THE ORDER OF CERTAIN CLASSES OF FUNCTIONS DEFINED IN THE UNIT DISK

D. C. RUNG

1. Introduction[1]

Let D denote the open unit disk in the complex plane and let C be the boundary of D. If, for a given complex-valued function $f(z)$ defined in D, the existence of a subset M of C is known, with the linear measure of M equal to 2π, as well as an estimate on the growth of $|f(z)|$ on sequences in D which tends to a point of M, then such a result will be called a "statistical" result on order. This terminology is due to Lelong-Ferrand [3].

Such statistical-type results are known, for example, if the function $f(z)$ is the derivative of a univalent, holomorphic function (Seidel and Walsh [5], p. 141.); or if $f(z)$ is holomorphic in D and omits two values there (Rung [4], p. 330). Both of these results depend upon first estimating the order of a holomorphic function $g(z)$ for which

$$\iint_D |g(z)|^2 dxdy < \infty. \tag{1.0}$$

In sections 3, 4, and 5 of this paper we replace the function $|g(z)|$ in (1.0) by several arbitrary real-valued functions defined in D and obtain statistical type results for these functions.

We conclude, in Section 6, by presenting examples of functions exhibiting this behavior.

2. Terminology

For $z_0 \in D \cup C$ and $r>0$ set $D(z_0, r) = \{z \in D \mid |z - z_0| < r\}$. We proceed to introduce an outer measure on the plane. For $r \geq 0$ let $h(r)$ be a real valued, non-decreasing, continuous function with $h(0) = 0$, $h(r) > 0$ for $r>0$, and $h(\infty) > 1$.

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DEFINITION 1. *For a given set E in the plane and fixed $\rho>0$ let $\{H(z_j, r_j)\}$ denote any denumerable family of open disks in the plane with center z_j and radius r_j, $r_j<\rho$, which cover E. If A_ρ is the* $\inf\left\{\sum_{j=1}^{\infty} h(r_j) \,\middle|\, \bigcup_{j=1}^{\infty} H(z_j, r_j) \supset E,\ r_j<\rho\right\}$ *define the h-measure of E to be* $h^*(E) = \lim_{\rho\to 0} A_\rho$.

Remark 1. In the case $h(r) = r^k$, $0<k<2$, this defines on the plane the usual k-dimensional outer measure.

3. Order of functions summable on D

The following results depend upon a theorem of the author [4, p. 324], which is closely related to a result of Lelong-Ferrand [3, pp. 20-23]. For completeness we state this theorem without proof.

THEOREM A. *Let $U(z)$ be a real-valued, non-negative, measurable function defined in D such that,*

$$\iint_D U(z)\,dx\,dy < \infty, \qquad z = x+iy.$$

Then

$$\lim_{r\to 0}\left[\frac{1}{h(r)}\iint_{D(e^{i\theta}, r)} U(z)\,dx\,dy\right] = 0,$$

except for at most a set of $e^{i\theta}$ of h-measure 0.

Remark 2. All integrals are to be considered as Lebesgue integrals.

In the following theorem an estimate is obtained on the order of such summable $U(z)$ on certain sequences in D.

THEOREM 1. *Let $U(z)$ satisfy the hypotheses of Theorem A. Then for every point of C, except possibly for a subset S of C of h-measure* 0, *the following behavior occurs. Let $\{z_n\}$ be any sequence in D tending to a point $e^{i\theta}$ of C not in S. For any fixed t, $0<t<1$, there exists a sequence of measurable sets $\{M_n(t)\}$ such that*

i) $M_n(t) \subset D(z_n, (1-|z_n|)t)$;

ii) $M_n(t)$ has positive two dimensional Lebesgue measure;

iii) if $\{\zeta_n\}$ is a sequence with

$$\zeta_n \in M_n(t), \quad n = 1, 2, \ldots,$$

$$\lim_{n\to\infty} \frac{U(\zeta_n)(1-|\zeta_n|)^2}{h(|\zeta_n - e^{i\theta}|)} = 0,$$

Remark 3. It is obvious that $\zeta_n \in M_n(t)$ implies $\zeta_n \to e^{i\theta}$ as $n \to \infty$.

