ON METRIC PROPERTIES OF SETS OF ANGULAR LIMITS OF MEROMORPHIC FUNCTIONS

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Let f be a nonconstant function meromorphic in the unit disc $D = \{|z| < 1\}$, with circumference C, and let E_z be a subset of C with positive (linear) measure. Suppose that at each $\zeta \in E_z$, f has an angular limit a_ζ, and let $E_w = \{a_\zeta : \zeta \in E_z\}$. It is known that E_w contains a closed set with positive harmonic measure (see Priwalow [6, p. 210] or Tsuji [7, p. 339]). Also known is that even when f is a schlicht function mapping D onto the interior of a Jordan curve, it may happen that E_w has linear measure zero (see Lavrentieff [2]) ; and a recent theorem of Matsumoto [4, p. 133] states, in effect, that if f is a schlicht function mapping D onto the interior of a Jordan curve, then E_w cannot have $\frac{1}{2}$-dimensional measure zero (For the definitions of (exterior) linear measure and α-dimensional measure zero ($\alpha > 0$), see [5, pp. 149, 150].). The purpose of the present paper is to prove a theorem that generalizes Matsumoto's theorem. As a corollary of our theorem, we obtain: *If each point of E_w is accessible (with a Jordan arc) through the complement of $f(D) = \{f(z) : z \in D\}$, then E_w contains a closed set that does not have $\frac{1}{2}$-dimensional measure zero.*

If E_w is all of the extended w-plane Ω, the desired conclusion already holds; so that we may, by first subjecting Ω to a linear transformation, *assume that* $\infty \notin E_w$. Our result is most conveniently expressed in terms of the Riemann surface S of f over Ω. For each $\zeta \in E_z$ and positive number h, let $S(\zeta, h)$ be the component of S over $\{|w - a_\zeta| < h\}$ such that if r is sufficiently near 1 ($r < 1$), then $r\zeta$ corresponds under f to a point of $S(\zeta, h)$; and let $PS(\zeta, h)$ be the projection of $S(\zeta, h)$ onto Ω.

We prove

THEOREM. *Suppose that to each $\zeta \in E_z$ there correspond a Jordan arc γ_ζ (contained in the finite w-plane) with one endpoint a_ζ and a positive number h_ζ such*

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that $PS(\zeta, h_\zeta)\cap\gamma_\zeta=\phi$. *Then* E_w *contains a closed set that does not have* $\frac{1}{2}$*-dimensional measure zero.*

Proof. Let $m(E)$ and $m_e(E)$ denote the (linear) measure and exterior (linear) measure of the set $E\subset C$. From Lusin's theorem, there exists a closed set $E_z^{(1)}\subset E_z$ such that $m(E_z^{(1)})>0$ and

(1) the restriction of a_ζ to $E_z^{(1)}$ is a continuous function.

For each $\zeta\in E_z^{(1)}$, let Δ_ζ be an open (Euclidean) disc with rational radius and center with two rational coordinates such that

$$a_\zeta\in\Delta_\zeta\subset\{|w-a_\zeta|<h_\zeta\},$$

and let S_ζ be the component of S over Δ_ζ such that if r is sufficiently near 1 ($r<1$), then $r\zeta$ corresponds under f to a point of S_ζ. Then $PS_\zeta\cap\gamma_\zeta=\phi$. Since there are only countably many distinct S_ζ, there exists $\zeta_0\in E_z^{(1)}$ such that the set

$$E_z^{(2)}=\{\zeta\in E_z^{(1)}:S_\zeta=S_{\zeta_0}\}$$

has positive exterior measure. Let $S_0=S_{\zeta_0}$ and $\Delta_0=\Delta_{\zeta_0}$. Then

(2) for each $\zeta\in E_z^{(2)}$, $PS_0\cap\gamma_\zeta=\phi$ and $a_\zeta\in\Delta_0$.

