COHERENT PAIRS OF EXTENSIONS OF ASSOCIATIVE ALGEBRAS

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Let K be a commutative ring with identity element and let Λ_i, $i=1, 2$ be two K-projective associative algebras with identity element such that the map $k \to k\cdot 1$ of K into Λ_i is a monomorphism of K onto a K-direct summand of Λ_i, $i=1, 2$. Let $\Lambda = \Lambda_1 \otimes \Lambda_2$ and A be a two sided (Λ_1, Λ_2)-bimodule. Let $(\Sigma): 0 \longrightarrow A \xrightarrow{\beta} \Gamma \xrightarrow{\alpha} \Lambda \longrightarrow 0$ be an extension over Λ with kernel A (abelian). This gives rise to a pair (Σ_1), (Σ_2) of extensions (Σ_i) : $0 \to A \to \alpha^{-1}(\Lambda_i) \to \Lambda_i \to 0$ over Λ_i with kernel A, $i=1, 2$. The object of this paper is to give a characterization of pairs of extensions over Λ_1 and Λ_2 respectively, with kernel A which arise in this way from an extension over $\Lambda_1 \otimes \Lambda_2$ with kernel A. This leads to the notion of coherent pairs of extensions (Def. 3.8). The corresponding problem for groups has been treated by F. Haimo and S. Maclane [2]. We define bicohomology groups $H^n(\Lambda_1, \Lambda_2 ; A)$ of the pair (Λ_1, Λ_2) and show (Prop. 3.14) that the set $\mathscr{C}(\Lambda_1, \Lambda_2 ; A)$ of all coherent pairs of equivalence classes of extensions forms a K-module which is a homomorphic image of $H^2(\Lambda_1, \Lambda_2 ; A)$. The kernel of this homomorphism is also determined (Prop. 3.16). The analogous problem for Lie algebras has also been treated by us and will appear elsewhere. We have followed the notation and terminology, as in [1] and [2]. All tensor products are over K.

1. Bicohomology of a pair of associative algebras

Let K be a commutative ring with identity element 1 and Λ_1 and Λ_2 be two K-projective associative algebras with identity element. Further, let the map $k \to k\cdot 1$ of K into Λ_i be monomorphism of K onto a K-direct summand of Λ_i, $i=1, 2$. Let A be a two sided (Λ_1, Λ_2)-bimodule i.e. A is a two sided Λ_i-module, $i=1, 2$ the operators from Λ_1 commuting with those from Λ_2. Let $S(\Lambda_i) = \sum_{n\geq 0} S_n(\Lambda_i)$ be the standard complex [1] of the associative algebra Λ_i where

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$$S_n(\Lambda_i) = \Lambda_i^e \otimes \tilde{S}_n(\Lambda_i), \quad \tilde{S}_0(\Lambda_i) = K$$

and $\tilde{S}_n(\Lambda_i)$ is the tensor product (over K) of the K-module Λ_i taken n-times, $n > 0$, Λ_i^e being the enveloping algebra $\Lambda_i \otimes \Lambda_i^*$ of Λ_i, $i = 1, 2$. It is known [1] (Since Λ_i is K-projective) that $S(\Lambda_i)$ is a Λ_i^e-projective resolution of Λ_i. Let $\mathscr{L}^n(\Lambda_i, A) = \mathrm{Hom}_K(\tilde{S}_n(\Lambda_i), A)$, $n \geq 0$ and $\mathscr{L}^{n,m}(\Lambda_1, \Lambda_2 ; A) = \mathrm{Hom}_K(\tilde{S}_n(\Lambda_1) \otimes \tilde{S}_m(\Lambda_2), A)$, $n \geq 0$, $m \geq 0$. Then the K-module

$$\mathscr{L}(\Lambda_i, A) = \textstyle\sum_{n\geq 0} \mathscr{L}^n(\Lambda_i, A)$$

is a cochain complex [1] the coboundary operator δ_i being given by

$$(1) \qquad \begin{aligned}(\delta_i f)(\lambda_1^{(i)}, \ldots, \lambda_{n+1}^{(i)}) &= \lambda_1^{(i)} f(\lambda_2^{(i)}, \ldots, \lambda_{n+1}^{(i)}) \\ &+ \textstyle\sum_{0<t<n+1} (-1)^t f(\lambda_1^{(i)}, \ldots, \lambda_t^{(i)} \cdot \lambda_{t+1}^{(i)}, \ldots, \lambda_{n+1}^{(i)}) \\ &+ (-1)^{n+1} f(\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \cdot \lambda_{n+1}^{(i)}\end{aligned}$$

for $f \in \mathscr{L}^n(\Lambda_i, A)$ and $\lambda_1^{(i)}, \ldots, \lambda_{n+1}^{(i)} \in \Lambda_i$. The K-modules $\mathscr{L}^n(\Lambda_i, A)$ and $\mathscr{L}^{n,m}(\Lambda_1, \Lambda_2 ; A)$ can be considered as two sided (Λ_1, Λ_2)-bimodules through A and, as usual, we can identify $\mathscr{L}^{n,m}(\Lambda_1, \Lambda_2 ; A)$ with $\mathscr{L}^n(\Lambda_1, \mathscr{L}^m(\Lambda_2, A))$ and $\mathscr{L}^m(\Lambda_2, \mathscr{L}^n(\Lambda_1, A))$. Let

$$\mathscr{L}^n(\Lambda_1, \Lambda_2 ; A) = \textstyle\sum_{0\leq r\leq n} \mathscr{L}^{r,n-r}(\Lambda_1, \Lambda_2 ; A), \quad n \geq 0.$$