Remark 4. In this theorem, and in the sequel, we assume $h(r)$ also satisfies

$$h(\alpha r) \leq K_\alpha h(r), \tag{3.0}$$

$0 \leq r \leq \infty$, $\alpha > 0$, and K_α is a positive constant depending only on α. This property is not essential for this group of theorems and the necessary changes, if (3.0) is not assumed, will be obvious. For example, $h(r) = r^k$ satisfies this property.

Proof. Theorem A yields

$$\lim_{r\to 0}\left[\frac{1}{h(r)}\iint_{D(e^{i\theta},\, r)} U(z)\,dx\,dy\right] = 0, \tag{3.1}$$

except possibly for a subset of C of h-measure 0. Let $e^{i\theta}$ be a point at which (3.1) holds and suppose $\{z_n\}$ is any sequence in D tending to $e^{i\theta}$. Further choose an arbitrary t, $0 < t < 1$, which remains fixed during the course of the proof.

For $\zeta \in D(z_n, (1-|z_n|)t)$, an easy calculation gives

$$(1-t)|z_n - e^{i\theta}| < |\zeta - e^{i\theta}| < 2|z_n - e^{i\theta}|, \tag{3.2}$$

and

$$(1-t)(1-|z_n|) < 1 - |\zeta| < (1+t)(1-|z_n|). \tag{3.3}$$

For the remainder of the proof set

$$D(z_n, (1-|z_n|)t) \equiv D_n, \quad n = 1, 2, \ldots .$$

Since $U(z) \geq 0$, the right side of (3.2) gives

$$\iint_{D_n} U(z)\,dx\,dy \leq \iint_{D(e^{i\theta},\, 2|z_n - e^{i\theta}|)} U(z)\,dx\,dy. \tag{3.4}$$

Theorem A, together with (3.0) and (3.4), enable us to conclude

$$\lim_{n\to\infty}\left[\frac{1}{h(|z_n - e^{i\theta}|)}\iint_{D_n} U(z)\,dx\,dy\right] = 0. \tag{3.5}$$

The existence of the sets $M_n(t)$ is now demonstrated. Fix a positive integer n and for this value of n let H denote the set of all points $\xi = u + iv$ contained in D_n for which

$$U(\xi)\pi(1-|z_n|)^2t^2 > \iint_{D_n} U(z)dxdy. \tag{3.6}$$

Since $U(z)$ is a measurable function H is a measurable set. Further if the measure of H were equal to $\pi(1-|z_n|)^2t^2$ integrating both sides of (3.6) over H would give

$$\iint_H U(\xi)dudv > \iint_{D_n} U(z)dxdy = \iint_H U(z)dxdy$$

which is impossible. Hence the measure of H is less than $\pi(1-|z_n|)^2t^2$. Setting $M_n(t)$ equal to the complement of H relative to D_n we have, for $\zeta_n \in M_n(t)$, $n = 1, 2, \ldots$

$$U(\zeta_n)\pi(1-|z_n|)^2t^2 \leq \iint_{D_n} U(z)dxdy. \tag{3.7}$$

We remark that the sequence of sets $\{M_n(t)\}$ depend upon the function $U(z)$, the sequence $\{z_n\}$, and the value t. In the sequel, if we introduce a function $U(z)$, a sequence $\{z_n\}$ and a value t, $0<t<1$, $\{M_n(t)\}$ will always represent the above sequence of sets.

The proof is nearly complete since combining (3.5) and (3.7) gives

$$\lim_{n\to\infty} \frac{U(\zeta_n)(1-|z_n|)^2}{h(|z_n - e^{i\theta}|)} = 0. \tag{3.8}$$

However (3.8) may be revised to give

$$\lim_{n\to\infty} \frac{U(\zeta_n)(1-|\zeta_n|)^2}{h(|\zeta_n - e^{i\theta}|)} = 0,$$

if we refer to the right side of both (3.2) and (3.3) together with (3.0). Since this limit holds at every point $e^{i\theta}$ at which (3.1) is valid, the proof of Theorem 1 is complete.

If we restrict the sequence $\{z_n\}$ to approach $e^{i\theta}$ within some Stolz domain Theorem 1 can be reformulated. To this end let $S(e^{i\theta}, \alpha)$, $0<\alpha<\frac{\pi}{2}$, denote the symmetric Stolz domain at $e^{i\theta}$ of opening 2α.

