d\vartheta_{n-1} \geqq \int_0^{2\pi} \sqrt[n]{|J|}\, r\, d\vartheta_{n-1} = 1(r) > 2|f(x_1)|$$

in E, i.e. for almost all r $(r_1 < r < 1)$, or for at least one k $(k = 1, \ldots, n-2)$ for almost all ϑ_k $(0 < \vartheta_k < \pi)$ and where $1(r)$ is the length of the image of

$$|x| = r,\ \vartheta_k = \text{const.} \qquad (k = 1, \ldots, n-2).$$

Inequalities (8), (9) and (10) imply

$$A_n > \frac{2|f(x_1)|^n}{(\pi Q)^{n-1}} \int_{r_1}^{r_0} \frac{dr}{r} \int_0^{\pi} \cdots \int_0^{\pi} \sin^{n-2}\vartheta_1 \cdots \sin\vartheta_{n-2}\, d\vartheta_1 \cdots d\vartheta_{n-2} >$$

$$> \frac{2|f(x_1)|^n}{(\pi Q)^{n-1}} \log \frac{r_0}{r_1} \int_{\pi/6}^{5\pi/6} \cdots \int_{\pi/6}^{5\pi/6} \sin^{n-2}\vartheta_1 \cdots \sin\vartheta_{n-2}\, d\vartheta_1 \cdots d\vartheta_{n-2} >$$

$$> \frac{2|f(x_1)|^n}{2^{(n-1)(n-2)/2}(\pi Q)^{n-1}} \log \frac{r_0}{r_1} \int_{\pi/6}^{5\pi/6} \cdots \int_{\pi/6}^{5\pi/6} d\vartheta_1 \cdots d\vartheta_{n-2} =$$

$$= \frac{|f(x_1)|^n}{2^{(n-1)(n-4)/2} 3^{n-2} \pi Q^{n-1}} \log \frac{r_0}{r_1}.$$

hence

$$|f(x_1)| < \left[\frac{2^{(n-1)(n-4)/2} 3^{n-2} \pi A_n Q^{n-1}}{\log \frac{r_0}{|x_1|}} \right]^{1/n},$$

which proves the second part of the inequality (7) for every $x_1 \in \{|x| \leqq r_0\}$. But we have seen that the inverse mapping $x = f^{-1}(x)$ is also Q-quasiconformal with the same Q and applying the inequality just established to $x = f^{-1}(\bar{x})$, we obtain

$$|x_1| < \left[\frac{2^{(n-1)(n-4)/2} 3^{n-2} \pi A_n Q^{n-1}}{\log \frac{R_0}{|f(x_1)|}} \right]^{1/n},$$

for every $x \in f^{-1}(|\bar{x}| \leqq R_0)$, hence

$$|f(x_1)| > R_0 e^{\frac{-2^{(n-1)(n-4)/2} 3^{n-2} \pi A_n Q^{n-1}}{|x_1|^n}},$$

which completes our proof.

Theorem 8. *Let $\bar{x} = f(x)$ be a Q-quasiconformal homeomorphism in D. Then*

$$\delta_L(x_0) = \overline{\lim_{r \to 0}} \frac{\max\limits_{|x-x_0|=r} |f(x) - f(x_0)|}{\min\limits_{|x-x_0|=r} |f(x) - f(x_0)|} < B(Q_1 + 2)^3 e^{[B(Q_1+2)^3]^n},$$

where B is the constant of the preceding theorem and Q_1 is a quantity so that $q(x) \leqq Q_1$ everywhere in D.

Proof. Let $x_0 \in D$ and let $\{\Sigma_\alpha\}$ $(0 < \alpha < 1)$ be a family of surfaces regular of parameter k relatively to x_0. The family $\{f(\Sigma_\alpha)\}$ $(0 < \alpha < 1)$ is regular of parameter k'. Suppose now we chose $\{\Sigma_\alpha\}$, so that $kk' < Q_1 + 1$ and $\Sigma_\alpha \subset D$. Then

$$\text{(11)} \qquad \frac{\max\limits_{x \in \Sigma_\alpha} |x - x_0|}{\min\limits_{x \in \Sigma_\alpha} |x - x_0|} < k + 1, \qquad \frac{\max\limits_{x \in \Sigma_\alpha} |f(x) - f(x_0)|}{\min\limits_{x \in \Sigma_\alpha} |f(x) - f(x_0)|} < k' + 1$$

