ON THE GROTHENDIECK RING OF AN ABELIAN p-GOUP

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Introduction

The Grothendieck ring of a finite group has been studied by Swan ([5], [6]). At the end of [6] he determined completely the structure of the Grothendieck ring $G(Z\mathfrak{G})$ of a cyclic p-group $\mathfrak{G}$ over the ring of rational integers Z.

In this paper we investigate the structure of $G(Z\mathfrak{G})$ of an abelian p-group $\mathfrak{G}$.

In the first section we consider some properties of the integral group ring of $\mathfrak{G}$. The results of this section are applied in the second section to investigate the additive structure of $G(Z\mathfrak{G})$. Let $\mathfrak{o}$ be a maximal order of the group ring $Q\mathfrak{G}$ over the rational number field Q and let $Co(\mathfrak{o})$ be the reduced projective class group of $\mathfrak{o}$ (Rim [4]). We show that $G(Z\mathfrak{G})$ is isomorphic to the splitting Z-algebra extension of $Co(\mathfrak{o})$ by $G(Q\mathfrak{G})$ (§2, §3). The latter half of the third section is devoted to study the action of $G(Q\mathfrak{G})$ to $Co(\mathfrak{o})$. Some examples are given in the final section.

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§1. The integral group ring of a finite abelian group

Let R be the ring of integers of an algebraic number field K. The group ring $K\mathfrak{G}$ of a finite abelian group $\mathfrak{G}$ over K decomposes into a direct sum of algebraic number fields K_i over K

$$K\mathfrak{G} = K_1 \oplus \cdots \oplus K_s, \tag{1.1}$$

and $K_1, \ldots, K_s$ are a full set of non-isomorphic irreducible $K\mathfrak{G}$-modules. This decomposition induces the decomposition of the maximal order $\mathfrak{o}$ of $K\mathfrak{G}$ into a direct sum of maximal orders $\mathfrak{o}_i$ of K_i, i.e. the ring of integers of K_i. Since

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$\mathfrak{o}$ contains $R\mathfrak{G}$, each projection π_i of $K\mathfrak{G}$ onto K_i induces a ring homomorphism of $R\mathfrak{G}$ into $\mathfrak{o}_i$. We will denote by Λ_i the kernel of this ring homomorphism and we will set $\Gamma_i = \prod_{j \neq i} \Lambda_j$.

PROPOSITION 1.1. *Let $\mathfrak{G}$ be a finite abelian group of order n and exponent n_0 and let $K = Q(\zeta_m)$ be a cyclotomic field, where ζ_m means a primitive m-th root of* 1. *Then*

(1) *in* (1.1), *each K_i is also a cyclotomic field $Q(\zeta_{m_i})$ for some m_i which divides L.C.M. (m, n_0),*

(2) *each projection π_i induces a surjection of $R\mathfrak{G}$ onto $\mathfrak{o}_i$.*

(3) *for each i, $\Lambda_i + \Gamma_i \supseteq n^{s-1} R\mathfrak{G}$, and*

(4) *there exists a positive integer l such that*

$$\Gamma_1 + \cdots + \Gamma_s \supseteq n^l R\mathfrak{G}.$$

Proof. Let $\mathfrak{G} = \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_t$ be the decomposition of $\mathfrak{G}$ into a direct product of cyclic subgroups $\mathfrak{G}_h$ and let g_h be the fixed generator of $\mathfrak{G}_h$.

Then we have $K_i = K(\pi_i(g_1), \ldots, \pi_i(g_t))$. But for each h $\pi_i(g_h)^{n_0} = 1$, which implies that $K_i = Q(\zeta_{m_i})$ for some m_i which divides $L.C.M.(m, n_0)$. This shows (1). Each π_i gives rise to the surjection of $R\mathfrak{G}$ onto $R[\pi_i(g_1), \ldots, \pi_i(g_t)] = Z[\zeta_{m_i}]$, which is the maximal order of $K_i = Q(\zeta_{m_i})$. This proves (2). (3) and (4) is proved by an induction on t. First, we suppose that $\mathfrak{G}$ is a cyclic group generated by an element g. We have a ring isomorphism $K\mathfrak{G} \cong K[x]/(x^n - 1)K[x]$, where $K[x]$ is the polynomial ring over K in an indeterminate x. If

$$x^n - 1 = f_1(x) \cdots f_s(x) \tag{1.2}$$

is the factorization of $x^n - 1$ into irreducible non-constant monic polynomials in $K[x]$, by the Chinese remainder theorem we have

$$K[x]/(x^n - 1)K[x] \cong K[x]/f_1(x)K[x] \oplus \cdots \oplus K[x]/f_s(x)K[x]. \tag{1.3}$$

Obviously every root of $f_i(x)$ is a primitive n_i-th root of 1 for some n_i which divides n. Let ζ_{n_i} be one of these roots and let $K_i = K(\zeta_{n_i})$. Then the map $g \to \zeta_{n_i}$ gives rise to the projection π_i of $K\mathfrak{G}$ onto K_i. This shows that the kernel of π_i is $f_i(g)K\mathfrak{G}$, so that Λ_i is just given by $R\mathfrak{G} \cap f_i(g)K\mathfrak{G} = f_i(g)R\mathfrak{G}$. By a simple calculation, we have from (1.2)

$$f_i(x)R[x]+f_j(x)R[x]\supseteq nR[x] \qquad (i\neq j). \tag{1.4}$$

Replacing x by g, we have

$$\Lambda_i+\Lambda_j\supseteq nR\mathfrak{G} \qquad (i\neq j). \tag{1.5}$$

This implies that $\Lambda_i+\prod_{j\neq i}\Lambda_j\supseteq n^{s-1}R\mathfrak{G}$, which shows (3). (1.2) yields also that

$$\prod_{j\neq 1}f_j(x)R[x]+\cdots+\prod_{j\neq s}f_j(x)R[x]\supseteq nR[x]. \tag{1.6}$$

Since $\prod_{j\neq i}f_j(g)R\mathfrak{G}=\Gamma_i$, this implies (4).