COROLLARY 1. *Let $U(z)$ satisfy the hypotheses of Theorem* 1, *and let $\{z_n\}$ be a sequence in D tending to a point $e^{i\theta}$ but with $z_n \in S(e^{i\theta}, \alpha)$, $n = 1, 2, \ldots,$ for some $0<\alpha<\frac{\pi}{2}$. Then for any fixed $0<t<1$, and any sequence $\{\zeta_n\}$, $\zeta_n \in M_n(t)$,*

$n = 1, 2, \ldots,$

$$\lim_{n\to\infty} \frac{U(\zeta_n)(1-|\zeta_n|)^2}{h(1-|\zeta_n|)} = 0 \tag{3.9}$$

except for at most a subset of C of h-measure 0.

Proof. An argument involving elementary geometry shows that, for any sufficiently small $\varepsilon > 0$, there exists a positive integer N_0, depending on ε, such that for $n > N_0$

$$D_n \subset S(e^{i\theta}, \alpha + \arcsin (t \cos \alpha) + \varepsilon). \tag{3.10}$$

This implies that for $n > N_0$ all ζ_n lie in the above Stolz domain, and, as is well known, then satisfy

$$1 \leq \frac{|\zeta_n - e^{i\theta}|}{1-|\zeta_n|} \leq C,$$

for suitable constant C. Again referring to (3.0) as well as to the monotonicity of $h(r)$ we see that Theorem 1 can be restated to give Corollary 1.

Remark 5. I am indebted to Professor W. Seidel for indicating (3.10).

Remark 6. Setting $h(r) = r$, which defines on C the ordinary outer linear measure, the conclusion of Corollary 1 now reads

$$\lim_{n\to\infty} U(\zeta_n)(1-|\zeta_n|) = 0,$$

and the exceptional subset of C has linear measure 0.

The question arises as to whether any estimate can be obtained for such summable $U(z)$ on the original sequence $\{z_n\}$. Several sufficient conditions are discussed in 4 and 5.

4. Sequentially subharmonic functions

If we return to the proof of Theorem 1 we see that (3.7) relates the values of $U(z)$ at certain points in D_n to the value of the integral of $U(z)$ over D_n. With this in mind we give

DEFINITION 2. *Let $U(z)$ be a real valued, non-negative measurable function defined in D. We say $U(z)$ is sequentially subharmonic in D if, for each sequence $\{z_n\}$ in D and for at least one value of t, $0<t<1$, there exists a positive constant K (which is a function of both the sequence and the value t) such that*

$$U(z_n)\pi(1-|z_n|)^2t^2K\leq\iint\limits_{D(z_n,\,(1-|z_n|)t)}U(z)dxdy, \tag{4.0}$$

for $n=1, 2, \ldots$

Remark 7. If $G(z)$ is a positive subharmonic function in D then it is sequentially subharmonic in D since K can be chosen identically 1 for each sequence and each $0<t<1$.

The conclusion of Theorem 1 can be revised so that the sequence $\{\zeta_n\}$ is replaced by the original sequence $\{z_n\}$ if (3.7) is replaced by (4.0). This gives

THEOREM 2. *Let $U(z)$ be sequentially subharmonic in D and suppose also*

$$\iint\limits_D U(z)dxdy<\infty.$$

If $\{z_n\}$ is a sequence in D which tends to a point $e^{i\theta}$ we have

$$\lim_{n\to\infty}\frac{U(z_n)(1-|z_n|)^2}{h(|z_n-e^{i\theta}|)}=0$$

except possibly for a set of $e^{i\theta}$ of h-measure 0.

COROLLARY 2. *Let the hypotheses of Theorem 2 be satisfied and in addition suppose the sequence $\{z_n\}$ approaches $e^{i\theta}$ within some Stolz domain at $e^{i\theta}$. Then*

$$\lim_{n\to\infty}\frac{U(z_n)(1-|z_n|)^2}{h(1-|z_n|)}=0$$

except possibly for a set of $e^{i\theta}$ of h-measure 0.

Remark 8. If $V(z)$ is a positive subharmonic function in D then Theorem 2 applies to the function $U(z)=V^p(z)$, $p\geq 1$, since $V^p(z)$ is also subharmonic. In the case $0<p<1$, $V^p(z)$ is still subharmonic provided $\log V(z)$ is. This generalizes a result of Gehring [1, p. 77].

