UNIQUE CONTINUATION FOR PARABOLIC EQUATIONS OF HIGHER ORDER

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1. Let $x=(x_1,\ldots,x_n)$ be a point in the n-dimensional Euclidean space and let $\mathscr{D}$ be the unit sphere $|x|=\left(\sum_{i=1}^{n}x_i^2\right)^{1/2}<1$. In the $(n+1)$-dimensional Euclidean space with coordinate (x,t), we put

$$\Omega=\Omega_{T',T''}=\{(x,t)\ ;\ x\in\mathscr{D},\ T'\leqq t\leqq T''\}$$

and

$$S=S_{T',T''}=\{(x,t)\ ;\ x\in\dot{\mathscr{D}},\ T'\leqq t\leqq T''\},$$

where $\dot{\mathscr{D}}$ denotes the boundary of $\mathscr{D}$. We also use the following notation:

$$\mathscr{D}_T=\{(x,t)\ ;\ x\in\mathscr{D},\ t=T\}.$$

For real-valued functions $h_1=h_1(x,t)$ and $h_2=h_2(x,t)$ square integrable in Ω, we put

$$(h_1,h_2)=(h_1,h_2)_\Omega=\iint_\Omega h_1h_2\,dxdt$$

and

$$\|h_1\|^2=\|h_1\|_\Omega^2=\iint_\Omega h_1^2\,dxdt.$$

We denote by $\mathfrak{V}$ the family of all the functions $v=v(x,t)\in C^{2s}(\Omega\cup S)$ which vanishes on $\mathscr{D}_{T'}$ and satisfies $D_x^\alpha v=0$ $(|\alpha|\leqq s-1)$ on S. Here $C^{2s}(\Omega\cup S)$ is the class of all functions $2s$-times continuously differentiable in (a neighbourhood of) $\Omega\cup S$ and $D_x^\alpha v$ is the derivative

$$\frac{\partial^{|\alpha|}v}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

of v for a multi-index $\alpha=(\alpha_1,\ldots,\alpha_n)$ $(\alpha_i\geqq 0)$ of integers with length $|\alpha|=\alpha_1+\cdots+\alpha_n$.

2. Consider a differential operator

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(1) $$L = A - (-1)^s \frac{\partial}{\partial t}$$

defined in $\Omega \cup S$, where A is of the form

$$A = \sum_{|\alpha| \leqq 2s} a_\alpha D_x^\alpha .$$

We assume that all the coefficients $a_\alpha = a_\alpha(x, t)$ are s-times continuously differentiable in $\Omega \cup S$ and are real-valued.

In this note, we shall prove the following theorem.

THEOREM. *Suppose that L is an operator of the form* (1) *and that A is uniformly elliptic in $\Omega \cup S$, that is, suppose that there exists a positive constant k_0 depending only on A and satisfying, at every point $(x, t) \in \Omega \cup S$,*

$$\sum_{|\alpha| = 2s} a_\alpha(x, t)\xi^\alpha \geqq k_0(\xi_1^2 + \cdots + \xi_n^2)^s$$

for any real vector $\xi = (\xi_1, \ldots, \xi_n)$, where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$.

If in Ω

(2) $$(Lu)^2 \leqq k_1 \sum_{|\alpha| \leqq s} |D_x^\alpha u|^2$$

for some constant k_1 and if $u = 0$ on $\mathscr{D}_{T''}$ and $D_x^\alpha u = 0$ ($|\alpha| \leqq s-1$) on S, then u vanishes in Ω.

In the case when s is even, our theorem gives a backward uniqueness property of a solution of the equation $\left(A - \frac{\partial}{\partial t}\right)u = 0$. If s is odd, our theorem gives a uniqueness of a solution of the boundary value problem for $\left(A - \frac{\partial}{\partial t}\right)u = 0$.

Analogous theorems were given by many authors, Ito-Yamabe [3], Mizohata [7], Yamabe [10], Lees-Protter [5], Protter [9] and Edmunds [1]. In abstract way, such results were stated by Yosida [11], Lions-Malgrange [6] and Lees [4].