We define a map δ from $\mathscr{L}^n(\Lambda_1, \Lambda_2 ; A)$ into $\mathscr{L}^{n+1}(\Lambda_1, \Lambda_2 ; A)$ (which is a K-homomorphism) as follows. Let $f \in \mathscr{L}^n(\Lambda_1, \Lambda_2 ; A)$, $f = f_{n,0} + \cdots + f_{0,n}$ $f_{r,n-r} \in \mathscr{L}^{r,n-r}(\Lambda_1, \Lambda_2 ; A)$. Let $\delta f = f' = f'_{n+1,0} + \cdots + f'_{0,n+1}$ where

$$(2) \qquad f'_{r,n+1-r} = \delta_1 f_{r-1,n-r+1} + (-1)^r \delta_2 f_{r,n-r}.$$

Since δ_1 and δ_2 commute and $\delta_i^2 = 0$, $i = 1, 2$ it follows that $\delta^2 = 0$. Thus the graded K-module $\mathscr{L}(\Lambda_1, \Lambda_2 ; A) = \sum_{n\geq 0} \mathscr{L}^n(\Lambda_1, \Lambda_2 ; A)$ is a cochain complex with δ as coboundary operator. Elements of $\mathscr{L}^n(\Lambda_1, \Lambda_2 ; A)$ are called n-bicochains of the pair (Λ_1, Λ_2) with values in A, those of $Z^n(\Lambda_1, \Lambda_2 ; A) =$ Kernel δ, $\delta : \mathscr{L}^n \to \mathscr{L}^{n+1}$ are called n-bicocycles and those of $B^n(\Lambda_1, \Lambda_2 ; A)$ $=$ image of δ, $\delta : \mathscr{L}^{n-1} \to \mathscr{L}^n$, $n \geq 1$ are called n-bicoboundaries. Further we define $B^0(\Lambda_1, \Lambda_2 ; A)$ to be 0. Since $\delta^2 = 0$ we have the bicohomology module $H^n(\Lambda_1, \Lambda_2 ; A) = Z^n(\Lambda_1, \Lambda_2 ; A)/B^n(\Lambda_1, \Lambda_2 ; A)$, $n \geq 0$. Let $\Lambda = \Lambda_1 \otimes \Lambda_2$. Then $S(\Lambda)$ is Λ^e-projective resolution of Λ. Since Λ_i is K-projective, $i = 1, 2$, it follows [1] that, the $\Lambda_1^e \otimes \Lambda_2^e (= \Lambda^e)$-module

$$S(\Lambda_1) \otimes S(\Lambda_2) = \textstyle\sum_{n\geq 0} S_r(\Lambda_1, \Lambda_2)$$

where

$$S_r(\Lambda_1, \Lambda_2) = \sum_{0\leq r\leq n} S_r(\Lambda_1)\otimes S_{n-r}(\Lambda_2)$$

considered as tensor product of the complexes $S(\Lambda_1)$ and $S(\Lambda_2)$, is a Λ^e-projective resolution of Λ. Thus $H^n(\Lambda, A)$, the n^{th} cohomology module of the K-algebra Λ, can also be considered as the n^{th} derived groups of the complex $\sum_{n\geq 0}\mathrm{Hom}_{\Lambda^e}(S_n(\Lambda_1, \Lambda_2), A)$. Comparing the coboundary operators in this complex and in $\mathscr{L}(\Lambda_1, \Lambda_2 ; A)$ we have

PROPOSITION 1.1. *$H^n(\Lambda_1, \Lambda_2 ; A)$ is isomorphic to $H^n(\Lambda, A)$ as K-modules,* $n\geq 0$.

Let $N(\Lambda_i)$, $i = 1, 2$ and $N(\Lambda)$ denote the normalized standard complexes [1] of the K-algebras Λ_i and Λ respectively. $N(\Lambda_i) = \sum_{n\geq 0} N_n(\Lambda_i)$, $N_n(\Lambda_i) = \Lambda_i^e \otimes \tilde{N}_n(\Lambda_i)$ where $\tilde{N}_0(\Lambda_i) = K$ and $\tilde{N}_n(\Lambda_i)$ is the tensor product (over K) of the K-module $\Lambda_i' = \text{cokernel } (K\to\Lambda_i)$, taken n-times, $n>0$ (Because of our assumption on Λ_i and K, this cokernel is a K-direct summand of Λ_i). It follows that $N(\Lambda_i)$ (resp. $N(\Lambda)$) is a Λ_i^e-(resp. Λ^e-) projective resolution of Λ_i (resp. Λ). Let $N^n(\Lambda_i, A) = \mathrm{Hom}_K(\tilde{N}_n(\Lambda_i), A)$. Thus $N^n(\Lambda_i, A)$ can be identified with the submodule of $\mathscr{L}^n(\Lambda_i, A)$ consisting of those elements f which take the value 0 if any one of the variables λ_i is 1. Such cochains are called normalized cochains. Then $N(\Lambda_i, A) = \sum_{n\geq 0} N^n(\Lambda_i, A)$ is a cochain complex with the coboundary operator, denoted again by δ_i, given by the formula (1). Similarly $N(\Lambda, A) = \sum_{n\geq 0} N^n(\Lambda, A)$ is a cochain complex (normalized). Thus $H^n(\Lambda_i, A)$, $i = 1, 2$ (resp. $H^n(\Lambda, A)$) can also be considered as the n^{th} derived groups of $N(\Lambda_i, A)$ (resp. $N(\Lambda, A)$). Let ξ be the map from $N(\Lambda)$ into $N(\Lambda_1)\otimes N(\Lambda_2)$ [1] defined by

$$\begin{aligned}(3)\quad &\xi((\lambda_0^{(1)}\otimes\lambda_0^{(2)})[\lambda_1^{(1)}\otimes\lambda_1^{(2)}, \dots, \lambda_n^{(1)}\otimes\lambda_n^{(2)}](\mu_0^{(1)}\otimes\mu_0^{(2)}))\\ &= \sum_{0\leq r\leq n}\lambda_0^{(1)}[\lambda_1^{(1)}, \dots, \lambda_r^{(1)}]\lambda_{r+1}^{(1)}\cdots\lambda_n^{(1)}\cdot\mu_0^{(1)}\otimes\lambda_0^{(2)}\cdots\lambda_r^{(2)}[\lambda_{r+1}^{(2)}, \dots, \lambda_n^{(2)}]\mu_0^{(2)}\end{aligned}$$

for $n\geq 0$, and

$$\lambda_0^{(i)}, \dots, \lambda_n^{(i)}, \ \mu_0^{(i)} \in \Lambda_i, \ i = 1, 2.$$