5. Complex-valued functions summable over D

Let $\phi(z)$ denote a complex-valued function defined in D. For any two points a, b of D set $\rho(a, b)$ equal to the non-euclidean (hyperbolic) distance between a and b, i.e. $\rho(a, b)=1/2\log\dfrac{|1-a\bar{b}|+|a-b|}{|1-a\bar{b}|-|a-b|}$.

DEFINITION 3. *Let $\{z_n\}$ be a sequence in D which tends to a point of C. A complex-valued function $\phi(z)$ defined in D is said to be close along $\{z_n\}$ if there* *exists a pair of positive real numbers* (δ, M) *such that if* $\rho(z, z_n) < \delta$ *then* $|\phi(z) - \phi(z_n)| < M$, *for each* $n = 1, 2, \ldots$. *If* $\phi(z)$ *is close along* $\{z_n\}$ *for all* $\{z_n\}$ *such that* $\lim_{n\to\infty} z_n = e^{i\theta}$, $e^{i\theta} \in C$ (*respectively* $e^{i\theta} \in B$, B *a subset of* C *with* $h^*(B) = h^*(C)$) *then we say* $\phi(z)$ *is close along all sequences* (*respectively close along almost all sequences in the h-measure*). *When* $h(r) = r$ *we omit the phrase "in the h-measure."*

For $z_0 \in D$ and $\delta > 0$ set $N(z_0, \delta) = \{z \mid \rho(z_0, z) < \delta\}$. This set of points is known to be an open Euclidean disk. Thus let z' denote the center and $(1 - |z'|)t'$ the radius (both in the Euclidean geometry) of $N(z, \delta)$. We now indicate a connection between the non-Euclidean radius δ and the value t'.

LEMMA 1. *Given the non-Euclidean disk* $N(z, \delta)$ *and its corresponding Euclidean representation* $D(z', (1 - |z'|)t')$ *then*

$$t' = \frac{K(1 + |z|)}{1 + K^2|z|}.$$

Hence as $|z| \to 1$, $t' \to \dfrac{2K}{1+K^2}$. $K = \dfrac{e^{2\delta} - 1}{e^{2\delta} + 1}$.

Proof. If $z = re^{i\theta}$, then the point z_2 on the boundary of $N(z, \delta)$ closest to z in the Euclidean sense, is of the form $z_1 = r_1 e^{i\theta}$, $r_1 > r$; the point z_2 furthest from z is $z_2 = r_2 e^{i\theta}$, $r_2 < r$. The point z' is also on the radius to $e^{i\theta}$ thus $z' = r'e^{i\theta}$, $r_2 < r' < r < r_1$. If we put $|z_1 - z| = (1 - |z|)t_1$ and $|z - z_2| = (1 - |z|)t_2$, an elementary calculation gives $t_1 = \dfrac{K(1 + |z|)}{1 + K|z|}$, and $t_2 = \dfrac{K(1 + |z|)}{1 - K|z|}$. Thus the Euclidean radius of $N(z, \delta)$ is $\dfrac{|z_1 - z_2|}{2} = (1 - |z|)\left(\dfrac{t_1 + t_2}{2}\right)$, and the Euclidean center $z' = \dfrac{z_2 + z_1}{2}$. Finally to find t' note that $(1 - |z'|)t' = (1 - |z|)\dfrac{(t_1 + t_2)}{2}$ and a straightforward calculation gives the value t' in the Lemma.

This enables us to view a sequence of non-Euclidean disks $N(z_n, \delta)$, with $\lim_{n\to\infty} |z_n| = 1$, as a sequence of Euclidean disks $D(z'_n(1 - |z'_n|)t'_n)$ where for $n > N_0$, $0 < \underline{t} \le t'_n \le \overline{t} < 1$, with N_0, $\underline{t}$ and $\overline{t}$ determined by Lemma 1.

THEOREM 3. *Let* $\phi(z)$ *be a complex-valued measurable function defined in* D *which is close along almost all sequences in the h-measure and also, with* $z = x + iy$,

$$\iint_D |\phi(z)|\,dx\,dy < \infty.$$

Then, if $z_n \to e^{i\theta}$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{|\phi(z_n)|(1-|z_n|)^2}{h(|z_n - e^{i\theta}|)} = 0,$$

except for at most a set of $e^{i\theta}$ *of h-measure* 0.