In the general case, let $\mathfrak{G}'=\mathfrak{G}_1\times\cdots\times\mathfrak{G}_{t-1}$ and let n' and n'' be the order of $\mathfrak{G}'$ and $\mathfrak{G}_t$, respectively. If $x^{n''}-1=f_1(x)\cdots f_s(x)$ is the factorization of $x^{n''}-1$ into irreducible monic polynomials in $K[x]$ and ζ_{n_i} is a root of $f_i(x)$, the map $g_t\to\zeta_{n_i}$ gives an isomorphism $K\mathfrak{G}/f_i(g_t)K\mathfrak{G}\cong K(\zeta_{n_i})\mathfrak{G}'$. Denoting $K(\zeta_{n_i})$ by K_i, we have $K\mathfrak{G}\cong K_1\mathfrak{G}'\oplus\cdots\oplus K_s\mathfrak{G}'$. On the other hand, (1) implies that each $K_i\mathfrak{G}'$ is a direct sum of cyclotomic fields $K_{i,j}$:

$$K_i\mathfrak{G}'=K_{i,1}+\cdots+K_{i,s_i}.$$

Let R_i and $\mathfrak{o}_{i,j}$ be the rings of integers of K_i and $K_{i,j}$, respectively, and let $\Lambda_{i,j}$ be the kernel of the surjection of $R\mathfrak{G}$ onto $\mathfrak{o}_{i,j}$. This surjection is given by the combined map $R\mathfrak{G}\to R_i\mathfrak{G}'\to\mathfrak{o}_{i,j}$. Since $f_i(g_t)R\mathfrak{G}$ is the kernel of the surjection $R\mathfrak{G}\to R_i\mathfrak{G}'$, we see that

$$\Lambda_{i,j}\supseteq f_i(g_t)R\mathfrak{G} \qquad (j=1,\ldots,s_i), \tag{1.7}$$

and that the image $\bar{\Lambda}_{i,j}$ in $R_i\mathfrak{G}'$ of $\Lambda_{i,j}$ is the kernel of $R_i\mathfrak{G}'\to\mathfrak{o}_{i,j}$. Now for any distinct $\Lambda_{i,j}$ and $\Lambda_{h,k}$, we will show that $\Lambda_{i,j}+\Lambda_{h,k}\supseteq nR\mathfrak{G}$. When $\mathfrak{G}$ is a cyclic group, this is given in (1.5). Then for any distinct k and k', the induction hypothesis shows that $\bar{\Lambda}_{i,k}+\bar{\Lambda}_{i,k'}\supseteq n'R_i\mathfrak{G}'$. Since n' divides n, this implies that $\Lambda_{i,k}+\Lambda_{i,k'}\supseteq nR\mathfrak{G}$. On the other hand, for any distinct i and i', we see easily that $f_i(g_t)R\mathfrak{G}+f_{i'}(g_t)R\mathfrak{G}\supseteq n''R$ similarly as in (1.4). Since n'' divides n, (1.7) shows that $\Lambda_{i,j}+\Lambda_{i',j'}\supseteq nR\mathfrak{G}$. Let $\Gamma_{i,j}$ be the product of all $\Lambda_{h,k}$ but $\Lambda_{i,j}$. Then a simple calculation shows that $\Lambda_{i,j}+\Gamma_{i,j}\supseteq n^{\Sigma s_k-1}R\mathfrak{G}$ from the above result, which proves (3). Let $\Delta_{i,j}=\prod_{k\neq j}\Lambda_{i,k}$. Then by the induction hypothesis, there exists a positive integer l_i such that $\bar{\Delta}_{i,1}+\cdots+\bar{\Delta}_{i,s_i}\supseteq n'^{l_i}R_i\mathfrak{G}'$, which shows that

$$\Delta_{i,1}+\cdots+\Delta_{i,s_i}\supseteq n'^{l_i}R\mathfrak{G}, \tag{1.8}$$

Since $\Lambda_{h,k}+\Lambda_{h,k'}\supseteq nR\mathfrak{G}$ for any distinct k and k', it follows that $\Lambda_{h,1}\cdots\Lambda_{h,s_h}\supseteq n^{s_h(s_h-1)/2}(\Lambda_{h,1}\cap\cdots\cap\Lambda_{h,s_h})$. But each $\Lambda_{h,k}$ contains $f_h(g_t)R\mathfrak{G}$ from (1.6), so that $\Lambda_{h,1}\cdots\Lambda_{h,s_h}\supseteq n^{s_h(s_h-1)/2}f_h(g_t)R\mathfrak{G}$. Let $l'=\text{Max.}\{l_1,\dots,l_s\}$ and $l''=\text{Max.}\left\{\frac{1}{2}\sum_{h\neq 1}s_h(s_h-1),\dots,\frac{1}{2}\sum_{h\neq s}s_h(s_h-1)\right\}$. Then we have from (1.8)

$$\sum_{i,j}\Gamma_{i,j}=\sum_{i,j}\Delta_{i,j}\prod_{h\neq i}(\Lambda_{h,1}\cdots\Lambda_{h,s_h})\supseteq n^{l''}n'^{l'}\sum_{i}\prod_{h\neq i}f_h(g_t)R\mathfrak{G}.$$

As in (1.5) we have $\sum_i\prod_{h\neq i}f_h(g_t)R\mathfrak{G}\supseteq n''R\mathfrak{G}$. Hence $l=l'+l''$ satisfies (4). This completes the proof of the proposition.

§ 2. The additive structure of $G(Z\mathfrak{G})$

We are now ready to investigate the additive structure of $G(Z\mathfrak{G})$ of an abelian p-group $\mathfrak{G}$. Let $\mathfrak{G}$ be of order p^e and exponent p^{e_0}. We denote by ζ_d a primitive p^d-*th* root of 1.

From Proposition 1.1, $Q\mathfrak{G}$ is a direct sum of cyclotomic fields $K_i=Q(\zeta_{d_i})$ for some d_i such that $0\leqq d_i\leqq e_0$ and the maximal order $\mathfrak{o}$ of $Q\mathfrak{G}$ is also a direct sum of the maximal orders $\mathfrak{o}_i=Z[\zeta_{d_i}]$ of K_i. Furthermore, the surjection of $Z\mathfrak{G}$ onto $\mathfrak{o}_i$ induced by π_i gives a ring isomorphism

$$Z\mathfrak{G}/\Lambda_i\cong\mathfrak{o}_i. \tag{2.1}$$

Let M be any regular (i.e. finitely generated and Z-torsion free) $Z\mathfrak{G}$-module and let

$$M_i=\{m\in M : \lambda_i m=0 \text{ for any } \lambda_i\in\Lambda_i\}.$$

Then M_i is a Z-pure submodule of M. Since Λ_i annihilates M_i, we may turn M_i into an $\mathfrak{o}_i$-module from (2.1). Clearly M_i is finitely generated and torsion free as an $\mathfrak{o}_i$-module, so that M_i is projective since $\mathfrak{o}_i$ is a Dedekind ring. Thus M_i is isomorphic to the direct sum of l_i-1 copies of $\mathfrak{o}_i$ and an ideal $\mathfrak{a}$ of $\mathfrak{o}_i$