for all $\alpha \leqq \alpha_0$ (α_0 sufficiently small). Let

$$\rho_0 = \max_{x \in \Sigma_{\alpha_0}} |x - x_0|, \qquad \rho_0' = \max_{x \in \Sigma_{\alpha_0}} |f(x) - f(x_0)|,$$

and

$$(12)\qquad \bar{x}' = \psi(x') = \frac{1}{\rho_0'}\varphi(\rho_0 x') = \frac{1}{\rho_0'}\left\{f\left[\frac{\rho_0(x-x_0)}{\rho_0}+x_0\right]-f(x_0)\right\} = \frac{1}{\rho_0'}[f(x)-f(x_0)],$$

where $x' = \frac{x-x_0}{\rho_0}$. Obviously $\psi(0)=0$, $|x-x_0|=\rho_0$ corresponds to $|x'|=1$, $|\bar{x}-\bar{x}_0|=\rho_0'$ to $|\bar{x}'|=1$ and Σ_α to Σ_α'. Finally let

$$r_0 = \min_{x'\in\Sigma'\alpha_0}|x'| > \frac{1}{k+1} > \frac{1}{Q_1+2}, \qquad R_0 = \min_{x'\in\Sigma'\alpha_0}|\psi(x')| > \frac{1}{k'+1} > \frac{1}{Q_1+2}.$$

Obviously the Q-quasiconformality of $\bar{x}=f(x)$ in D implies the Q-quasiconformality of $\bar{x}'=\psi(x')$ in the domain bounded by Σ_{α_0}', because we obtain x' from x and $\bar{x}'$ from $\bar{x}$ by a product (in functional sense) of translations and homotheties. Hence, as for $\bar{x}'=\psi(x')$, the hypotheses of the preceding theorem hold, we see that

$$(13)\qquad \frac{e^{-(B/|x'|)^n}}{Q_1+2} < R_0 e^{-(B/|x'|)^n} < |\psi(x')| < \frac{B}{\left(\log\frac{r_0}{|x'|}\right)^{1/n}} < \frac{B}{\left[\log\frac{1}{(Q_1+2)|x'|}\right]^{1/n}}$$

holds for every $x' \in \left[\psi^{-1}(|\bar{x}'| \leq R_0) \cap \left(|x'| \leq \frac{1}{Q_1+2}\right)\right]$.

Let $r_0^* = \min_{|\bar{x}'|=R_0}|\psi^{-1}(\bar{x}')|$. We distinguish two cases: 1°: $r_0^* \geqq \frac{1}{(Q_1+2)^2}$ and 2°: $r_0^* < \frac{1}{(Q_1+2)^2}$.

1°: $r_0^* \geqq \frac{1}{(Q_1+2)^2}$. In this case, (13) and $e<Q_1+2$ imply

$$\frac{e^{-[B(Q_1+2)^3]^n}}{Q_1+2} \leqq \frac{e^{-(B/|x'|)^n}}{Q_1+2} < |\psi(x')| < \frac{B}{\left[\log\frac{1}{(Q_1+2)|x'|}\right]^{1/n}} < \frac{B}{[\log(Q^1+2)]^{1/n}} < B$$

for every x' in

$$(14)\qquad \frac{1}{(Q_1+2)^3} \leqq |x'| \leqq \frac{1}{(Q_1+2)^2}.$$

Hence

$$(15)\qquad \frac{|\psi(x_1')|}{|\psi(x_2')|} < B(Q_1+2)e^{[B(Q_1+2)^3]^n},$$

for all x_1', x_2' in (14).

2°: $r_0^* < \frac{1}{(Q_1+2)^2}$. In this case we distinguish two possibilities which we examine successively: A. $r_0^* \leqq \frac{1}{(Q_1+2)^4}$ and B. $\frac{1}{(Q_1+2)^4} < r_0^* < \frac{1}{(Q_1+2)^2}$.