Proof. Set $U(z) = |\phi(z)|$ and let S denote the exceptional set of Theorem 1 for this $U(z)$. If B is the set of all $e^{i\theta}$ relative to which $\phi(z)$ is close along sequences, $h^*(B) = h^*(C)$; hence setting $S_1 = C - B$, $h^*(S_1) = 0$, and $h^*(S_2) = 0$ where $S_2 = S \cup S_1$.

Let $\{z_n\}$ be any sequence in D tending to a point $e^{i\theta}$ not in S_2. Since $\phi(z)$ is close along $\{z_n\}$ there exists a pair of positive real numbers (δ, M) such that $\rho(z, z_n) < \delta$ implies $|\phi(z) - \phi(z_n)| < M$, $n = 1, 2, \ldots$.

Referring to Lemma 1 we consider the sequence $\{N(z_n, \delta)\}$ as a sequence of Euclidean disks $\{D(z'_n, (1-|z'_n|)t'_n)\}$. Since $t_n > \underline{t}$, $n > N_0$, we apply Theorem 1 with $t = \underline{t}$.

Thus for any sequence
$\{\zeta_n\}$, $\zeta_n \in M_n(t)$, $n = 1, 2, \ldots$,

$$\lim_{n\to\infty} \frac{|\phi(\zeta_n)|(1-|\zeta_n|)^2}{h(|\zeta_n - e^{i\theta}|)} = 0. \tag{5.0}$$

Since $M_n(t) \subset D(z'_n, (1-|z'_n|)t')$, $n = 1, 2, \ldots$, application of Lemma 1 gives $M_n(t) \subset N(z_n, \delta)$, $n > N_0$. Since $\phi(z)$ is close along $\{z_n\}$.

$$|\phi(z_n)| \leq M + |\phi(\zeta_n)|, \quad n > N_0 \tag{5.1}$$

Combining (3.0), (3.2), (3.3) and (5.1)

$$0 \leq \frac{|\phi(z_n)|(1-|z_n|)^2}{h(|z_n - e^{i\theta}|)} \leq \frac{M(1-|z_n|)^2}{h(|z_n - e^{i\theta}|)} + \frac{|\phi(\zeta_n)|(1-|\zeta_n|)^2}{h(|\zeta_n - e^{i\theta}|)}. \tag{5.2}$$

Now under the assumption that the h-measure of C is positive (otherwise the statement of the theorem is vacuous)

$$\lim_{z\to e^{i\theta}} \frac{(1-|z|)^2}{h(|z - e^{i\theta}|)} = 0, \tag{5.3}$$

for all $e^{i\theta} \in C$. This follows by setting $U(z) \equiv 1$ in Theorem 2 and observing that if (5.3) holds for some $e^{i\theta}$, it holds for all $e^{i\theta}$.

The proof of Theorem 3 is completed by combining (5.0) and (5.3) with

(5.2).

As Corollary 1 follows from Theorem 1 so also does the following Corollary follow from Theorem 3.

COROLLARY 3. *Under the hypotheses of Theorem* 3, *and supposing also that* $\{z_n\}$ *tends to* $e^{i\theta}$ *within some Stolz domain at* $e^{i\theta}$,

$$\lim_{n\to\infty}\frac{|\phi(z_n)|(1-|z_n|)^2}{h(1-|z_n|)}=0$$

except possibly for a set of $e^{i\theta}$ of h-measure 0.

6. Examples to Theorem 1

The following examples to Theorem 1 are constructed with $h(r)=r$.

EXAMPLE 1. *Given an arbitrary countable subset* P *of* C *there exists a function* $U_1(z)$ *satisfying the hypotheses of Theorem* 1 *for which the exceptional set* S (*of Theorem* 1) *contains* P.

Let $\{e^{ip_j}\}$ be some enumeration of the points of P. For each point $e^{ip_j}\in P$ we will consider a sequence $\{z_n^{(j)}\}$ tending radially to e^{ip_j}; a sequence of disks $D(z_n^{(j)},\ (1-|z_n^{(j)}|)t_j)$; and a function $U_1(z)$ which takes the value $\frac{1}{(1-|z|)^{4/3}}$ in each disk and which is summable over D. If $\zeta_n^{(j)}\in D(z_n^{(j)},\ (1-|z_n^{(j)}|)t_j)$, $n=1$, 2, . . .

$$\lim_{n\to\infty}U_1(\zeta_n^{(j)})(1-|\zeta_n^{(j)}|)=\infty.$$

Referring to Corollary 1 of Theorem 1 we see that the sets $M_n(t_j)$ do not exist for the sequece $\{z_n^{(j)}\}$; and P is therefore a subset of the exceptional set S. We proceed to the details.