$$M_i\cong\mathfrak{o}_i\oplus\cdots\oplus\mathfrak{o}_i\oplus\mathfrak{a}, \tag{2.2}$$

where the $\mathfrak{o}_i$-rank l_i of M_i and the ideal class $C_i(\mathfrak{a})$ of $\mathfrak{a}$ are complete invariants of M_i (Curtis and Reiner [3]). By Proposition 1.1, (3), we have $M_i\cap(M_1+\cdots+M_{i-1}+M_{i+1}+\cdots+M_s)=0$. This shows that the sum of M_i is a direct sum. Now we denote by $\overline{M}$ the quotient $M/\sum\oplus M_i$. Since $\Lambda_i\Gamma_i=0$, $\overline{M}$ is annihilated by $\Gamma_1+\cdots+\Gamma_s$. Then Proposition 1.1, (4) implies that $\overline{M}$ may be regarded as a module over $Z/(p^{el})\mathfrak{G}$ for some positive integer l. But the only irreducible $Z/(p^{el})\mathfrak{G}$-module is $Z/(p)$ on which $\mathfrak{G}$ acts trivially. Hence $\overline{M}$ has a composition series with factors $Z/(p)$. The sequence

$$0 \longrightarrow Z \xrightarrow{p} Z \longrightarrow Z/(p) \longrightarrow 0$$

shows that $[Z/(p)]=0$ in $G(Z\mathfrak{G})$, where $[Z/(p)]$ means the element of $G(Z\mathfrak{G})$ associated with $Z/(p)$, so that $[\overline{M}]=0$ in $G(Z\mathfrak{G})$. This implies that $[M]=\sum[M_i]$. For any ideal $\mathfrak{a}$ of $\mathfrak{o}_i$ we denote by $\mathfrak{a}_i^*$ the element $[\mathfrak{a}]-[\mathfrak{o}_i]$ of $G(Z\mathfrak{G})$. The map $\mathfrak{a}\to\mathfrak{a}_i^*$ defines a homomorphism of the ideal class group of $\mathfrak{o}_i$ to $G(Z\mathfrak{G})$, and from (2.2), any element x of $G(Z\mathfrak{G})$ may be written in the form

$$x=\textstyle\sum_i(l_i[\mathfrak{o}_i]+\mathfrak{a}_i^*) \qquad (l_i\in Z).$$

The uniqueness of this expression follows immediately from the following proposition.

Proposition 2.1. *For any exact sequence of regular $Z\mathfrak{G}$-modules*

$$0 \longrightarrow M' \longrightarrow M \xrightarrow{\psi} M'' \longrightarrow 0, \tag{2.3}$$

we have $C_i(\mathfrak{a})=C_i(\mathfrak{a}')\cdot C_i(\mathfrak{a}'')$, where $C_i(\mathfrak{a})$, $C_i(\mathfrak{a}')$ and $C_i(\mathfrak{a}'')$ are ideal class invariants of M_i, M_i' and M_i'', respectively.

Proof. The sequence (2.3) induces an exact sequence

$$0\to \mathrm{Hom}_{Z\mathfrak{G}}(\mathfrak{o}_i, M')\to \mathrm{Hom}_{Z\mathfrak{G}}(\mathfrak{o}_i, M)\to \mathrm{Hom}_{Z\mathfrak{G}}(\mathfrak{o}_i, M'')\to \mathrm{Ext}^1_{Z\mathfrak{G}}(\mathfrak{o}_i, M').$$

But $\mathrm{Hom}_{Z\mathfrak{G}}(\mathfrak{o}_i, M)$ is isomorphic to M_i by the map $f\to f(1)$. Hence we have an exact sequence

$$0\to M_i'\to M_i\to M_i''\to \mathrm{Ext}^1_{Z\mathfrak{G}}(\mathfrak{o}_i, M').$$

Since the order p^e of $\mathfrak{G}$ annihilates $\mathrm{Ext}^1_{Z\mathfrak{G}}(\mathfrak{o}_i, M')$ (Cartan and Eilenberg [2]), we see that

$$p^e M_i''\subseteq\psi(M_i)\subseteq M_i'', \tag{2.4}$$

where $\psi(M_i)$ is also a projective $\mathfrak{o}_i$-module whose $\mathfrak{o}_i$-rank is equal to that of M_i''. Thus by Invariant factor theorem ([3]), there exist elements $u_1, \ldots, u_{l_i}$ of M_i'' and ideals $\mathfrak{b}_1, \ldots, \mathfrak{b}_{l_i}$ of $\mathfrak{o}_i$ such that

$$M_i''=\mathfrak{o}_i u_1\oplus\cdots\oplus\mathfrak{o}_i u_{l_i-1}\oplus\mathfrak{a}'' u_{l_i}$$
$$\psi(M_i)=\mathfrak{b}_1 u_1\oplus\cdots\oplus\mathfrak{b}_{l_i-1}u_{l_i-1}\oplus\mathfrak{b}_{l_i}\mathfrak{a}'' u_{l_i}.$$

Then the inclusion (2.4) shows that each $\mathfrak{b}_k$ divides (p^e). But p is a power of the principal prime ideal $(1-\zeta_{d_i})$ of $\mathfrak{o}_i$, which implies that $\mathfrak{b}_k$ is also a principal ideal. Then $C_i(\mathfrak{b}_1\cdots\mathfrak{b}_{l_i}\mathfrak{a}'')=C_i(\mathfrak{a}'')$. Furthermore, M_i is isomorphic to the direct sum of M_i' and $\psi(M_i)$ since $\psi(M_i)$ is projective. Therefore $C_i(\mathfrak{a})=C_i(\mathfrak{a}')\cdot C_i(\mathfrak{b}_1\cdots\mathfrak{b}_{l_i}\mathfrak{a}'')$, which coincides with $C_i(\mathfrak{a}')\cdot C_i(\mathfrak{a}'')$. This completes the proof.