Let $S(\zeta, r)$ denote the sector ($\zeta=e^{i\tau}$, $0<r<1$)

$$\left\{\zeta+\rho e^{i\theta}:0<\rho<r,\ \tau+\frac{3\pi}{4}<\theta<\tau+\frac{5\pi}{4}\right\},$$

and for each $\zeta\in E_z^{(2)}$, let r_ζ be a positive number such that

(3) $f(S(\zeta, r_\zeta))\subset\Delta_0$.

Let r be a positive number and $E_z^{(3)}$ a subset of $E_z^{(2)}$ such that $m_e(E_z^{(3)})>0$, and for each $\zeta\in E_z^{(3)}$, $r\le r_\zeta$. Let r' $(0<r'<1)$ be such that $\{|z|=r'\}$ intersects the rectilinear segments on the boundary of $S(1, r)$, and let I be a component of $\{r'<|z|<1\}\cap\cup S(\zeta, r)$, the union being taken over all $\zeta\in E_z^{(3)}$, such that the set

$$E_z^{(4)}=\{\zeta\in E_z^{(3)}:S(\zeta, r)\cap I\neq\phi\}$$

has positive exterior measure. Then

$$I=\{r'<|z|<1\}\cap\cup S(\zeta, r),$$

where the union is taken over all $\zeta \in \overline{E}_z^{(4)}$ (the bar denotes closure). Thus I is the interior of a rectifiable Jordan curve Γ, and

$$\overline{E}_z^{(4)} = \Gamma \cap C \subset E_z^{(1)}.$$

From (3) we have $f(I) \subset \Delta_0$, and it follows that I corresponds under f to a subset of S_0. Thus $f(I) \subset PS_0$, and from (2) we have

(4) for each $\zeta \in E_z^{(4)}$, $f(I) \cap \gamma_\zeta = \phi$.

Let l be a positive constant, and let $E_z^{(5)}$ be a subset of $E_z^{(4)}$ such that $m_e(E_z^{(5)}) > 0$ and

(5) for each $\zeta \in E_z^{(5)}$, the diameter of γ_ζ is greater than or equal to $2\,l$.

By making suitable linear transformations, we may suppose that

$$0 \in I \quad \text{and} \quad f(0) = \infty. \tag{6}$$

Let γ be an arbitrary Jordan arc joining $(0<r<l,\ a \in \Omega - \{\infty\})$ $\{|w-a| = r\}$ to $\{|w-a| = l\}$ and lying, except for its endpoints, in $\{r<|w-a|<l\}$. Let $\omega(w\ ;\ a,\ r,\ \gamma)$ denote the harmonic measure of $\{|w-a| = r\}$ with respect to $\Omega - [\{|w-a| \leq r\} \cup \gamma]$. Using Matsumoto's argument [4, pp. 134, 135], we now prove that there exist positive constants h and M (which are independent of a, r and γ) such that

$$\omega(\infty\ ;\ a,\ r,\ \gamma) \leq M\sqrt{r} \qquad (0<r<h). \tag{7}$$

By letting γ' denote the image of γ under the translation $w-a$ and noting that

$$\omega(\infty\ ;\ a,\ r,\ \gamma) = \omega(\infty\ ;\ 0,\ r,\ \gamma'),$$

we see that we need only prove (7) under the assumption that $a = 0$. We assume then that $a = 0$, and write

$$D_r = \{|w| < r\}, \qquad C_r = \{|w| = r\}.$$

Let $\omega_r(w)$ be the harmonic measure of C_r with respect to

$$\Omega - [\overline{D}_r \cup \{u + iv\ :\ r \leq u \leq l,\ v = 0\}].$$

Then from Matsumoto's Lemma 2 [4, p. 132], there exist positive constants h and M such that $(h<l)$