For a fixed $e^{ip_j}\in P$ consider a Stolz domain $S(e^{ip_j},\ \alpha_j)$ of opening $2\alpha_j$, $0<\alpha_j<\frac{\pi}{2}$, where $\{\alpha_j\}$ is any decreasing sequence of positive numbers satisfying

$$\sum_{j=1}^{\infty}\alpha_j<\infty. \tag{6.0}$$

Let $\{z_n^{(j)}\}$ be a sequence approaching e^{ip_j} radially with $|z_n^{(j)}|=1-\frac{1}{n^2}$, $n=1$, 2, ... If $t_j=\sin\alpha_j$ the disks $D(z_n^{(j)},\ (1-|z_n^{(j)}|)t_j)$, $n=1, 2, \ldots$, are easily seen to lie inside $S(e^{ip_j},\ \alpha_j)$ (see figure 1). Hence-forth we use $D_n^{(j)}$ to denote these disks.

New define

$$U_1(z)=\begin{cases}\dfrac{1}{(1-|z|)^{4/3}}, & z\in\bigcup_{j=1}^{\infty}\bigcup_{n=1}^{\infty}D_n^{(j)};\\ 0, & z\in D-\bigcup_{j=1}^{\infty}\bigcup_{n=1}^{\infty}D_n^{(j)}.\end{cases}$$

In order to compute the double integral of $U(z)$ over D some estimates of the integral of $U(z)$ over each $D_n^{(j)}$ are required.

Let $\theta_n^{(j)}$ be that positive angle formed by the radius to e^{ip_j} and the line segment from the origin tangent to the circumference of $D_n^{(j)}$ (See figure 1.).

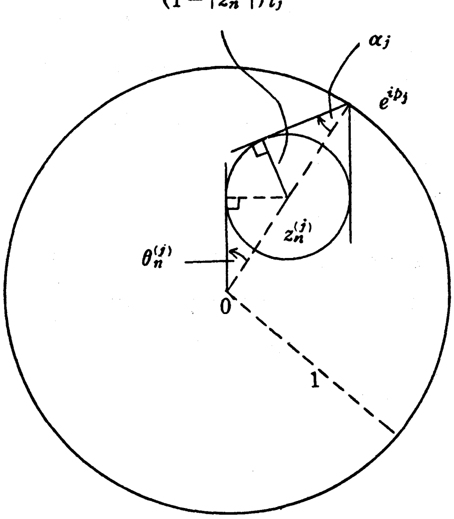

FIGURE 1

Since

(6.1) $$\sin\theta_n^{(j)}=\frac{(1-|z_n^{(j)}|)t_j}{|z_n^{(j)}|}=\frac{t_j}{n^2-1}.$$

and

(6.2) $$\theta_n^{(j)}<\frac{\pi}{2}\sin\theta_n^{(j)},$$

then for $n\geq 2$, and all j,

(6.3) $$\theta_n^{(j)}<\frac{\pi}{2}\,\frac{t_j}{n^2-1}<\frac{\pi t_j}{n^2}.$$

Since the quadrilateral

$$Q_n^{(j)}=\left\{re^{i\theta}\,|\,p_j-\theta_n^{(j)}\leq\theta\leq p_j+\theta_n^{(j)},\ 1-\left(\frac{1+t_j}{n^2}\right)\leq r\leq 1-\left[\frac{1-t_j}{n^2}\right]\right\}$$

contains $D_n^{(j)}$ for each value n and j, setting $z=re^{i\theta}$

$$(6.5)\qquad \iint_{D_n^{(j)}} U_1(z)\,r\,dr\,d\theta = \iint_{D_n^{(j)}} \frac{1}{(1-r)^{4/3}}\,r\,dr\,d\theta \le \iint_{Q_n^{(j)}} \frac{r}{(1-r)^{4/3}}\,dr\,d\theta < \frac{6\,\theta_n^{(j)} n^{2/3}}{(1-t_1)^{1/3}}.$$

By (6.3)

$$(6.6)\qquad \frac{\theta_n^{(j)} n^{2/3}}{(1-t_1)^{1/3}} < \frac{\pi t_j}{n^{4/3}(1-t_1)^{1/3}}.$$

Thus (6.5) and (6.6) combine to give

$$\iint_D U_1(z)\,r\,dr\,d\theta \le \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\iint_{D_n^{(j)}} U_1(z)\,r\,dr\,d\theta < \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{6\,\pi t_j}{(1-t_1)^{1/3} n^{4/3}} = \frac{6\,\pi}{(1-t_1)^{1/3}}\sum_{j=1}^{\infty} t_j \sum_{n=1}^{\infty}\frac{1}{n^{4/3}}.$$

That this expression is finite follows from (6.0); hence $U(z)$ is summable over D.