THEOREM 2.1. *If $\mathfrak{G}$ is an abelian p-group, $G(Z\mathfrak{G})$ is isomorphic to the direct sum of $C_0(\mathfrak{o})$ and $G(Q\mathfrak{G})$ as an additive group*

$$G(Z\mathfrak{G})\cong C_0(\mathfrak{o})\oplus G(Q\mathfrak{G}). \tag{2.5}$$

Proof. Since $\mathfrak{o}$ is the direct sum of the $\mathfrak{o}_i$, $C_0(\mathfrak{o})=\sum\oplus C_0(\mathfrak{o}_i)$ and each $C_0(\mathfrak{o}_i)$ is isomorphic to the ideal class group of $\mathfrak{o}_i$ (Rim [4]). Then the map $C_i(\mathfrak{a})\to\mathfrak{a}_i^*$ defines a homomorphism $\phi: C_0(\mathfrak{o})\to G(Z\mathfrak{G})$, where the action of $\mathfrak{G}$ on $\mathfrak{a}$ is given by setting $g\cdot\alpha=\pi_i(g)\alpha$, $g\in\mathfrak{G}$, $\alpha\in\mathfrak{a}$. On the other hand, $[K_1],\dots,[K_s]$ make a base for $G(Q\mathfrak{G})$. We define a linear map $\varphi: G(Q\mathfrak{G})\to G(Z\mathfrak{G})$ by $\varphi([K_i])=[\mathfrak{o}_i]$. Then we have an additive isomorphism $C_0(\mathfrak{o})\oplus G(Q\mathfrak{G})\to G(Z\mathfrak{G})$ by $(x, y)\to\phi(x)+\varphi(y)$ because the image $\phi(x)+\varphi(y)$ in $G(Z\mathfrak{G})$ is uniquely determined by Proposition 2.1. This proves Theorem 2.1.

§ 3. Ring structure

We will now study the multiplicative structure of $G(Z\mathfrak{G})$. In (2.5), Swan [6] showed that $\phi(C_0(\mathfrak{o}))^2=0$. Hence $G(Z\mathfrak{G})$ is a Z-algebra extension over an abelian kernel, and is determined by the action of $G(Q\mathfrak{G})$ to $C_0(\mathfrak{o})$ and the associated 2-cohomology class of $H^2(G(Q\mathfrak{G}), C_0(\mathfrak{o}))$.

In this section we denote by p^{e_h} the order of a cyclic factor $\mathfrak{G}_h$ of $\mathfrak{G}$. As in § 2, each $\pi_i(g_h)$ is of the form $\zeta_{d_i}^{i_h}$ for some integer i_h such that $0\leq i_h\leq e_0$, which satisfies $i_hp^{e_h}\equiv 0 \pmod{p^{d_i}}$. In general, given a t-tuple $(\xi_1,\dots,\xi_t)$ of integers which satisfy that $\xi_hp^{e_h}\equiv 0 \pmod{p^{d_i}}$ for each h, we may construct a regular $Z\mathfrak{G}$-module as follows. Let $\mathfrak{a}$ be an ideal of $Z[\zeta_{d_i}]$. We turn $\mathfrak{a}$ into a regular $Z\mathfrak{G}$-module by defining

$$g_h\cdot\alpha=\zeta_{d_i}^{\xi_h}\alpha,\quad \alpha\in\mathfrak{a}.$$

We denote this module by $(\mathfrak{a};\xi_1,\dots,\xi_t)$. In particular, for the t-tuple $(i_1,\dots,i_t)$, i_h being as above, we denote $(\mathfrak{a};i_1,\dots,i_t)$ by $\mathfrak{a}_i$. Then the element $\mathfrak{a}_i^*$ of $G(Z\mathfrak{G})$ can be written in the form $[\mathfrak{a}_i]-[\mathfrak{o}_i]$.

PROPOSITION 3.1. *For any ideal* $\mathfrak{a}$ *of* $Z[\zeta_{d_i}]$, $(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ *is reducible if and only if every* ξ_h *is divisible by* p.

Proof. $(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ is reducible if and only if $Q\otimes_Z(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ is reducible. Let $Q\otimes_Z(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ be reducible. Then this contains, as a direct summand, K_j for some j such that $d_j < d_i$ and each g_h acts on K_j as the multiplication of $\zeta_{d_i}^{\xi_h}$. This shows that every ξ_h is divisible by p. Conversely let every ξ_h be divisible by p and let $p^{d_i-d_j}$ be the highest power of p which divides every ξ_h. Set $\xi_h = \xi'_h \cdot p^{d_i-d_j}$. Then $Q\otimes_Z(Z[\zeta_{d_j}]\ :\ \xi'_1, \ldots, \xi'_t)$ is obviously a direct summand of $Q\otimes_Z(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$. This proves the proposition.

PROPOSITION 3.2. *Let* $\mathfrak{a}$ *be any ideal of* $Z[\zeta_{d_i}]$. *If* $(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ *is irreducible, there exist some* j *and an ideal* $\mathfrak{b}$ *of* $Z[\zeta_{d_j}]$ *such that* $d_j = d_i$ *and* $(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t) \cong \mathfrak{b}_j$ *as* $Z\mathfrak{G}$*-modules. Otherwise, there exist some* j *and an ideal* $\mathfrak{b}$ *of* $Z[\zeta_{d_j}]$ *such that* $d_j < d_i$ *and* $(\mathfrak{a}\ ;\ \zeta_1, \ldots, \xi_t) \cong \mathfrak{o}_j \oplus \cdots \oplus \mathfrak{o}_j \oplus \mathfrak{b}_j$ $(p^{d_i-d_j}$ *summands) as* $Z\mathfrak{G}$*-modules.*

Proof. Let $(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ be irreducible. Then this is annihilated by only one A_j, so that this can be regarded as an $\mathfrak{o}_j$-module as in § 2. By the irreducibility, $(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ is, then, isomorphic to some $\mathfrak{b}_j$. Hence the Z-rank of $\mathfrak{o}_j$ is equal to that of $\mathfrak{o}_i$, and we have $d_j = d_i$. This proves the first assertion.

Let $(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ be reducible. Then each ξ_h is divisible by p (Proposition 3.1.). Let $p^{d_i-d_{j'}}$ be the highest power of p which divides every ξ_h and let $\xi_h = \xi'_h \cdot p^{d_i-d_{j'}}$. Then each g_h acts on $(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t)$ as the multiplication of $\xi_{d_i}^{\xi_h} = \zeta_{d_{j'}}^{\xi'_h}$. Since $\mathfrak{a}$ is, as a $Z[\zeta_{d_{j'}}]$-module, finitely generated and projective, $\mathfrak{a}$ is isomorphic to the direct sum of $p^{d_i-d_{j'}}-1$ copies of $Z[\zeta_{d_{j'}}]$ and an ideal $\mathfrak{b}'$ of $Z[\zeta_{d_{j'}}]$. Then we have a $Z\mathfrak{G}$-isomorphism

$$(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t) \cong (Z[\zeta_{d_{j'}}]\ ;\ \xi'_1, \ldots, \xi'_t) \oplus \cdots \oplus (Z[\zeta_{d_{j'}}]\ ;\ \xi'_1, \ldots, \xi'_t) \oplus (\mathfrak{b}'\ ;\ \xi'_1, \ldots, \xi'_t), \tag{3.1}$$

where each summand is irreducible. Hence, there exist some j and ideals $\mathfrak{c}$ and $\mathfrak{d}$ such that $d_j = d_{j'}$, $(Z[\zeta_{d_{j'}}]\ :\ \xi'_1, \ldots, \xi'_t) \cong \mathfrak{c}_j$ and $(\mathfrak{b}'\ ;\ \xi'_1, \ldots, \xi'_t) \cong \mathfrak{d}_j$ (the first assertion). Setting $\mathfrak{b} = \mathfrak{c}^{p^{d_i-d_j}-1} \cdot \mathfrak{d}$, we have

$$(\mathfrak{a}\ ;\ \xi_1, \ldots, \xi_t) \cong \mathfrak{o}_j \oplus \cdots \oplus \mathfrak{o}_j \oplus \mathfrak{b}_j.$$

This proves the second assertion and completes the proof of the proposition.