$$\omega_r(\infty) \leq M\sqrt{r} \qquad (0<r<h). \tag{8}$$

Now let r be a fixed number satisfying $0<r<h$. For each r' satisfying $r<r'<l$, let $\gamma_{r'}$ be the subarc of γ that joins $C_{r'}$ to C_l and lies, except for its end-points, in $\{r'<|w|<l\}$. And let $\{J_n\}$ be a sequence of Jordan curves such that $\overline{D}_r$ is contained in the exterior of J_n, J_{n+1} is contained in the interior I_n of J_n, and $\gamma_{r'}=\bigcap_{n=1}^{\infty} I_n$. Then the harmonic measure $\omega_n(w)$ of C_r with respect to $\Omega-[\overline{D}_r \cup J_n \cup I_n]$ and the harmonic measure $\omega'(w)$ of C_r with respect to $\Omega-[\overline{D}_r \cup \gamma_{r'}]$ satisfy

$$\omega_n(\infty)\uparrow \omega'(\infty). \tag{9}$$

For a fixed n, we choose rectilinear segments

$$L_j=\{w : r_j\leq |w| \leq r_{j+1},\ \text{argument } w=\theta_j\}$$

$(j=1, \ldots, k;\ r'=r_1<r_2<\cdots<r_{k+1}=l)$ that are contained in I_n. Then the harmonic measure $\omega_n^*(w)$ of C_r with respect to $\Omega-\left[\overline{D}_r \cup \bigcup_{j=1}^{k} L_j\right]$ satisfies the relation $\omega_n(\infty)\leq \omega_n^*(\infty)$; and from Matsumoto's Lemma 1 [4, p. 131], the harmonic measure $\tilde{\omega}(w)$ of C_r with respect to

$$\Omega-[\overline{D}_r \cup \{u+iv : r'\leq u \leq l,\ v=0\}]$$

satisfies the relation $\omega_n^*(\infty)\leq\tilde{\omega}(\infty)$. Thus $\omega_n(\infty)\leq\tilde{\omega}(\infty)$, and from (9) we have the relation $\omega'(\infty)\leq\tilde{\omega}(\infty)$; and letting $r'\downarrow r$, we see that $\omega(\infty;0,r,\gamma)\leq\omega_r(\infty)$. Thus from (8) the proof of (7) is complete.

We now suppose that the set $E_w^{(1)}=\{a_\zeta : \zeta\in E_z^{(1)}\}$ (which is closed and bounded because $E_z^{(1)}$ is closed, $\infty\notin E_w$, and (1)) has $\frac{1}{2}$-dimensional measure zero. We wish to prove that this assumption leads to a contradiction. Let $E_w^*=\overline{E}_w^{(5)}$, where $E_w^{(5)}=\{a_\zeta : \zeta\in E_z^{(5)}\}$. Then $E_w^*\subset E_w^{(1)}$. Let $E_z^*=\{\zeta\in E_z^{(1)} : a_\zeta\in E_w^*\}$. Then from (1), E_z^* is closed relative to the closed set $E_z^{(1)}$, and is therefore closed.

Let ε be a positive number. Since E_w^* is closed and bounded and has $\frac{1}{2}$-dimensional measure zero, there exists a finite number of discs $\Delta_j=\{|w-a_j|<r_j\}$ $(j=1, \ldots, n)$ such that

$$0<r_j<h \qquad (j=1, \ldots, n), \tag{10}$$

$$\sum_{j=1}^{n}\sqrt{r_j}<\frac{\varepsilon}{M}, \tag{11}$$

$$E_w^*\subset\bigcup_{j=1}^{n}\Delta_j, \tag{12}$$

and

(13) $$\Delta_j \cap E_w^* \neq \phi \qquad (j=1, \ldots, n).$$

It follows from (13) that for each j $(j=1, \ldots, n)$ there exists $\zeta_j \in E_z^{(5)}$ such that $a_{\zeta_j} \in \Delta_j$; and from (5) we see that $\gamma_{\zeta_j} \cap \{|w-a_j|=l\} \neq \phi$. Thus we may let γ_j be a subarc of γ_{ζ_j} that joins $\{|w-a_j|=r_j\}$ to $\{|w-a_j|=l\}$ and lies, except for its endpoints, in $\{r_j<|w-a_j|<l\}$. Let U be the component of $\Omega - \bigcup_{j=1}^{n} [\bar{\Delta}_j \cup \gamma_j]$ that contains ∞, and let

$$\omega(w) = \sum_{j=1}^{n} \omega(w\,;\, a_j, r_j, \gamma_j) \qquad (w \in U).$$