3. To prove the theorem, we prepare two lemmas which are analogous to Lees-Protter's estimates.

LEMMA 1. *Assume that A in* (1) *is uniformly elliptic in $\Omega \cup S$. If v is in $\mathfrak{B}$, if $f = f(t)$ is in $C^1([T', T''])$ and if $g = g(t)$ continuous in $[T', T'']$ has no zero, then there exist two positive constants k_2 and k_3 depending only on A such that*

$$k_2\|fv\|_s^2 \leqq \|fgLv\|^2 + ((k_3f^2 - 2ff' + f^2g^{-2})v, v) + \int_{\mathscr{D}_{T''}} f^2v^2dx,$$

where $\|v\|_s^2 = \sum_{|\alpha|\leqq s} \|D_x^\alpha v\|^2$.

Proof. It is obvious that

$$(3) \qquad (-1)^s 2(fv, fLv) \leqq \|fgLv\|^2 + \|fg^{-1}v\|^2.$$

Since A is uniformly elliptic in $\Omega \cup S$, it is easily proved in a manner quite similar to Nirenberg's [8] that Gårding's inequality [2] holds, that is, there exist two constants k_2 and k_3 depending only on A such that

$$k_2\|fv\|_s^2 \leqq (-1)^s 2(fv, fAv) + k_3\|fv\|^2.$$

So we have

$$(4) \qquad k_2\|fv\|_s^2 \leqq (-1)^s 2(fv, fLv) + k_3\|fv\|^2 + 2\left(fv, f\frac{\partial v}{\partial t}\right).$$

As to the last term of the right hand side of this inequality, we see by integration by parts

$$2\left(fv, f\frac{\partial v}{\partial t}\right) = -2(fv, f'v) + \int_{\mathscr{D}_{T''}} f^2v^2dx.$$

Here we have used the assumption $v \in \mathfrak{V}$. From (3), (4) and this, we have our lemma.

LEMMA 2. *Suppose that v is in $\mathfrak{V}$ and that $f = f(t) \in C^\infty([T', T''])$ and $g = g(t)$ continuous in $[T', T'']$ have no zero. Then for a given operator L in* (1), *there exists a constant k_4 depending only on A such that*

$$(fv, f''v) \leqq \|fLv\|^2 + k_4(\|fgv\|_s^2 + \|f'g^{-1}v\|_s^2) + \int_{\mathscr{D}_{T''}} ff'v^2dx.$$

Proof. Putting $u = fv$, we see easily

$$(5) \qquad -2\left(\frac{\partial u}{\partial t}, f'v\right) \leqq \|fLv\|^2 - 2(-1)^s(Au, f'v) - \|f'v\|^2.$$

Obviously u is in $\mathfrak{V}$. Integrating by parts we get

$$(6) \qquad -2\left(\frac{\partial u}{\partial t}, f'v\right) = ((ff'' - f'^2)v, v) - \int_{\mathscr{D}_{T''}} ff'v^2dx.$$

Now we estimate the integral $(a_\alpha D_x^\alpha v, f'v)$. Repeated use of integration by parts and Leibniz' formula gives us

$$|(a_\alpha D_x^\alpha u,\ f'v)| = |(D_x^\beta u,\ D_x^\gamma(a_\alpha f'v))|$$
$$\leqq Mk_5\|fgD_x^s v\|\,\|f'g^{-1}v\|_s,$$

where $\alpha=\beta+\gamma$, $|\beta|\leqq s$, $|\gamma|\leqq s$ and k_5 is a constant depending only on s and n and further the constant M depends only on L. Hence it holds that

$$(7)\qquad -2(-1)^s(Au,\ fv)\leqq k_4(\|fgv\|_s^2+\|f'g^{-1}v\|_s^2)$$

for a constant k_4 depending only on A. From (5), (6) and (7) we obtain the required.

4. Now we give the proof of Theorem.

Take two numbers $\eta(>T'')$ and T_1 $(T'<T_1<T'')$ such that

$$(8)\qquad k_1\left(1+\frac{K}{2}\right)(\eta-T_1)<\frac{k_2}{4},$$

where $K=k_3(\eta-T_1)+1$ and k_1, k_2 and k_3 are constants appearing in Lemma 1 and the assumption of Theorem.

It is sufficient to show that u vanishes in $\Omega_{T_1,T''}$.