COROLLARY 3.1. *If* $(Z[\xi_{d_i}]\ ;\ \xi_1,\ldots,\xi_t)$ *is irreducible, there exists some* j *such that* $d_j=d_i$ *and* $(Z[\zeta_{d_i}]\ ;\ \xi_1,\ldots,\xi_t)\cong\mathfrak{o}_j$. *Otherwise, there exists some* j *such that* $d_j<d_i$ *and* $(Z[\zeta_{d_j}]\ ;\ \xi_1,\ldots,\xi_t)\cong\mathfrak{o}_j\oplus\cdots\oplus\mathfrak{o}_j$ $(p^{d_i-d_j}$ *summands*$)$.

Proof. According to Artin [1] $(D/\Delta)^{1/2}$ is the ideal class invariant of $Z[\zeta_{d_i}]$ as a $Z[\zeta_{d_{j'}}]$-module, where D is the discriminant of $Z[\zeta_{d_i}]$ over $Z[\zeta_{d_{j'}}]$ and Δ is the discriminant of any equation defining the extension of $Q(\zeta_{d_i})$ over $Q(\zeta_{d_{j'}})$. But it is easily checked that $(D/\Delta)^{1/2}$ divides some power of p. Then $(D/\Delta)^{1/2}$ is a principal ideal. Hence, by Proposition 3.2, it is sufficient to prove that $\mathfrak{b}$ is a principal ideal. Let τ be the isomorphism $(Z[\zeta_{d_i}]\ ;\ \xi_1,\ldots,\xi_t)\cong\mathfrak{b}_j$. Since $Z[\zeta_{d_i}]$ is generated by 1, $\mathfrak{b}$ is generated by $\tau(1)$. This shows that $\mathfrak{b}$ is a principal ideal, which completes the proof.

PROPOSITION 3.3. *Let* $\mathfrak{a}$ *be any ideal of* $Z[\zeta_{d_i}]$ *and let* σ *be a Galois automorphism of* $Q(\zeta_{d_i})$. *If* $\zeta_{d_i}^{\sigma}=\zeta_{d_i}^{\nu}$, *then*

$$(\mathfrak{a}\ ;\ \xi_1,\ldots,\xi_t)\cong(\mathfrak{a}^{\sigma}\ ;\ \xi_1\nu,\ldots,\xi_t\nu).$$

Proof. This follows immediately from the comparison of actions of $\mathfrak{G}$ to the both sides.

LEMMA 3.1. *If* $d_i\geqq d_j$, *then for any ideal* $\mathfrak{a}$ *of* $Z[\zeta_{d_i}]$ *we have*

$$[\mathfrak{o}_j][\mathfrak{a}_i]=\sum_{\sigma_\nu\in G_{d_j}}[(\mathfrak{a}\ :\ i_1+j_1\nu p^{d_i-d_j},\ \ldots,\ i_t+j_t\nu p^{d_i-d_j})],$$

where G_{d_j} *denotes the Galois group of* $Q(\zeta_{d_j})$ *and* σ_ν *denotes an element of* G_{d_j} *such that* $\zeta_{d_j}^{\sigma_\nu}=\zeta_{d_j}^{\nu}$.

Proof. Let $\Phi_{d_j}(x)$ be the cyclotomic polynomial of index p^{d_j}. Then we have $\mathfrak{o}_j\cong Z[x]/\Phi_{d_j}(x)Z[x]$. This implies the isomorphism

$$\mathfrak{o}_j\otimes_Z\mathfrak{a}_i\cong\mathfrak{a}[x]/\Phi_{d_j}(x)\mathfrak{a}[x].$$

Let $M=\mathfrak{a}_i[x]/\Phi_{d_j}(x)\mathfrak{a}_i[x]$. $\mathfrak{G}$ operates on M by $g_hm=\zeta_{d_i}^{i_h}x^{j_h}\cdot m$, $m\in M$. The assumption $d_i\geqq d_j$ implies that $\Phi_{d_j}(x)$ factorizes into $\prod_{\sigma_\nu\in G_{d_j}}(x-\zeta_{d_j}^{\nu})$ in $\mathfrak{o}_i[x]$. Let $\sigma_{\nu_1},\ldots,\sigma_{\nu_l}$ be the elements of G_{d_j} and let $M_k=(x-\zeta_{d_j}^{\nu_1})\cdots(x-\zeta_{d_j}^{\nu_k})M$. Then we have a series of submodules of M

$$M\supseteq M_1\supseteq\cdots\supseteq M_l=0.$$

Each quotient M_{k-1}/M_k is $\mathfrak{a}_i[x]/(x-\zeta_{d_j}^{\nu_k})\mathfrak{a}_i[x]$, which is isomorphic to $\mathfrak{a}$ by the map $x \to \zeta_{d_j}^{\nu_k}$. But this map carries $\zeta_{d_i}^{i_h} x^{j_h}$ into $\zeta_{d_i}^{i_h}\zeta_{d_j}^{j_h\nu_k} = \zeta_{d_i}^{i_h + j_h\nu_k p^{d_i-d_j}}$. Then each M_{k-1}/M_k is, as a $Z\mathfrak{G}$-module, isomorphic to $(\mathfrak{a} : i_1 + j_1\nu_k p^{d_i-d_j}, \ldots, i_t + j_t\nu_k p^{d_i-d_j})$. Since M is composed from these modules by forming extensions, we conclude that

$$[M] = \sum_{\sigma_\nu \in G_{d_j}} [(\mathfrak{a} : i_1 + j_1\nu p^{d_i-d_j}, \ldots, i_t + j_t\nu p^{d_i-d_j})].$$

This proves the lemma.

Now we will prove that Z-algebra extension (2.5) splits.