A. $r_0^* \leqq \frac{1}{(Q_1+2)^4}$. Let Σ'_{α_1} be the surface of the family $\{\Sigma'_\alpha\}$ which lies inside the surface $\psi^{-1}(|\bar{x}'| = R_0)$ and is tangent to it. In this case

$$\min_{x' \in \Sigma'_{\alpha_1}} |x'| \leqq \frac{1}{(Q_1+2)^4},$$

hence (11) implies on the one hand

$$\max_{x' \in \Sigma'_{\alpha_1}} |x'| < \frac{1}{(Q_1+2)^3},$$

and on the other hand, since $\psi(\Sigma'_{\alpha_1})$ is inside $|\bar{x}'| = R_0 > \frac{1}{Q_1+2}$ and tangent to it,

$$\min_{x' \in \Sigma'_{\alpha_1}} |\psi(x')| > \frac{1}{(Q_1+2)^2}.$$

Thus, in this case

$$\frac{|\psi(x_1')|}{|\psi(x_2')|} < (Q_1+2)^2 \tag{16}$$

for all x_1', x_2' in the domain between Σ'_{α_0} and Σ'_{α_1}. But this ring contains the ring (14) so that (16) holds for all x_1', x_2' in (14).

B. $\frac{1}{(Q_1+2)^4} < r_0^* < \frac{1}{(Q_1+2)^2}$. Arguing as in A of 2°, we conclude that in this case (16) holds for all x_1', x_2' in the intersection of the ring (14) with the closed ring E bordered by Σ'_{α_0} and Σ'_{α_1}. Besides, (13) implies that (15) holds for all x_1', x_2' in the intersection of (14) with $\overline{CE}$ (CE = the complement of E and $\overline{E}$ = the closure of E).

Let now x_1', x_2' be in the ring (14), but $x_1' \in E$ and $x_2' \in CE$. Let also x_3' be in

$$\Sigma'_{\alpha_1} \cap \left\{ \frac{1}{(Q_1+2)^3} \leqq |x'| \leqq \frac{1}{(Q_1+2)^2} \right\}.$$

Then (16) and (15) yield

$$\frac{|\psi(x_1')|}{|\psi(x_2')|} = \frac{|\psi(x_1')|}{|\psi(x_3')|} \cdot \frac{|\psi(x_3')|}{|\psi(x_2')|} \leqq (Q_1+2)^3 B e^{[B(Q_1+2)^3]^n}. \tag{17}$$

But

$$(Q_1+2)^2,\ B(Q_1+2)e^{[B(Q_1+2)^3]^n} \leqq B(Q_1+2)^3 e^{[B(Q_1+2)^3]^n},$$

then (17) holds for all x'_1, x'_2 in (14) for both cases 1° and 2°.

But (17) also holds for all x'_1, x'_2 in any ring of the following sequence:

$$\frac{1}{(Q_1+2)^{m+3}} \leqq |x'| \leqq \frac{1}{(Q_1+2)^{m+2}} \qquad (m=0,1,2,\dots). \tag{18}$$

Indeed let

$$S_m = \left\{ |x'| = \frac{1}{(Q_1+2)^m} \right\}, \quad r_m = \min_{x' \in \Sigma'_{\alpha_m}} |x'|, \quad R_m = \min_{x' \in \Sigma'_{\alpha_m}} |\psi(x')|, \quad r_m^* = \min_{|\bar{x}'| = R_m} |\psi^{-1}(\bar{x}')|,$$

where Σ'_{α_m} is the surface of the family $\{\Sigma'_\alpha\}$ which lies inside S_m and is tangent to it. Let us consider, as above, the cases $r_m^* \geqq \frac{1}{(Q_1+2)^{m+2}}$ and $r_m^* < \frac{1}{(Q_1+2)^{m+2}}$. Let $x'' = (Q_1+2)^m x'$ be a homothetic tramsformation with the ratio of similitude $(Q_1+2)^m$ and put

$$\psi(x') = \psi\left[\frac{x''}{(Q_1+2)^m}\right] = \chi(x'').$$

Then

$$\frac{|\psi(x'_1)|}{|\psi(x'_2)|} = \frac{|\chi(x''_1)|}{|\chi(x''_2)|} < B(Q_1+2)^3 e^{[B(Q_1+2)^3]^n}$$

holds for all x'_1, x'_2 in the corresponding ring of the sequence (18), because x''_1, x''_2 lie in (14) and for $\chi(x'')$ hold the conditions of the case 1° for $r_m^* \geqq \frac{1}{(Q_1+2)^{m+2}}$ and of the case 2° for $r_m^* < \frac{1}{(Q_1+2)^{m+2}}$.