To conclude, we see that for any fixed value j, and any sequence $\{\zeta_n^{(j)}\}$, $\zeta_n^{(j)} \in D_n^{(j)}$, $n=1, 2, \dots$, $U_1(\zeta_j)(1-|\zeta_j|) \to \infty$ as $n\to\infty$. Thus the sets $M_n^{(t)}$ do not exist for the sequence $\{z_n^{(j)}\}$ and any $t \le t_j$. Since this behavior is true for all values j the exceptional set for $U_1(z)$ contains the set of points e^{ip_j}, $j=1, 2, \dots$.

Our next example concerns the behavior of a function $U_2(z)$ whose integral over D diverges in some specified manner but for each $e^{i\theta}\in C$, there is a sequence tending to $e^{i\theta}$ and a value t, $0<t<1$, for which the sets $M_n^{(t)}$ fail to exist.

For $0\le r<1$ let $\Psi(r)$ denote any real-valued function such that

(6.7)
I) $\Psi(r)$ is a non-decreasing function of r, $0\le r<1$;
II) $\Psi(0)=0$, $\lim_{r\to 1}\Psi(r)=\infty$.

Example 2. *Let* $\Psi(r)$ *be an arbitrary function satisfying* (6.7). *Then there* *exists a real-valued, non-negative, measurable function* $U_2(z)$ *defined in D with the property that the double integral of* $U_2(z)$ *over D is infinite but, setting* $z = re^{i\theta}$,

$$\int_0^{\rho}\int_0^{2\pi} U_2(z) r d\theta dr < \Psi(\rho)$$

for $0 \leq \rho < 1$. *Further for any* $\theta \in [0, 2\pi]$ *there is a sequence* $\{z_j\}$ *in D which tends to* $e^{i\theta}$ *such that, if* $\zeta_j \in D(z_j, (1-|z_j|)1/8)$, *then*

$$\lim_{j\to\infty} U_2(\zeta_j)(1-|\zeta_j|) = \infty.$$

Define a sequence of concentric, disjoint rings in D as follows: let a sequence of positive numbers $\{n_j\}$ be chosen so that

$$n_{j+1} \geq 2 n_j, \tag{6.8}$$

and

$$2\pi j^2 \log 2 \leq \Psi\left(1 - \frac{1}{n_j}\right), \quad j = 1, 2, \ldots. \tag{6.9}$$

That such a sequence exists follows from the monotonicity of $\Psi(r)$. Next let R_j be the ring $\left\{z \in D \mid 1 - \frac{1}{n_j} \leq |z| < 1 - \frac{1}{2n_j}\right\}$ $j = 1, 2, \ldots$. Note that these rings are disjoint by (6.8). Lastly select any function $\Psi^*(r)$ satisfying (6.7) and in addition

$$1 \leq \Psi^*\left(1 - \frac{1}{2n_j}\right) \leq j \qquad j = 1, 2, \ldots. \tag{6.10}$$

The desired function is

$$U_2(z) = \begin{cases} \dfrac{\Psi^*(|z|)}{1-|z|}, & z \in \bigcup_{j=1}^{\infty} R_j; \\ 0, & z \in D - \bigcup_{j=1}^{\infty} R_j. \end{cases}$$

To demonstrate that the integral of $U_2(z)$ over D diverges in the proper fashion fix a value r_0, $0 < r_0 < 1$, and let j_0 be chosen so that

$$1 - \frac{1}{n_{j_0}} \leq r_0 < 1 - \frac{1}{n_{j_0+1}}. \tag{6.11}$$

Since $U_2(z)$ vanishes except on the ring R_j, (6.9), (6.10) and (6.11) imply

$$\int_0^{r_0}\int_0^{2\pi} U_2(z) r d\theta dr \leq \sum_{j=1}^{j_0} \iint_{R_j} \frac{\Psi^*(r) r d\theta dr}{1-|z|}$$

$$< 2\pi j_0^2 \log 2$$
$$< \Psi(r_0).$$

A short calculation yields that the integral of $U_2(z)$ over D is infinite.