THEOREM 3.1. *The linear map φ defined in the proof of Theorem 2.1 is a ring homomorphism. Hence the Z-algebra extension (2.5) splits.*

Proof. Take any two generators $[K_i]$ and $[K_j]$ of $G(Q\mathfrak{G})$. We may assume that $d_i \geqq d_j$. From Lemma 3.1, we have

$$[\mathfrak{o}_j][\mathfrak{o}_i] = \sum_{\sigma_\nu \in G_{d_j}} [(Z[\zeta_{d_i}] : i_1 + j_1\nu p^{d_i-d_j}, \ldots, i_t + j_t\nu p^{d_i-d_j})].$$

But each term of the right hand is equal to either $[\mathfrak{o}_k]$ for some k such that $d_k = d_i$ or a direct sum of $p^{d_i-d_{k'}}$ copies of $[\mathfrak{o}_{k'}]$ for some k' such that $d_{k'} < d_i$ (Corollary 3.1). Then we have

$$[\mathfrak{o}_j][\mathfrak{o}_i] = \sum_{\substack{k \\ d_k = d_i}} [\mathfrak{o}_k] + \sum_{\substack{k' \\ d_{k'} < d_i}} p^{d_i-d_{k'}}[\mathfrak{o}_{k'}].$$

This shows that φ is a ring homomorphism, and this completes the proof of Theorem 3.1.

LEMMA 3.2. *If $d_i \leqq d_j$, then for any ideal $\mathfrak{a}$ of $Z[\zeta_{d_i}]$*

$$[\mathfrak{o}_j][\mathfrak{a}_i] = \sum_{\sigma_\nu \in G_{d_i}} [(\tilde{\mathfrak{a}} : i_1 p^{d_j-d_i} + j_1\nu, \ldots, i_t p^{d_j-d_i} + j_t\nu)],$$

where $\tilde{\mathfrak{a}}$ denotes $\mathfrak{a}Z[\zeta_{d_j}]$.

Proof. Notice that if $d_i \leqq d_j$, the cyclotomic polynomial $\Phi_{d_j}(x)$ factorizes into $\prod_{\sigma_\nu \in G_{d_i}} (x^{p^{d_j-d_i}} - \zeta_{d_i}^{\nu})$ in $\mathfrak{o}_i[x]$ and $\zeta_{d_j}^{\nu}$ is a root of $x^{p^{d_j-d_i}} - \zeta_{d_i}^{\nu}$. Then the lemma is proved by the same method as the proof of Lemma 3.1.

Let $\mathfrak{a}$ be any ideal of $Z[\zeta_{d_i}]$ and $(\xi_1, \ldots, \xi_t)$ be any t-tuple of integers such that $\xi_h p^{e_h} \equiv 0 \pmod{p^{d_i}}$. We denote the element $[(\mathfrak{a} ; \xi_1, \ldots, \xi_t)] - [(Z[\zeta_{d_i}] ; \xi_1, \ldots, \xi_t)]$ by $(\mathfrak{a} ; \xi_1, \ldots, \xi_t)^*$. Then $(\mathfrak{a} ; \xi_1, \ldots, \xi_t)^*$ is obviously contained in $\phi(C_0(\mathfrak{o}))$.

THEOREM 3.2. *For any* $\mathfrak{a}_i^*$ *of* $\phi(C_0(\mathfrak{o}_i))$, *each generator* $[K_j]$ *of* $G(Q\mathfrak{G})$ *acts on* $\mathfrak{a}_i^*$ *as follows.*

$$[K_j]\mathfrak{a}_i^* = \begin{cases} \sum_{\sigma_\nu \in G_{d_j}} (\mathfrak{a} : i_1 + j_1 \nu p^{d_i - d_j}, \ldots, i_t + j_t \nu p^{d_i - d_j})^* & \text{if } d_i \geqq d_j. \\ \sum_{\sigma_\nu \in G_{d_i}} (\tilde{\mathfrak{a}} : i_1 p^{d_j - d_i} + j_1 \nu, \ldots, i_t p^{d_j - d_i} + j_t \nu)^* & \text{if } d_i \leqq d_j. \end{cases}$$

Proof. The action of $[K_j]$ on $\phi(C_0(\mathfrak{o}))$ is given by the multiplication of $\varphi([K_j]) = [\mathfrak{o}_j]$. Then this theorem follows immediately from preceding two lemmas.

§ 4. Example

Let $\mathfrak{G}$ be an abelian group of type (p, p^e), that is, $\mathfrak{G}$ be a direct product of cyclic groups $\mathfrak{G}_1 = (g_1)$ and $\mathfrak{G}_2 = (g_2)$ of order p and p^e, respectively. In this case we can describe more explicitly the action of $G(Q\mathfrak{G})$ to $(C_0(\mathfrak{o}))$. In this section we denote by ζ_i a primitive p^i-th root of 1 for any integer i such that $1 \leqq i \leqq e$.

Let $\mathfrak{a}$ be any ideal of $Z[\zeta_i]$ and let ν be any integer such that $0 \leqq \nu \leqq p-1$. We denote $(\mathfrak{a} : p^{i-1}\nu, 1)$ by $\mathfrak{a}_{i,\nu}$. Put $\mathfrak{o}_{i,\nu} = (Z[\zeta_i])_{i,\nu}$ and $K_{i,\nu} = Q \otimes_Z \mathfrak{o}_{i,\nu}$. Furthermore; for any ideal $\mathfrak{a}$ of $Z[\zeta_1]$ we denote $(\mathfrak{a} : 1, 0)$ by $\mathfrak{a}_0$. Put $\mathfrak{o}_0 = (Z[\zeta_1])_0$ and $K_0 = Q \otimes_Z \mathfrak{o}_0$. Then we see that

$$Q\mathfrak{G} = Q \oplus K_0 \oplus \textstyle\sum_{i=1}^{e} \sum_{\nu=0}^{p-1} K_{i,\nu}$$

and that

$$C_0(\mathfrak{o}) = C_0(\mathfrak{o}_0) \oplus \textstyle\sum_{i=1}^{e} \sum_{\nu=0}^{p-1} C_0(\mathfrak{o}_{i,\nu}).$$

1. $[Q]$ acts on $\phi(C_0(\mathfrak{o}))$ trivially.