But for every $r' < \frac{1}{(Q_1+2)^2}$ the sphere $|x'| = r'$ is contained in one of the rings (18), thus

$$\frac{\max_{|x'|=r'} |\psi(x')|}{\min_{|x'|=r'} |\psi(x')|} < B(Q_1+2)^3 e^{[B(Q_1+2)^3]^n}$$

and according to the notations (12)

$$\frac{\max_{|x-x_0|=r} |f(x)-f(x_0)|}{\min_{|x-x_0|=r} |f(x)-f(x_0)|} = \frac{\max_{|x'|=r'} |\psi(x')|}{\min_{|x'|=r'} |\psi(x')|} < B(Q_1+2)^3 e^{[B(Q_1+2)^3]^n},$$

where $r = \rho_0 r'$. As this inequality holds for every $r < \frac{1}{(Q_1+2)^2}$, it holds also for $r \to 0$ and this completes our proof.

3. The equivalence of the definition of quasiconformal homeomorphisms in Pesin's sense with the other definitions

THEOREM 9. *The definition of quasiconformal homeomorphisms in Pesin's sense is equivalent with those of Gehring and Väisälä.*

Proof. The preceding theorem implies that a quasiconformal homeomorphism in Pesin's sense has $\delta_L(x)$ uniformly bounded in D and thus it is quasiconformal according to Gehring's metric definition.

The converse inclusion is obvious, because $\delta_L(x) \leqq K$ means that the family of the spheres (which is a regular family of parameter 1) is mapped in regular family of parameter $k'(x) \leqq K$. Thus the definition of quasiconformal homeomorphisms in Pesin's sense is equivalent with Gehring's definition of quasiconformal homeomorphisms and hence is equivalent also with Väisälä's definitions and with Gehring's other definition.

COROLLARY 2. *Let $\bar{x} = f(x)$ be a quasiconformal homeomorphism in Pesin's sense in $D \subset R^n$. Then $J(x) \neq 0$ a.e. Moreover, if E is a measurable set in D, then $E^* = f(E)$ is also measurable and*

$$mE^* = \int_E |J(x)|\, d\tau.$$

This is a consequence of Gehring's theorem 6 in [7] or Väisälä's theorems 6.9 and 6.10 in [14].

We recall that an A-point or point of affinity (see [2]) of a mapping $\bar{x} = f(x)$ is a point x_0, in a neibourhood of which $f(x)$ is continuous and one-to-one and is differentiable and $J(x_0) \neq 0$.

LEMMA 2. *Let $\bar{x} = f(x)$ be a regular homeomorphism. Then any regular family of surfaces Σ_α $(0 < \alpha < 1)$ of parameter k $(1 \leqq k < \infty)$ relatively to an A-point x_0 is mapped in a regular family of parameter k' $(1 \leqq k' < \infty)$ if and only if the affine transformation $f_1(x, x_0)$ given by (4) maps the family $\{\Sigma_\alpha\}$ $(0 < \alpha < 1)$ in a regular family of parameter k'.*

Proof. For the sufficiency, let

$$\frac{\max\limits_{x \in \Sigma_\alpha} |x - x_0|}{\min\limits_{x \in \Sigma_\alpha} |x - x_0|} = k + \varepsilon_\alpha,$$

where $\lim_{\alpha\to 0}\varepsilon_\alpha = 0$. In the A-point x_0, (3) imply

$$\sqrt[n]{\frac{p_1(x_0)\cdots p_{n-1}(x_0)|J(x_0)|}{p_1^{n-1}(x_0)}} = \lambda_f(x_0) \leqq \frac{|f_1(x,x_0)-f(x_0)|}{|x-x_0|} = \left|\frac{\partial f(x_0)}{\partial s}\right| \leqq \Lambda_f(x_0) =$$
$$= \sqrt[n]{p_1(x_0)\cdots p_{n-1}(x_0)|J(x_0)|},$$

where s is the direction of the vector $x-x_0$. Hence

$$\overline{\lim_{\alpha\to 0}}\frac{\max\limits_{x\in\Sigma_\alpha}|f(x)-f(x_0)|}{\min\limits_{x\in\Sigma_\alpha}|f(x)-f(x_0)|} \leqq \overline{\lim_{\alpha\to 0}}\frac{\max\limits_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|+\max\limits_{x\in\Sigma_\alpha}[|\varepsilon(x,x_0)||x-x_0|]}{\min\limits_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|-\max\limits_{x\in\Sigma_\alpha}[|\varepsilon(x,x_0)||x-x_0|]} \leqq$$