Then it is known [1] that ξ is a 'map' over the identity of Λ. ξ gives rise to a K-homomorphism ξ^* from $\mathrm{Hom}_{\Lambda_1^e\otimes\Lambda_2^e}(\sum_{0\leq r\leq n} N_r(\Lambda_1)\otimes N_{n-r}(\Lambda_2), A)$ into $\mathrm{Hom}_{\Lambda^e}(N_n(\Lambda), A)$ for $n\geq 0$. In particular for $n = 2$ we have the following formula

$$(4)\qquad \xi^*(\omega_1+r+\omega_2)(\lambda_1^{(1)}\otimes\lambda_1^{(2)},\ \lambda_2^{(1)}\otimes\lambda_2^{(2)})$$
$$=\omega_1(\lambda_1^{(1)},\ \lambda_2^{(1)})\cdot\lambda_1^{(2)}\cdot\lambda_2^{(2)}+\lambda_1^{(2)}r(\lambda_1^{(1)},\ \lambda_2^{(2)})\cdot\lambda_2^{(1)}+\lambda_1^{(1)}\cdot\lambda_2^{(1)}\omega_2(\lambda_1^{(2)},\ \lambda_2^{(2)}).$$

Where $\omega_1\in\mathrm{Hom}_{\Lambda_1^e\otimes\Lambda_2^e}(N_2(\Lambda_1)\otimes\Lambda_2^e,\ A)=\mathrm{Hom}_K(\tilde{N}_2(\Lambda_1),\ A)$

$$r\in\mathrm{Hom}_{\Lambda_1^e\otimes\Lambda_2^e}(N_1(\Lambda_1)\otimes N_1(\Lambda_2),\ A)=\mathrm{Hom}_K(\tilde{N}_1(\Lambda_1)\otimes\tilde{N}_1(\Lambda_2),\ A)$$

and

$$\omega_2\in\mathrm{Hom}_{\Lambda_1^e\otimes\Lambda_2^e}(\Lambda_1^e\otimes N_2(\Lambda_2),\ A)=\mathrm{Hom}_K(\tilde{N}_2(\Lambda_2),\ A).$$

If $\omega=\xi^*(\omega_1+r+\omega_2)$ then ω is a 2-cochain (normalized) of Λ with values in A and ω is a 2-cocycle if and only if (i) ω_i is a 2-cocycle and (ii) $\delta_i\omega_j=(-1)^j\delta_j r$, $i\neq j$, $i=1, 2$ (The formula (4) will be used later in Prop. 3.10).

2. Stability Lie algebra

Let Γ be an associative K-algebra (with identity) and M a two-sided ideal of Γ and Γ/M the quotient algebra.

DEFINITION 2.1. The *stability Lie algebra* of the chain $\Gamma\supset M\supset 0$ denoted by $S(\Gamma\supset M\supset 0)$ is the set of all derivations of Γ which are trivial on M and which induce the trivial derivation on Γ/M. $S(\Gamma\supset M\supset 0)$ is an abelian subalgebra of the Lie algebra $\mathfrak{D}(\Gamma)$ of all derivations of Γ. Let Λ be a K-projective associative algebra (with identity) and A a two sided Λ-module. Let $(\Sigma): 0\longrightarrow A\xrightarrow{\beta}\Gamma\xrightarrow{\alpha}\Lambda\longrightarrow 0$ be an extension over Λ with kernel A (abelian).

LEMMA 2.2. *There exists a K-isomorphism of* $S(\Gamma\supset\beta(A)\supset 0)$ *onto* $Z^1(\Lambda, A)$ (*the K-module of* 1-*cocycles of* Λ *with values in* A).

Proof. Let $\theta: S(\Gamma\supset\beta(A)\supset 0)\to Z^1(\Lambda, A)$ be the mapping defined by the relation $\beta(\theta(s)(\lambda))=s(\gamma)$, $\gamma\in\Gamma$, $s\in S$, and $\lambda=\alpha(\gamma)\in\Lambda$. Then θ is a K-isomorphism.

3. Coherent pairs of extensions of associative algebras

Let Λ_1 and Λ_2 be two K-projective associative algebras with identity such that the map $k\to k\cdot 1$ is a monomorphism of K onto a K-direct summand of Λ_i, $i=1, 2$ and A be a two sided (Λ_1, Λ_2)-bimodule.

Let (Σ_1), (Σ_2)

$$(5)\qquad \begin{array}{l}(\Sigma_1): 0\longrightarrow A\xrightarrow{\beta_1}\Gamma_1\xrightarrow{\alpha_1}\Lambda_1\longrightarrow 0\\ (\Sigma_2): 0\longrightarrow A\xrightarrow{\beta_2}\Gamma_2\xrightarrow{\alpha_2}\Lambda_2\longrightarrow 0\end{array}$$

be a pair of extensions over Λ_1 and Λ_2 respectively with kernel A (abelian). Let $L(\lambda_i)$ (resp. $R(\lambda_i)$), $\lambda_i \in \Lambda_i$ denote the operator on A corresponding to λ_i with respect to the left Λ_i-module (resp. right Λ_i-module) structure of A. i.e. $L(\lambda_i)\cdot a = \lambda_i \cdot a$: $R(\lambda_i)\cdot a = a\cdot\lambda_i$, $a \in A$, $\lambda_i \in \Lambda_i$.

DEFINITION 3.1. A *complementary extending derivation* for an element $\lambda_i \in \Lambda_i$, $i=1, 2$ is an extension $\zeta_i(\lambda_i)$ of the operator $L(\lambda_i) - R(\lambda_i)$ as a derivation of Γ_j, $i \neq j$ such that $\zeta_i(\lambda_i)$ induces the trivial derivation on Λ_j.