2. The action of $[K_0]$ on $\phi(C_0(\mathfrak{o}_0))$.

For any element $\mathfrak{a}_0^*$ of $\phi(C_0(\mathfrak{o}_0))$ it follows immediately from Theorem 3.2 and Proposition 3.3 that

$$\begin{aligned}[K_0]\mathfrak{a}_0^* &= \sum_{\sigma_\mu \in G_1} (\mathfrak{a} ; 1+\mu, 0)^* \\ &= \sum_{\substack{\sigma_\mu \in G_1 \\ \mu \not\equiv -1 \pmod p}} (\mathfrak{a}^{\sigma_{1+\mu}^{-1}} ; 1, 0)^* + (\mathfrak{a} ; 0, 0)^* = \sum_{\substack{\sigma_\mu \in G_1 \\ \mu \not\equiv -1 \pmod p}} (\mathfrak{a}^{\sigma_{1+\mu}^{-1}})_0^*\end{aligned}$$

since $(\mathfrak{a} : 0, 0)^* = (Z : 0, 0)^* = 0$ by Proposition 3.2. On the other hand, $\sigma_{1+\mu}^{-1}$ such that $\mu \not\equiv -1 \pmod p$ ranges over all elements of G_1 but σ_1. Then

$\prod_{\substack{\sigma_\mu \in G_1 \\ \mu \not\equiv -1 \pmod p}} \mathfrak{a}^{\sigma_{1+\mu}^{-1}} = N_{1/0}(\mathfrak{a})\mathfrak{a}^{-1}$, where $N_{i/0}$ means the norm of $Z[\zeta_i]$ over Z. Since $N_{i/0}(\mathfrak{a})$ is a principal ideal, $(N_{i/0}(\mathfrak{a}))_0^* = 0$. Hence we conclude that

$$[K_0]\mathfrak{a}_0^* = -\mathfrak{a}_0^*.$$

3. The action of $[K_0]$ on $\phi(C_0(\mathfrak{o}_i))$.

It follows immediately from Theorem 3.2 that

$$[K_0]\mathfrak{a}_{i,\nu}^* = \sum_{\sigma_\mu \in G_1} (\mathfrak{a} : p^{i-1}(\nu + \mu),\ 1)^*,$$

where $\nu + \mu$ ranges over $0, 1, \ldots, \nu - 1, \nu + 1, \ldots, p - 1 \bmod p$. Hence,

$$[K_0]\mathfrak{a}_{i,\nu}^* = \sum_{\mu=0,\ \mu \neq \nu}^{p-1} \mathfrak{a}_{i,\mu}^*.$$

4. The action of $[K_{i,\nu}]$ on $\phi(C_0(\mathfrak{o}_0))$.

Let x_μ be an integer such that $\mu x_\mu \equiv 1 \pmod{p^i}$. Then Theorem 3.2 and Proposition 3.3 imply that

$$[K_{i,\nu}]\mathfrak{a}_0^* = \sum_{\sigma_\mu \in G_1} (\tilde{\mathfrak{a}}\ ;\ p^{i-1}(1 + \nu\mu), \mu)^* = \sum_{\sigma_\mu \in G_1} (\tilde{\mathfrak{a}}^{\sigma_\mu^{-1}}\ ;\ p^{i-1}(x_\mu + \nu),\ 1)^*.$$

But we can easily check that $x_\mu + \nu$ ranges over $0, 1, \ldots, p - 1 \bmod p$. Hence we have

$$[K_{i,\nu}]\mathfrak{a}_0^* = \sum_{\sigma_\mu \in G_1} (\tilde{\mathfrak{a}}^{\sigma_\mu})_{i,\nu+\mu}^*.$$

5. The action of $[K_{j,\nu}]$ on $\phi(C_0(\mathfrak{o}_{i,\nu'}))$.

The case $i > j$. Let y_μ be an integer such that $(1 + p^{i-j}\mu)y_\mu \equiv 1 \pmod{p^i}$. Then Theorem 3.2 and Proposition 3.3 imply that

$$\begin{aligned}[K_{j,\nu}]\mathfrak{a}_{i,\nu'}^* &= \sum_{\sigma_\mu \in G_j} (\mathfrak{a}\ ;\ p^{i-1}(\nu' + \nu\mu),\ 1 + p^{i-j}\mu)^* \\ &= \sum_{\sigma_\mu \in G_j} (\mathfrak{a}^{\sigma y_\mu}\ ;\ p^{i-1}(\nu' + \nu\mu),\ 1)^*\end{aligned}$$

because $y_\mu \equiv 1 \pmod p$. In general we denote by $G_{i/j}$ the Galois group of $Q(\zeta_i)$ over $Q(\zeta_j)$. Then $G_j = \bigcup_{\lambda=1}^{p-1} G_{j/1}\cdot\sigma_\lambda$ and $\nu' + \nu\mu \equiv \nu' + \nu\lambda \pmod p$ for any element σ_μ of $G_{j/1}\cdot\sigma_\lambda$. This shows that

$$[K_{j,\nu}]\mathfrak{a}_{i,\nu'}^* = \sum_{\lambda=1}^{p-1} \Big(\prod_{\sigma_\mu \in G_{j/1}\cdot\sigma_\lambda} \mathfrak{a}^{\sigma_{y_\mu}} \Big)_{i,\nu'+\nu\lambda}^*.$$

The case $i = j$. For each μ such that $\mu \not\equiv -1 \pmod p$, let x_μ be an integer such that $(1 + \mu)x_\mu \equiv 1 \pmod{p^i}$. Then Theorem 3.2 and Proposition 3.3 imply that

$$[K_{i,\nu}]\mathfrak{a}_{i,\nu'}^{*} = \sum_{\substack{\sigma_\mu\in G_i \\ \mu\not\equiv -1 \pmod p}} (\mathfrak{a}^{\sigma_{x_\mu}} \,;\, p^{i-1}(\nu'+\nu\mu)x_\mu,\ 1)^* + \sum_{\substack{\sigma_\mu\in G_i \\ \mu\equiv -1 \pmod p}} (\mathfrak{a} \,;\, p^{i-1}(\nu'+\nu\mu),\ 1+\mu)^*. \qquad (4.1)$$

In the first term of the right hand side, σ_{x_μ} ranges over $\bigcup_{\lambda=2}^{p-1} G_{i/1}\cdot\sigma_\lambda$ and $(\nu'+\nu\mu)x_\mu \equiv (\nu'-\nu)\lambda+\nu \pmod p$ for any σ_{x_u} of $G_{i/1}\cdot\sigma_\lambda$. Then the first term of (4.1) is equal to

$$\sum_{\lambda=2}^{p-1}\Big(\prod_{\sigma\in G_{i/1}\cdot\sigma_\lambda}\mathfrak{a}^{\sigma}\Big)_{i,(\nu'-\nu)\lambda+\nu}^{*} = \sum_{\lambda=2}^{p-1}(N_{i/1}(\mathfrak{a})^{\sigma_\lambda})_{i,(\nu'-\nu)\lambda+\nu}^{*}.$$