Finally to exhibit sequences which have the desired properties consider, for any $0 \leq \theta < 2\pi$, the sequence $z_j = \left(1 - \frac{3}{4}\frac{1}{n_j}\right)e^{i\theta}$, $j = 1, 2, \ldots$. The width of the ring R_j is $\frac{1}{2n_j}$, while the diameter of $D(z_j, (1-|z_j|)1/8)$ is $\frac{3}{16}\frac{1}{n_j}$; and since the point z_j is equidistant from the boundary circles of R_j

$$D(z_j, (1-|z_j|)1/8) \subset R_j, \quad j = 1, 2, \ldots.$$

From the definition of $U_2(z)$ and $\Psi^*(|z|)$ if $\zeta_j \in D(z_j, (1-|z_j|)1/8)$, $j = 1, 2, \ldots$, then

$$\lim_{j\to\infty} U_2(\zeta_j)(1-|\zeta_j|) = \infty.$$

The last example indicates that the rate of growth demonstrated in Theorem 1 cannot be improved.

EXAMPLE 3. *Let $\Psi(r)$ be any function satisfying (6.7). Then there exists a real valued, nonnegative, measurable function $U_3(z)$ defined in D for which*

$$\iint_D U_3(z)\,dx\,dy < \infty;$$

further for any $0 \leq \theta < 2\pi$ there is a sequence $\{z_j\}$ in D tending radially to $e^{i\theta}$ with the property that if $\zeta_j \in D(z_j, (1-|z_j|)1/8)$, $j = 1, 2, \ldots$, then

$$\lim_{j\to\infty} U_3(\zeta_j)(1-|\zeta_j|)\Psi(|\zeta_j|) = \infty.$$

Let $\Psi^*(r)$ be any function satisfying (6.7) and such that as $r \to 1$, $\Psi(r)/\Psi^*(r) \to \infty$. As before, define a sequence of concentric, disjoint rings R_j in D by first specifying a sequence of positive integers $\{n_j\}$ with

$$\text{(6.12)} \qquad \text{I) } \Psi^*\left(1 - \frac{1}{n_j}\right) \geq j^2; \qquad \text{II) } n_{j+1} \geq 2n_j$$

for $j = 1, 2, \ldots$;

then set $R_j = \left\{z \mid 1 - \frac{1}{n_j} \leq |z| < 1 - \frac{1}{2n_j}\right\}$, $j = 1, 2, \ldots$. Define

$$U_3(z) = \begin{cases} \dfrac{1}{(1-|z|)\Psi^*(|z|)}, & z \in \bigcup_{j=1}^{\infty} R_j; \\ 0, \ z \in D - \bigcup_{j=1}^{\infty} R_j. \end{cases}$$

The finiteness of the double integral of $U_3(z)$ over D follows from

$$\iint_D U_3(r)\, rdrd\theta = \sum_{j=1} \iint_{R_j} \frac{rdrd\theta}{(1-r)\Psi^*(|z|)}$$

$$< \sum_{j=1} \frac{2\pi \log 2}{\Psi^*\left(1 - \dfrac{1}{n_j}\right)}.$$

By (6.12) the last term is less than or equal to

$$\sum_{j=1}^{\infty} \frac{2\pi \log 2}{j^2}.$$

For arbitrary $e^{i\theta}$ let $z_j = \left(1 - \dfrac{3}{4}\dfrac{1}{n_j}\right)e^{i\theta}$, $j = 1, 2, \ldots.$ As in the preceding example $D(z_j, (1-|z_j|)1/8) \subset R_j$ for all values j. Thus if $\zeta_j \in D(z_j, (1-|z_j|)1/8)$

$$\lim_{j\to\infty} U_3(\zeta_j)(1-|\zeta_j|)\Psi^*(|\zeta_j|) = 1,$$

then the definition of $\Psi^*(r)$ gives

$$\lim_{j\to\infty} U_3(\zeta_j)(1-|\zeta_j|)\Psi(|\zeta_j|) = \infty.$$

This completes the proof of example 3.

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Pennsylvania State University
University Park, Pennsylvania