DEFINITION 3.2. A pair (Σ_1), (Σ_2) of extensions over Λ_1 and Λ_2 respectively with kernel A is called a *complementary pair of extensions* if, for $i=1, 2$ there exists a K-homomorphism $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j)$, $i \neq j$, as K-modules, such that $\zeta_i(\lambda_i)$ is a complementary extending derivation for λ_i and $\zeta_i(1) = 0$.

Since Λ_i is K-projective, the exact sequence (Σ_i), $i=1, 2$ splits as a sequence of K-modules. Further, as $K\cdot 1$ is a K-direct summand of Λ_i, we can assume that $\Lambda_i = K\cdot 1 \oplus \Lambda_i'$ and the splitting $u_i : \Lambda_i \to \Gamma_i$ can be chosen such that $u_i(1) = 1$. Let ω_i be the 2-cocycle of Λ_i with values in A associated to the extension (Σ_i) by means of the splitting u_i. i.e.

$$u_i(\lambda_1^{(i)})\cdot u_i(\lambda_2^{(i)}) - u_i(\lambda_1^{(i)}\cdot\lambda_2^{(i)}) = \beta_i(\omega_i(\lambda_1^{(i)}, \lambda_2^{(i)})), \ \lambda_1^{(i)}, \lambda_2^{(i)} \in \Lambda_i.$$

ω_i is then a normalized 2-cocycle and the elements γ_i of Γ_i can be expressed uniquely in the form $\gamma_i = \beta_i(a_i) + u_i(\lambda_i)$, $\lambda_i = \alpha_i(\gamma_i)$ for some $a_i \in A$.

PROPOSITION 3.3. *(Σ_1), (Σ_2) is a complementary pair if and only if there exists a pair of functions r_1, r_2 such that $r_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_1) \otimes \tilde{N}_1(\Lambda_2), A)$ satisfying the following conditions:*

$$\beta_j(r_i(\lambda_1, \lambda_2)) = (\zeta_i(\lambda_i))(u_j(\lambda_j)) \tag{6}$$

$$\delta_j r_i = \delta_i \omega_j, \ i \neq j, \ i=1, 2. \tag{7}$$

Proof. Let (Σ_1), (Σ_2) be a complementary pair. Then there exists a K-homomorphism $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j)$, $i \neq j$ (as K-modules) such that $\zeta_i(\lambda_i)$ is a complementary extending derivation for λ_i, $i=1, 2$ and $\zeta_i(1)=0$. As $\zeta_i(\lambda_i)$ induces the trivial derivation on Λ_j, $i \neq j$ $(\zeta_i(\lambda_i))(\gamma_j) \in \beta_j(A)$ for $\gamma_j \in \Gamma_j$. In particular for $\lambda_j \in \Lambda_j$, $(\zeta_i(\lambda_i))(u_j(\lambda_j)) \in \beta_j(A)$. ($u_j$ is a splitting of (Σ_j)). Let us denote by $\beta_j(r_i(\lambda_1, \lambda_2))$ the element $(\zeta_i(\lambda_i))(u_j(\lambda_j))$. Thus we have function r_i, $i=1, 2$, $r_i : \Lambda_1 \times \Lambda_2 \to A$ such that $\beta_j(r_i(\lambda_1, \lambda_2)) = (\zeta_i(\lambda_i))(u_j(\lambda_j))$, $i \neq j$. Since $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j)$ is a K-homomorphism and $\zeta_i(1) = 0$ it follows that $r_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_1) \otimes$

$\tilde{N}_1(\Lambda_2),\ A)$. As $\zeta_i(\lambda_i)$ is a derivation of Γ_j it can be seen that the 2-cocycles ω_i (normalized) corresponding to the extension (Σ_i) satisfy the relation (7). Conversely, we define $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j),\ i \neq j$ as follows:

$$\gamma_j \in \Gamma_j,\ \gamma_j = \beta_j(a_j) + u_j(\lambda_j),\ \lambda_j = \alpha_j(\gamma_j)$$
$$(\zeta_i(\lambda_i))(\gamma_j) = (\zeta_i(\lambda_i))(\beta_j(a_j)) + (\zeta_i(\lambda_i))(u_j(\lambda_j))$$
$$= \beta_j((L(\lambda_i) - R(\lambda_i))(a_j) + r_i(\lambda_1,\ \lambda_2)).$$

The relation (7) shows that $\zeta_i(\lambda_i)$ is a complementary extending derivation and, as $r_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_1) \otimes \tilde{N}_1(\Lambda_2),\ A)$, ζ_i is a homomorphism of K-modules.

LEMMA 3.4. *Let* (Σ_1), (Σ_2) *be a complementary pair and* u_i, u_i' *be two splittings of* (Σ_i) *as K-modules such that* $u_i(1) = u_i'(1) = 1$ *and* r_i, r_i' *be the associated functions. Then there exists* $c_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_i),\ A)$ *such that* $r_i' = r_i + \delta_i c_j$, $i \neq j$, $i = 1, 2$.

LEMMA 3.5. *Let* (Σ_1), (Σ_2) *be a complementary pair. Any two complementary extending derivations* $\zeta_i(\lambda_i)$, $\zeta_i'(\lambda_i)$ *for* $\lambda_i \in \Lambda_i$ *differ by an element* $s(\lambda_i) \in S(\Gamma_j \supset \beta_j(A) \supset 0)$ $i \neq j$ *and conversely.*

The proofs of these two lemmas are straight forward verifications.

DEFINITION 3.6. A *partial cocycle* z is an element $z \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_1) \otimes \tilde{N}_1(\Lambda_2),\ A)$ such that z belongs to $\mathrm{Hom}_K(\tilde{N}_1(\Lambda_j),\ Z^1(\Lambda_i,\ A))$ for $i = 1$ or 2, $i \neq j$ (In this case, we denote z by z_i).