Let $\varphi=\varphi(t)$ be a function infinitely many times differentiable in $[T',\ T'']$ such that

$$\varphi=\begin{cases}1, & T_2<t<T''\\ 0, & T'<t<T_1(<T_2)\end{cases}$$

for some T_2 fixed. Put $w=\varphi u$. It is evident that w is in $\mathfrak{B}$ and $w=0$ on $\mathscr{D}_{T''}$. Taking an integer $m(>0)$ and applying Lemma 1 for $v=w$, $f=(\eta-t)^{-m-1/2}$ and $g=(\eta-t)^{1/2}$, we have

$$(9)\qquad k_2\|(\eta-t)^{-m-1/2}w\|_s^2\leqq\|(\eta-t)^{-m}Lw\|^2+K\|(\eta-t)^{-m-1}w\|^2.$$

Next we apply Lemma 2 for $v=w$, $f=(\eta-t)^{-m}$ and $g=m^{1/2}(\eta-t)^{-1/2}$ and we get

$$\|(\eta-t)^{-m-1}w\|^2\leqq\frac{1}{m(m+1)}[\|(\eta-t)^{-m}Lw\|^2+2k_4m\|(\eta-t)^{-m-1/2}w\|_s^2].$$

Substituting this into (9), we get

$$k_2\|(\eta-t)^{-m-1/2}w\|_s^2$$
$$\leqq\left(1+\frac{K}{2}\right)\|(\eta-t)^{-m}Lw\|^2+\frac{2k_4K}{m+1}\|(\eta-t)^{-m-1/2}w\|_s^2.$$

The function w is identical with u in $\Omega_{T_2,T''}$ and the assumption (2) implies that

$$\|(\eta-t)^{-m}Lw\|^2 = \|(\eta-t)^{-m}Lu\|^2_{\Omega_{T_2,T''}} + \|(\eta-t)^{-m}Lw\|^2_{\Omega_{T_1,T_2}}$$
$$\leqq k_1\|(\eta-t)^{-m}w\|_s^2 + \|(\eta-t)^{-m}Lw\|^2_{\Omega_{T_1,T_2}},$$

whence follows that

$$k_2\|(\eta-t)^{-m-1/2}w\|_s^2 \leqq \left(1+\frac{K}{2}\right)(\eta-T_2)^{-2m}\|Lw\|^2_{\Omega_{T_1,T_2}}$$
$$+\left[k_1\left(1+\frac{K}{2}\right)(\eta-T_1)+\frac{2k_4K}{m+1}\right]\|(\eta-t)^{-m-1/2}w\|_s^2.$$

This is valid for any positive integer m. We can choose an m_0 such that

$$\frac{2k_4K}{m+1} < \frac{k_2}{4}$$

for any $m \geqq m_0$. From this and (8), we have

$$\frac{k_2}{2}\|(\eta-t)^{-m-1/2}w\|_s^2 \leqq \left(1+\frac{K}{2}\right)(\eta-T_2)^{-2m}\|Lw\|^2_{\Omega_{T_1,T_2}}$$

for $m \geqq m_0$. Restricting the integral of the left hand side over $\Omega_{T_3,T''}$ for such a T_3 as $T_2 < T_3 < T''$, we get

$$\frac{k_2}{2}(\eta-T_3)^{-2m-1}\|u\|^2_{s,\Omega_{T_3,T''}} \leqq \left(1+\frac{K}{2}\right)(\eta-T_2)^{-2m}\|Lw\|^2_{\Omega_{T_1,T_2}}$$

or

$$\|u\|^2_{s,\Omega_{T_3,T''}} \leqq \frac{2}{k_2}\left(1+\frac{K}{2}\right)(\eta-T_3)\left(\frac{\eta-T_3}{\eta-T_2}\right)^{2m}\|Lw\|^2_{\Omega_{T_1,T_2}}$$

for $m \geqq m_0$. Making m tend to infinity, we see $u=0$ in $\Omega_{T_3,T''}$. Since T_3 is arbitrary as far as $T_2 < T_3 < T''$, it is seen that u vanishes in $\Omega_{T_2,T''}$. Further, T_2 is arbitrary as far as $T_1 < T_2 < T''$. So u vanishes throughout $\Omega_{T_1,T''}$. Thus our theorem is proved.

Remark. It is not difficult to see that, in our theorem, we can replace the assumption $u=0$ on $\mathscr{D}_{T''}$ by the condition

$$\lim_{t\to T''}\int_{\mathscr{D}_t}\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 dx = 0.$$

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