In particular, if $\nu'=\nu$, this is equal to $-(N_{i/1}(\mathfrak{a}))_{i,\nu}^{*}$. In the second term of (4.1), let p^h be the highest power of p which divides $1+\mu$ and set $1+\mu=\mu_h\cdot p^h$. Then (3.1) implies that

$$(\mathfrak{a} \,;\, p^{i-1}(\nu'+\nu\mu),\ 1+\mu)^* = (N_{i/i-h}(\mathfrak{a}) \,;\, p^{i-h-1}(\nu'-\nu),\ \mu_h)^*$$

since the ideal class of $\mathfrak{a}$ as a $Z[\zeta_{i-h}]$-module is the norm $N_{i/i-h}(\mathfrak{a})$ of $\mathfrak{a}$ from $Z[\zeta_i]$ to $Z[\zeta_{i-h}]$ ([1]). When σ_μ ranges over elements of G_i such that $1+\mu\equiv 0 \pmod{p^h}$ and $1+\mu\not\equiv 0 \pmod{p^{h+1}}$, $\sigma_{\mu_h}^{-1}$ obviously ranges over the elements of $G_{i-h}=\bigcup_{\lambda=1}^{p-1} G_{i-h/1}\cdot\sigma_\lambda$ and $(\nu'-\nu)\gamma\equiv(\nu'-\nu)\lambda \pmod p$ for any σ_γ of $G_{i-h/1}\cdot\sigma_\lambda$. Hence the second term of (4.1) is equal to

$$\sum_{h=1}^{i-1}\sum_{\lambda=1}^{p-1}\Big(\prod_{\sigma\in G_{i-h/1}\cdot\sigma_\lambda} N_{i/i-h}(\mathfrak{a})^{\sigma}\Big)_{i-h,(\nu'-\nu)\lambda}^{*} + (N_{i/1}(\mathfrak{a}) \,;\, \nu'-\nu,\ 0)^*$$
$$= \sum_{h=1}^{i-1}\sum_{\lambda=1}^{p-1}(N_{i/1}(\mathfrak{a})^{\sigma_\lambda})_{i-h,(\nu'-\nu)\lambda}^{*} + (N_{i/1}(\mathfrak{a}) \,;\, \nu'-\nu,\ 0)^*,$$

where if $\nu'\not\equiv\nu$, $(N_{i/1}(\mathfrak{a}) \,;\, \nu'-\nu,\ 0)^* = (N_{i/1}(\mathfrak{a})^{\sigma_{\nu'-\nu}^{-1}})_0^*$ and if $\nu'=\nu$, $(N_{i/1}(\mathfrak{a}) \,;\, \nu'-\nu,\ 0)^*=0$ and $\sum_{\lambda=1}^{p-1}(N_{i/1}(\mathfrak{a})^{\sigma_\lambda})_{i-h,(\nu'-\nu)\lambda}^{*} = (N_{i/0}(\mathfrak{a}))_{i-h,0}^{*}=0$ since $N_{i/0}(\mathfrak{a})$ is a principal ideal. The case $i<j$. From Theorem 3.2 we have

$$[K_{j,\nu}]\mathfrak{a}_{i,\nu'}^{*} = \sum_{\sigma_\mu\in G_i}(\tilde{\mathfrak{a}} \,;\, p^{j-1}(\nu'+\nu\mu),\ p^{j-i}+\mu)^*.$$

Let x_μ be an integer such that $(p^{j-i}+\mu)x_\mu\equiv 1 \pmod{p^j}$. Then $(\tilde{\mathfrak{a}} \,;\, p^{j-1}(\nu'+\nu\mu),\ p^{j-i}+\mu)^* = (\tilde{\mathfrak{a}}^{\sigma_{x_\mu}})_{j,\nu'x_\mu+\nu}^{*}$ by Proposition 3.3, σ_{x_μ} ranges over the elements of G_i, and $\nu'x_\mu+\nu\equiv\nu'\lambda+\nu \pmod p$ for any σ_{x_μ} of $G_{i/1}\cdot\sigma_\lambda$. This shows that

$$[K_{j,\nu}]\mathfrak{a}_{i,\nu'}^{*} = \sum_{\lambda=1}^{p-1}\Big(\prod_{\sigma\in G_{i/1}\cdot\sigma_\lambda}\tilde{\mathfrak{a}}^{\sigma}\Big)_{j,\nu'\lambda+\nu}^{*} = \sum_{\lambda=1}^{p-1}(\widetilde{N_{i/1}}(\mathfrak{a})^{\sigma_\lambda})_{j,\nu'\lambda+\nu}.$$

Summalizing, we have

PROPOSITION 4.1. *Let $\mathfrak{G}$ be an abelian group of type (p, p^e). Then $G(Q\mathfrak{G})$ acts on $\phi(C_0(\mathfrak{o}))$ as follows.*

1. $[Q]$ *acts trivially.*
2. $[K_0]\mathfrak{a}_0^* = -\mathfrak{a}_0^*$.
3. $[K_0]\mathfrak{a}_{i,\nu}^* = \sum_{\mu=0,\, \neq \nu}^{p-1} \mathfrak{a}_{i,\mu}^*$.
4. $[K_{i,\nu}]\mathfrak{a}_0^* = \sum_{\lambda=1}^{p-1} (\tilde{\mathfrak{a}}^{\sigma\lambda})_{i,\nu+\lambda}^*$.
5. $[K_{j,\nu}]\mathfrak{a}_{i,\nu'}^*$

$$= \begin{cases} \sum_{\lambda=1}^{p-1} \Big(\prod_{\sigma_\mu \in G_{i/1} \cdot \sigma\lambda} \mathfrak{a}^{\sigma y_\mu} \Big)_{i,\nu'+\nu\lambda}^*, \quad \textit{where } \sigma_{y_\mu} = \sigma_{1+p^i-j\mu}^{-1} & (i>j). \\ \begin{cases} \sum_{\lambda=2}^{\nu-1} (N_{i/1}(\mathfrak{a})^{\sigma\lambda})_{i,(\nu'-\nu)\lambda+\nu}^* + \sum_{h=1}^{i-1}\sum_{\lambda=1}^{p-1} (N_{i/1}(\mathfrak{a})^{\sigma\lambda})_{i-h,(\nu'-\nu)\lambda}^* + (N_{i/1}(\mathfrak{a})^{\sigma_{\nu'-\nu}^{-1}})_0^*, & (\nu' \neq \nu) \\ -(N_{i/1}(\mathfrak{a}))_{i,\nu}^*, \quad (\nu'=\nu) & \end{cases} & (i=j). \\ \sum_{\lambda=1}^{p-1} (\widetilde{N_{i/1}}(\mathfrak{a})^{\sigma\lambda})_{j,\nu'\lambda+\nu}^* & (i<j). \end{cases}$$

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