PROPOSITION 3.7. *Let* (Σ_1), (Σ_2) *be a complementary pair. Then for* (i) *different splittings* u_i' *of* (Σ_i) *such that* $u_i'(1) = 1$ (ii) *complementary extending derivations of* Γ_j *for elements of* Λ_i, $i \neq j$ *for different homomorphisms* $\zeta_i' : \Lambda_i \to \mathfrak{D}(\Gamma_j)$ *as K-modules, the sum* $r_1 + r_2$ *(of the associated functions* r_i *as in* Prop. 3.3) *gets changed into* $r_1 + r_2 + z_1 + z_2$ *where* z_i, $i = 1, 2$ *are partial cocycles,* $z_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_j),\ Z^1(\Lambda_i,\ A))$.

The proof is a straightforward verification.

DEFINITION 3.8. A complementary pair (Σ_1), (Σ_2) of extensions over Λ_1 and Λ_2 respectively with kernel A is called a *coherent pair* if, for some splittings u_i of (Σ_i) such that $u_i(1) = 1$ and some K-homomorphism $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j)$ $i \neq j$ (as K-modules) such that $\zeta_i(1) = 0$ and $\zeta_i(\lambda_i)$ is a complementary extending derivation of Γ_j for λ_i, the sum $r_1 + r_2$ of the associated functions r_i (as in Prop. 3,3) is equal to the sum z_1+z_2 for some partial cocycles

$$z_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_j),\ Z^1(\Lambda_i,\ A)),\ i \neq j,\ i=1,2.$$

From Prop. 3.7, we see that this property of being coherent depends only on the extensions (Σ_1), (Σ_2). We give a criterion for coherence in the following.

PROPOSITION 3.9. *A complementary pair* (Σ_1), (Σ_2) *is a coherent pair, if and only if, for some choice of splittings* u_i *of* (Σ_i) *with* $u_i(1)=1$ *and K-homomorphism* $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j)$, $i \neq j$ *(as K-modules) for* $i=1, 2$ *such that* $\zeta_i(1)=0$ *and* $\zeta_i(\lambda_i)$ *is complementary extending derivation for* λ_i, *the sum* $\rho_1+\rho_2$ *of the associated functions* ρ_i *(as given by Prop.* 3.3) *is* 0.

Proof. Let (Σ_1), (Σ_2) be a coherent pair. Then for some splitting u_i of (Σ_i) such that $u_i(1)=1$ and homomorphism $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j)$, $i \neq j$, $i=1, 2$ the sum r_1+r_2 of the associated functions r_i is equal to the sum z_1+z_2, z_i being partial cocycle belonging to $\mathrm{Hom}_K(\tilde{N}_1(\Lambda_j), Z^1(\Lambda_i, A))$, $i=1, 2$ i.e. $r_1+r_2=z_1+z_2$. With the same splittings u_i but with the homomorphism $\zeta_i' : \Lambda_i \to \mathfrak{D}(\Gamma_j)$, $\zeta_i' = \zeta_i + \theta_j^{-1}(-z_j)$ (θ_j is the K-isomorphism of $S(\Gamma_j \supset \beta_j(A) \supset 0)$ onto $Z^1(\Lambda_j, A)$ of Lemma 2.2), we see that the sum $\rho_1+\rho_2$ of the associated functions ρ_i is equal to $r_1+r_2-z_1-z_2=0$. Conversely if $\rho_1+\rho_2=0$ with $z_1=z_2=0$ we see that the pair (Σ_1), (Σ_2) is coherent.

An example of a coherent pair

Let $\Lambda=\Lambda_1 \otimes \Lambda_2$ (with the same assumptions on Λ_i and K, as earlier). Then A is a two sided Λ-module. Let $(\Sigma) : 0 \longrightarrow A \xrightarrow{\beta} \Gamma \xrightarrow{\alpha} \Lambda \longrightarrow 0$ be an extension over Λ with kernel A (abelian). Further let $\Gamma_i = \alpha^{-1}(\Lambda_i)$, $i=1, 2$ (Λ_i is identified as a sub-algebra of Λ). Then we have a pair (Σ_1), (Σ_2) of extensions, $(\Sigma_i) : 0 \longrightarrow A \xrightarrow{\beta_i} \Gamma_i \xrightarrow{\alpha_i} \Lambda_i \longrightarrow 0$, $\alpha_i = \alpha \mid \Gamma_i$ and $\beta_i=\beta$. Let $u_i : \Lambda_i \to \Gamma_i$ be a splitting of (Σ_i) as K-modules such that $u_i(1)=1$. We shall define a complementary extending derivation $\zeta_i(\lambda_i)$ for λ_i, $i=1, 2$ as follows:

$$(8)\qquad (\zeta_i(\lambda_i))(\gamma_j) = [u_i(\lambda_i),\ \gamma_j] = u_i(\lambda_i)\cdot\gamma_j - \gamma_j\cdot u_i(\lambda_i)$$
$$i \neq j,\ i=1, 2 : \gamma_j \in \Gamma_j \text{ and } \lambda_i \in \Lambda_i.$$

Then $\zeta_i(\lambda_i)$ is a derivation of Γ_j, $i \neq j$, $\zeta_i(1)=0$ and induces trivial derivation on Λ_j and $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j)$ is a K-homomorphism. Thus (Σ_1), (Σ_2) is a complementary pair. $\beta_j(r_i(\lambda_1, \lambda_2)) = (\zeta_i(\lambda_i))(u_j(\lambda_j)) = [u_i(\lambda_i), u_j(\lambda_j)]$, $i \neq j$. Hence $r_1+r_2=0$ i.e. (Σ_1), (Σ_2) is a coherent pair.