$$\leqq \overline{\lim_{\alpha\to 0}}\frac{\max\limits_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\min\limits_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}\,\overline{\lim_{\alpha\to 0}}\frac{1+\dfrac{\max\limits_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{\max\limits_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\max\limits_{x\in\Sigma_\alpha}|x-x_0|}}}{1-\dfrac{\max\limits_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{\min\limits_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\max\limits_{x\in\Sigma_\alpha}|x-x_0|}}} \leqq$$

$$\leqq k'\,\overline{\lim_{\alpha\to 0}}\frac{1+\dfrac{\max\limits_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{\max\limits_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\min\limits_{x\in\Sigma_\alpha}|x-x_0|(k+\varepsilon_\alpha)}}}{1-\dfrac{\max\limits_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{\min\limits_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\min\limits_{x\in\Sigma_\alpha}|x-x_0|(k+\varepsilon_\alpha)}}} \leqq k'\,\overline{\lim_{\alpha\to 0}}\frac{1+\dfrac{(k+\varepsilon_\alpha)\max\limits_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{|f_1(x',x_0)-f(x_0)|}{|x'-x_0|}}}{1-\dfrac{(k+\varepsilon_\alpha)\max\limits_{\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{|f_1(x'',x_0)-f(x_0)|}{|x''-x_0|}}} \leqq$$

$$\leqq k'\,\overline{\lim_{\alpha\to 0}}\frac{1+\dfrac{(k+\varepsilon_\alpha)\max\limits_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\sqrt[n]{\dfrac{p_1(x_0)\cdots p_{n-1}(x_0)|J(x_0)|}{p_1^{n-1}(x_0)}}}}{1-\dfrac{(k+\varepsilon_\alpha)\max\limits_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\sqrt[n]{p_1(x_0)\cdots p_{n-1}(x_0)|J(x_0)|}}} = k',$$

where

$$|f_1(x',x_0)-f(x_0)| = \max_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|,$$
$$|f_1(x'',x_0)-f(x_0)| = \min_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|.$$

With the same notations and arguing as above, we have

$$\overline{\lim_{\alpha\to 0}}\frac{\max_{x\in\Sigma_\alpha}|f(x)-f(x_0)|}{\min_{x\in\Sigma_\alpha}|f(x)-f(x_0)|}\geqq\overline{\lim_{\alpha\leftarrow 0}}\frac{\max_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\min_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}\overline{\lim_{\alpha\to 0}}\frac{1-\dfrac{\max_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\max_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}}{\dfrac{\max_{x\in\Sigma_\alpha}|x-x_0|}{1+\dfrac{\max_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\min_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}}}\geqq$$

$$\geqq k'\overline{\lim_{\alpha\to 0}}\frac{1-\dfrac{(k+\varepsilon_\alpha)\max_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{\max_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\min_{x\in\Sigma_\alpha}|x-x_0|}}}{1+\dfrac{(k+\varepsilon_\alpha)\max_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{\min_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\min_{x\in\Sigma_\alpha}|x-x_0|}}}\geqq k'\overline{\lim_{\alpha\to 0}}\frac{1-\dfrac{(k+\varepsilon_\alpha)\max_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{|f_1(x',x_0)-f(x_0)|}{|x'-x_0|}}}{1-\dfrac{(k+\varepsilon_\alpha)\max_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\dfrac{|f_1(x'',x_0)-f(x_0)|}{|x''-x_0|}}}\geqq$$

$$\geqq k'\overline{\lim_{\alpha\to 0}}\frac{1-\dfrac{(k+\varepsilon_\alpha)\max_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\sqrt[n]{\dfrac{p_1(x_0)\cdots p_{n-1}(x_0)|J(x_0)|}{p_1^{n-1}(x_0)}}}}{1+\dfrac{(k+\varepsilon_\alpha)\max_{x\in\Sigma_\alpha}|\varepsilon(x,x_0)|}{\sqrt[n]{\dfrac{p_1(x_0)\cdots p_{n-1}(x_0)|J(x_0)|}{p_1^{n-1}(x_0)}}}}=k'.$$

Thus

$$\overline{\lim_{\alpha\to 0}}\frac{\max_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\min_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}=k' \tag{19}$$

implies

$$\overline{\lim_{\alpha\to 0}}\frac{\max_{x\in\Sigma_\alpha}|f(x)-f(x_0)|}{\min_{x\in\Sigma_\alpha}|f(x)-f(x_0)|}=k'. \tag{20}$$