Then from (7), (10) and (11), we have

(14) $$\omega(\infty) < \varepsilon.$$

Let $z(z')$ be a conformal mapping of $D' = \{|z'|<1\}$ onto I such that $z(0)=0$ (recall (6)). Since E_z^* is closed and $E_z^{(5)} \subset E_z^*$, $m(E_z^*)>0$; and it follows that E_z^* corresponds under $z=z(z')$ to a closed set $E_{z'}^*$ on $C'=\{|z'|=1\}$; and since Γ is rectifiable, $m(E_{z'}^*)>0$ [6, p. 127]. Let $u(z')$ be the harmonic measure of $E_{z'}^*$ with respect to D'. Let $F(z')=f(z(z'))$, let D_0 be the component of $\{z' \in D' : F(z') \in U\}$ that contains 0 (recall (6)), and let B denote the boundary of D_0.

We wish now to establish the relation

(15) $$u(z') \leq \omega(F(z')) \qquad (z' \in D_0).$$

From (4) we see that

(16) $$F(B \cap D') \subset \bigcup_{j=1}^{n} \{|w-a_j|=r_j\} - \bigcup_{j=1}^{n} \gamma_j,$$

so that in particular,

$$\lim_{z' \to \zeta,\, z' \in D_0} \omega(F(z')) \geq 1 \quad \text{for each } \zeta \in B \cap D'.$$

It follows from (4) and a theorem of MacLane [3, p. 10] that for each j $(j=1, \ldots, n)$, the level set $\{z' \in D' : |F(z')-a_j|=r_j\}$ "ends at points of C'" [3, p. 8]. Thus it follows from (16) that each point of $B \cap C'$ is accessible through D_0 (that is, for each $\zeta \in B \cap C'$ there exists a Jordan arc that is, except for the one endpoint ζ, contained in D_0). Since at each point of $E_{z'}^*$, F has an asymptotic value that is in E_w^*, we have from (12) that each point of $E_{z'}^*$ is accessible through $D' - D_0$. Thus, each point of $E^*_{z'} \cap B$ is accessible through both D_0 and $D' - D_0$, and from a theorem of Bagemihl [1, Theorem 1], the set $E^*_{z'} \cap B$ is countable. But for each $\zeta \in C' - E^*_{z'}$, $\lim_{z' \to \zeta} u(z') = 0$, so that (15) follows from an extension of the maximum principle.

From (15) and (14) we have

$$\frac{1}{2\pi} m(E^*_{z'}) = u(0) \leq \omega(F(0)) = \omega(\infty) < \varepsilon,$$

and since ε is arbitrary, we have a contradiction; and the proof of the theorem is complete.

Remark. Let $E = E(p_0 p_1 \cdots)$, where $p_n = n$, be the Cantor-type set defined by Nevanlinna [5, p. 154]. Then E has positive harmonic measure [5, p. 155] and for each positive number α, since $2^n/(n!)^\alpha \to 0$ $(n \to \infty)$, E has α-dimensional measure zero. Let F be a holomorphic function that maps D one-to-one and conformally onto the universal covering surface of $\Omega - [E \cup \{\infty\}]$. It follows from theorems of Nevanlinna [5, pp. 208, 213] that F has angular limits at almost all (except for a set of measure zero) points of C; and from a theorem of Lusin and Priwalow [6, p. 212], at almost every point of C the angular limit value of F is in E. Applying now an argument of Lusin and Priwalow (see [6, p. 210]) we see that *there exists a nonconstant function f bounded and analytic in D such that for each positive number* α, E_w *has* α*-dimensional measure zero.*

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