PROPOSITION 3.10. *Let* (Σ_1), (Σ_2)

(9)
$$(\Sigma_1) : 0 \longrightarrow A \overset{\beta_1}{\longrightarrow} \Gamma_1 \overset{\alpha_1}{\longrightarrow} \Lambda_1 \longrightarrow 0$$
$$(\Sigma_2) : 0 \longrightarrow A \overset{\beta_2}{\longrightarrow} \Gamma_2 \overset{\alpha_2}{\longrightarrow} \Lambda_2 \longrightarrow 0$$

be a coherent pair of extensions (with the same assumptions on Λ_i *and* K, *as earlier). Then* (i) *there exists an extension* (Σ)

(10)
$$(\Sigma) : 0 \longrightarrow A \overset{\beta}{\longrightarrow} \Gamma \overset{\alpha}{\longrightarrow} \Lambda_1 \otimes \Lambda_2 \longrightarrow 0$$

over $\Lambda_1 \otimes \Lambda_2$ *with kernel* A

(ii) *there exists operator* K*-isomorphisms* $\varphi_i : \alpha^{-1}(\Lambda_i) \to \Gamma_i$, $i=1, 2$ (*with respect to operators from* Λ_j, $i \neq j$ *and*

(iii) $\alpha_i \cdot \varphi_i = \alpha | \alpha^{-1}(\Lambda_i)$, $i=1, 2$.

Proof. Let $(\Sigma_i) : 0 \longrightarrow A \overset{\beta_i}{\longrightarrow} \Gamma_i \overset{\alpha_i}{\longrightarrow} \Lambda_i \longrightarrow 0$ be a coherent pair. Let us choose (i) splitting $u_i : \Lambda_i \to \Gamma_i$ such that $u_i(1)=1$ with ω_i as associated 2-cocycles (normalized) and (2) K-homomorphism $\zeta_i : \Lambda_i \to \mathfrak{D}(\Gamma_j)$, $i \neq j$ so that the sum $r_1 + r_2 = 0$ (of the associated functions r_i (Prop. 3.9)). Let $r = r_1 = -r_2$. Then the 2-bicochain $\omega_1 + r + \omega_2$ is a 2-bicocycle and let $\omega = \xi^*(\omega_1 + r + \omega_2)$ (ξ^* given by (4)). Then ω is a 2-cocycle (normalized) of $\Lambda_1 \otimes \Lambda_2$ with values in A. Let $\Gamma = A \oplus (\Lambda_1 \otimes \Lambda_2)$ (direct sum as K-modules). We shall, as usual, denote by the pairs $(a, \Sigma_i \lambda_{1i} \otimes \lambda_{2i})$ the elements of Γ, $a \in A$, $\lambda_{1i} \in \Lambda_1$, $\lambda_{2i} \in \Lambda_2$. Let $\beta : A \to \Gamma$, $\alpha : \Gamma \to \Lambda_1 \otimes \Lambda_2$ be defined by $\beta(a) = (a, 0)$ and $\alpha((a, \sum \lambda_{1i} \otimes \lambda_{2i})) = \sum \lambda_{1i} \otimes \lambda_{2i}$. We define a multiplication in Γ, as usual, by means of the 2-cocycle ω.

(11)
$$\begin{aligned}(a_1, \lambda_{11} \otimes \lambda_{21}) \cdot (a_2, \lambda_{12} \otimes \lambda_{22}) \\ = (a_1 \cdot \lambda_{12} \cdot \lambda_{22} + \lambda_{11} \cdot \lambda_{12} \cdot a_2 + \omega_1(\lambda_{11}, \lambda_{12}) \lambda_{21} \cdot \lambda_{22} \\ + \lambda_{21} r(\lambda_{11}, \lambda_{22}) \lambda_{12} + \lambda_{11} \cdot \lambda_{12} \omega_2(\lambda_{21}, \lambda_{22}), \lambda_{11} \cdot \lambda_{12} \otimes \lambda_{21} \cdot \lambda_{22})\end{aligned}$$

(from (4)). Then Γ is an associative K-algebra and $(\Sigma) : 0 \longrightarrow A \overset{\beta}{\longrightarrow} \Gamma \overset{\alpha}{\longrightarrow} \Lambda_1 \otimes \Lambda_2 \longrightarrow 0$ is an extension over $\Lambda_1 \otimes \Lambda_2$ with kernel A. As in the example given above, this gives rise to a coherent pair (Σ_1'), (Σ_2') of extensions over Λ_1 and Λ_2 respectively with kernel A where

(12)
$$(\Sigma_i') : 0 \longrightarrow A \overset{\beta_i'}{\longrightarrow} \alpha^{-1}(\Lambda_i) \overset{\alpha_i'}{\longrightarrow} \Lambda_i \longrightarrow 0$$

where $\alpha_i' = \alpha | \alpha^{-1}(\Lambda_i)$ and $\beta_i' = \beta$. Let us define the mapping $\varphi_i : \alpha^{-1}(\Lambda_i) \to \Gamma_i$ as follows.

Case $i=1$: $\varphi((a, \lambda_1\otimes 1)) = \beta_1(a) + u_1(\lambda_1)$
Case $i=2$: $\varphi_2((a, 1\otimes \lambda_2)) = \beta_2(a) + u_2(\lambda_2)$.

Then φ_i, $i=1, 2$ is an isomorphism of K-algebras. We shall verify here only that the multiplication is preserved and this is operator isomorphism.