The necessity is a consequence of the sufficiency. Indeed, suppose that (20) holds and that instead of (19) we have

$$\overline{\lim_{\alpha\to 0}}\frac{\max_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}{\min_{x\in\Sigma_\alpha}|f_1(x,x_0)-f(x_0)|}=k''\neq k'.$$

But then, by the sufficiency, we should have also

$$\overline{\lim_{\alpha\to 0}}\frac{\max_{x\in\Sigma_\alpha}|f(x)-f(x_0)|}{\min_{x\in\Sigma_\alpha}|f(x)-f(x_0)|}=k^n\neq k'.$$

Which would contradict (20). This completes our proof.

THEOREM 10. *The definition of the class of Q-quasiconformal homeomorphisms in Pesin's sense is equivalent with that of the class of Q-quasiconformal homeomorphisms according to Gehring's metric definition.*

Proof. Let $\bar{x}=f(x)$ be a Q-quasiconformal homeomorphism in Pesin's sense in D. By Theorem 8, $\delta_L(x)$ is uniformly bounded in D.

Let $q(x_0)=\inf k(x_0)k'(x_0)$ be the characteristic of $f(x)$ in an A-point x_0, where the infimum is taken over all regular relatively to x_0 families of surfaces. Let $q_1(x_0)=\inf k_1(x_0)k_1'(x_0)$ be the characteristic of the affine transformation $f_1(x, x_0)$ given by (4). The preceding lemma implies $q(x_0)=q_1(x_0)$ in every A-point x_0. But Lemma 1 implies $q(x_0)=q_1(x_0)=p_1(x_0)$, where $p_1(x_0)$ is the principal characteristic of the affine transformation $f_1(x, x_0)$. Hence, the inequality $q(x)\leqq Q$ a.e. in D implies that $p_1(x)\leqq Q$ a.e. in the set of all A-points of D. Then, by Theorem 6 and corollary 2, $p_1(x)\leqq Q$ a.e. in D. Thus the relation

$$\delta_L(x)=\overline{\lim_{r\to 0}}\frac{\dfrac{|f(x')-f(x_0)|}{|x'-x_0|}}{\dfrac{|f(x'')-f(x_0)|}{|x''-x_0|}}\leqq\frac{\overline{\lim\limits_{x\to x_0}}\dfrac{|f(x)-f(x_0)|}{|x-x_0|}}{\underline{\lim\limits_{x\to x_0}}\dfrac{|f(x)-f(x_0)|}{|x-x_0|}}=\frac{\Lambda_f(x_0)}{\lambda_f(x_0)},$$

where we put

$$\max_{|x-x_0|=r}|f(x)-f(x_0)|=|f(x')-f(x_0)|,$$

$$\min_{|x-x_0|=r}|f(x)-f(x_0)|=|f(x'')-f(x_0)|$$

and the relation (25) of [3]:

$$\frac{\Lambda_f(x_0)}{\lambda_f(x_0)}=p_1(x_0),$$

both holding in every A-point of D, imply

$$\delta_L(x)\leqq\frac{\Lambda_f(x)}{\lambda_f(x)}\leqq p_1(x)=q(x)\leqq Q$$

a.e. in D. Hence $\bar{x}=f(x)$ is Q quasiconformal according to Gehring's metric definition.

Conversely, let $\bar{x}=f(x)$ be a K-quasiconformal homeomorphism according to Gehring's metric definition. Then

$$q(x)\leqq\delta_L(x)$$

holds, because, by definition, $q(x) = \inf k(x)k'(x)$, while $\delta_L(x)$ is the only product $k(x)k'(x)$ where $\{\Sigma_\alpha\}$ reduces to the family of spheres $\{|x'-x|=r\}$. Thus $q(x)$ is uniformly bounded in D and $q(x) \leqq K$ a.e. in D.

Remark. This theorem and the relationships (from the introduction) between Väisälä's class of K-quasiconformal homeomorphisms and both Gehring's classes imply the corresponding relationship between the class of K-quasiconformal homeomorphisms in Pesin's sense (particularly the class of K-quasiconformal homeomorphisms with one or two sets of characteristics) and the other classes of K-quasiconformal homeomorphisms.

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