Case $i=1$, $j=2$:

$$\begin{aligned}&\varphi_1((a, \lambda_{11}\otimes 1)\cdot(a_2, \lambda_{12}\otimes 1))\\ &= \beta_1(a_1\cdot\lambda_{12} + \lambda_{11}\cdot a_2 + \omega_1(\lambda_{11}, \lambda_{12})) + u_1(\lambda_{11}\cdot\lambda_{12})\\ &= \varphi_1((a_1, \lambda_{11}\otimes 1))\cdot\varphi_1((a_2, \lambda_{12}\otimes 1))\end{aligned}$$

Let $\lambda_2 \in \Lambda_2$:

$$\begin{aligned}&\varphi_1([(0, 1\otimes\lambda_2), (a, \lambda_1\otimes 1)])\\ &= \varphi_1((L(\lambda_2) - R(\lambda_2))(a) - r(\lambda_1, \lambda_2), 0)\\ &= \beta_1((L(\lambda_2) - R(\lambda_2))(a) - r(\lambda_1, \lambda_2))\\ &= \zeta_2(\lambda_2)\cdot\varphi_1((a, \lambda_1\otimes 1)).\end{aligned}$$

i.e. $\varphi_1 : \alpha^{-1}(\Lambda_1) \to \Gamma_1$ is operator isomorphism with operators from Λ_2. Similarly for $i=2$ we show φ_2 is operator isomorphism of $\alpha^{-1}(\Lambda_2)$ onto Γ_2. Further $(\alpha_1\cdot\varphi_1)((a, \lambda_1\otimes 1)) = \lambda_1 = \alpha((a, \lambda_1\otimes 1))$ i.e. $\alpha_1\cdot\varphi_1 = \alpha\,|\,\alpha^{-1}(\Lambda_1)$. Similarly $\alpha_2\cdot\varphi_2 = \alpha\,|\,\alpha^{-1}(\Lambda_2)$. We summarize together the Prop. 3.10 and the example given earlier in

THEOREM 3.11. *Let* $(\Sigma) : 0 \longrightarrow A \xrightarrow{\beta} \Gamma \xrightarrow{\alpha} \Lambda \longrightarrow 0$ *be an extension over* $\Lambda = \Lambda_1 \otimes \Lambda_2$ *with kernel* A, Λ_i *being* K*-projective associative algebra with identity such that the map* $k \to k\cdot 1$ *is a monomorphism of* K *onto a* K*-direct summand of* Λ_i, $i=1, 2$ *and* A *a two sided* (Λ_1, Λ_2)*-bimodule. Then* $(\Sigma_i') : 0 \to A \to \alpha^{-1}(\Lambda_i) \to \Lambda_i \to 0$ $i=1, 2$ *is a coherent pair of extensions over* Λ_i *with kernel* A, *the complementary extending derivations on* $\alpha^{-1}(\Lambda_j)$ *for* $\lambda_i \in \Lambda_i$, $i \neq j$ *being given by inner derivation of coset representatives* $u_i(\lambda_i)$ *of elements of* Λ_i *in* $\alpha^{-1}(\Lambda_i)$, $u_i : \Lambda_i \to \alpha^{-1}(\Lambda_i)$ *being a splitting of* (Σ_i') *as* K*-modules with* $u_i(1) = 1$. *Conversely if* $(\Sigma_i) : 0 \longrightarrow A \xrightarrow{\beta_i} \Gamma_i \xrightarrow{\alpha_i} \Lambda_i \longrightarrow 0$ $(i=1, 2)$ *is a coherent pair of extensions over* Λ_i *with kernel* A *then* (1) *there exists an extension* $(\Sigma) : 0 \longrightarrow A \xrightarrow{\beta} \Gamma \xrightarrow{\alpha} \Lambda_1 \otimes \Lambda_2 \longrightarrow 0$ *over* $\Lambda_1 \otimes \Lambda_2$ *with kernel* A (2) *there exists a pair of operator isomorphisms* φ_i *from* $\alpha^{-1}(\Lambda_i)$ *onto* Γ_i, $i=1, 2$ (*as* K*-algebras*) *and* (3) $\alpha_i\cdot\varphi_i = \alpha\,|\,\alpha^{-1}(\Lambda_i)$, $i=1, 2$.

Let $\mathscr{C}(\Lambda_1, \Lambda_2 ; A)$ denote the set of all coherent pairs of equivalence classes of extensions over Λ_1 and Λ_2 with kernel A (with the same assumptions on Λ_i, A and K, as earlier). A coherent pair (Σ_1), (Σ_2) of extensions determines at least one quadruple $\{\omega_1, r_1, r_2, \omega_2\}$ where $\omega_i \in Z^2(\Lambda_i, A)$ (normalized 2-cocycle) and $r_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_1) \otimes \tilde{N}_1(\Lambda_2), A)$ such that (1) $\delta_i\omega_j = \delta_j r_i$, $i \neq j$, $i = 1, 2$ and (2) $r_1 + r_2 = z_1 + z_2$ where z_i is a partial cocycle, $z_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_j), Z^1(\Lambda_i, A))$, $i \neq j$ we shall call such a quadruple a *standard quadruple.* Conversely, any standard quadruple $\{\omega_1, r_1, r_2, \omega_2\}$ determines a coherent pair of extensions.

DEFINITION 3.12. Two standard quadruples $\{\omega_1, r_1, r_2, \omega_2\}$ $\{\omega_1', r_1', r_2', \omega_2'\}$ are called equivalent if (1) there exists a 1-cochain c_i (normalized) such that $\omega_i' - \omega_i = \delta_i c_i$ for $i = 1, 2$ and (2) there exists partial cocycle z_i such that $r_i' - r_i = \delta_i c_j + z_j$. A standard quadruple of the form $\{\delta_1 c_1, \delta_1 c_2 + z_2, \delta_2 c_1 + z_1, \delta_2 c_2\}$ is called a *trivial quadruple.*

Let $\mathcal{S}(\Lambda_1, \Lambda_2 ; A)$ denote the set of all equivalence classes of standard quadruples. Then $\mathcal{S}(\Lambda_1, \Lambda_2 ; A)$ is a K-module under componentwise addition and scalar multiplication of representatives. It is also clear that there exists a bijection of $\mathcal{S}(\Lambda_1, \Lambda_2 ; A)$ onto $\mathscr{C}(\Lambda_1, \Lambda_2 ; A)$. Hereafter we shall identify $\mathscr{C}$ with $\mathcal{S}$. $\mathscr{C}$ is not empty because the pair (Σ_1), (Σ_2) where (Σ_i) is the inessential extension over Λ_i with kernel A, is a coherent pair. Let us now define the following maps

$$
\begin{aligned}
\Omega_i &: H^2(\Lambda_1, \Lambda_2 ; A) \to H^2(\Lambda_i, A), \ i = 1, 2 \\
\Omega &: H^2(\Lambda_1, \Lambda_2 ; A) \to H^2(\Lambda_1, A) \oplus H^2(\Lambda_2, A) \\
\Pi &: H^2(\Lambda_1, \Lambda_2 ; A) \to \mathcal{S}(\Lambda_1, \Lambda_2 ; A) \\
\Delta &: \mathcal{S}(\Lambda_1, \Lambda_2 ; A) \to H^2(\Lambda_1, A) \oplus H^2(\Lambda_2, A).
\end{aligned} \tag{13}
$$

Let $\omega_1 + r + \omega_2$ be a representative 2-bicocycle of an element of $H^2(\Lambda_1, \Lambda_2 ; A)$

$\Omega_i(\omega_1 + r + \omega_2) =$ the cohomology class of ω_i in $H^2(\Lambda_i, A)$

$\Omega(\omega_1 + r + \omega_2) = \Omega_1(\omega_1 + r + \omega_2) + \Omega_2(\omega_1 + r + \omega_2)$.

$\Pi(\omega_1 + r + \omega_2) =$ equivalence class of the standard quadruple $\{\omega_1, r, -r, \omega_2\}$

$\Delta(\{\omega_1, r_1, r_2, \omega_2\}) =$ (class of ω_1, class of ω_2).

These are well defined K-homomorphisms. Let $\mathfrak{K}_i$ be the kernel of Ω_i and $\mathfrak{K}$ the kernel of Ω. Then $\mathfrak{K} = \mathfrak{K}_1 \cap \mathfrak{K}_2$.

PROPOSITION 3.13. *$\mathfrak{R}$ consists of the classes of elements of the form* $(0, r, 0)$ *where* $r \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_1) \otimes \tilde{N}_1(\Lambda_2), A)$ *such that* $\delta_i r = 0$, $i = 1, 2$.

The proof is straightforward verification.

PROPOSITION 3.14. *Π is an epimorphism.*

Proof: Let $\{\omega_1, r_1, r_2, \omega_2\}$ be a representative of an element in $\mathscr{S}$. As this standard quadruple corresponds to a coherent pair of extensions, we can assume that $r_1 = r = -r_2$. Then $\omega_1 + r + \omega_2$ is a 2-bicocycle and the image under Π of the bicohomology class of $\omega_1 + r + \omega_2$ is the element in $\mathscr{S}$ which we started with.

PROPOSITION 3.15. *Δ is a monomorphism.*

Proof. Enough to verify that kernel Δ is 0. Let $\{\omega_1, r_1, r_2, \omega_2\}$ be a representative of an element in Ker Δ. Then

$$\Delta(\{\omega_1, r_1, r_2, \omega_2\}) = 0 \in H^2(\Lambda_1, A) \oplus H^2(\Lambda_2, A)$$

i.e. there exists a 1-cochain $c_i \in \mathrm{Hom}_K(\tilde{N}_1(\Lambda_i), A)$ such that $\omega_i = \delta_i c_i$, $i = 1, 2$. Hence the equivalence class of $\{\omega_1, r_1, r_2, \omega_2\}$ corresponds to the coherent pair of inessential extensions and this is the zero of $\mathscr{S}$ i.e. Ker $\Delta = 0$.

PROPOSITION 3.16. *The kernel of Π is $\mathfrak{R}$ (the kernel of Ω).*

Proof. Let $\omega_1 + r + \omega_2$ be a representative of an element in Ker Π. Then $\Pi(\omega_1 + r + \omega_2) =$ class of $\{\omega_1, r, -r, \omega_2\} = 0$ in $\mathscr{S}$. $\{\omega_1, r, -r, \omega_2\}$ is equivalent to a quadruple $\{\delta_1 c_1, \delta_1 c_2 + z_2, \delta_2 c_1 + z_1, \delta_2 c_2\}$ for a (normalized) 1-chain c_i, $i = 1, 2$ and partial cocycle z_i, $i = 1, 2$. Thus $\omega_i = \delta_i c_i$ and $r = r_1 = -r_2 = \delta_1 c_2 + z_2 = -\delta_2 c_1 - z_1$. Hence $(\omega_1 + r + \omega_2)$ is bicohomologous to $(0, r', 0)$ where $r' = \delta_2 c_1 + z_2 = -\delta_1 c_2 - z_1$.

$\delta_1 r' = \delta_2 r' = 0$ i.e. Ker $\Pi \subset \mathfrak{R}$. Let $\omega_1 + r + \omega_2$ be a representative of an element in $\mathfrak{R}$. Then $\omega_1 + r + \omega_2$ is bicohomologous to a bicocycle of the form $(0, r', 0)$ where $\delta_i r' = 0$, $i = 1, 2$. Then $\Pi(\omega_1 + r + \omega_2) = \Pi$(class of $(0, r', 0)$) = class of $\{0, r', -r', 0\} = 0 \in \mathscr{S}$ i.e. $\mathfrak{R} \subset$ Ker Π. Hence Ker $\Pi = \mathfrak{R}$.

We summarize these together in the following:

THEOREM 3.17. *Let Λ_1 and Λ_2 be two associative K-algebras (with all assumptions as earlier) and let A be a two sided (Λ_1, Λ_2)-bimodule. Then we have the following commutative diagram in which rows and columns are exact:*

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 \longrightarrow & \mathfrak{K} & \longrightarrow & H^2(\Lambda_1, \Lambda_2 ; A) & \xrightarrow{\Pi} & & \mathscr{S}(\Lambda_1, \Lambda_2 ; A) & \longrightarrow 0 & \\
 & \| & & \| & & & \downarrow{\scriptstyle \Delta} & & \\
0 \longrightarrow & \mathfrak{K} & \longrightarrow & H^2(\Lambda_1, \Lambda_2 ; A) & \xrightarrow{\Omega} & & H^2(\Lambda_1, A) \oplus H^2(\Lambda_2, A). & &
\end{array}
$$

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