CHERN CLASSES OF PROJECTIVE MODULES

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Introduction. In topology, one can define in several ways the Chern class of a vector bundle over a certain topological space (Chern [2], Hirzebruch [7], Milnor [9], Steenrod [15]). In algebraic geometry, Grothendieck has defined the Chern class of a vector bundle over a non-singular variety. Furthermore, in the case of differentiable vector bundles, one knows that the set of differentiable cross-sections to a bundle forms a finitely generated projective module over the ring of differentiable functions on the base manifold. This gives a one to one correspondence between the set of vector bundles and the set of f.g.-projective modules (Milnor [10]). Applying Grauert's theorems (Grauert [5]), one can prove that the same statement holds for holomorphic vector bundles over a Stein manifold.[2)]

The purpose of the present paper is to give the Chern class of a f.g.- projective module as an element of the de Rham cohomology of the ring. Thus we establish a completely algebraic treatment of the above cases. Our method of defining the Chern class is the same as that used in differential geometry; thus we obtain a differential geometric approach to the study of projective modules.

In Section 1, we introduce the notion of the trace and its symmetrized form on a finitely generated projective module. For each finitely generated projective module v with constant rank, we construct an exact sequence:

$$0 \to \mathrm{End}\,(v) \to N(v) \to D(R) \to 0$$

where $D(R)$ is the set of derivations of the ring R and $N(v)$ is the set of differential operators. This sequence will play a fundamental role in this paper in achieving a differential-geometric approach to the study of projective

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2) See also J. P. Serre, "Modules projectifs et espaces fibres a fibre vectorielle" Séminaire P. Dubreil, 1957-58.

modules. Section 3, Section 4 and Section 5 are devoted to a study of the de Rham cohomology of certain types of Lie d-algebras (Palais [13]). Since we deal with projective modules over an arbitrary commutative ring, special types of multiplications on alternating forms are needed to avoid the use of divistion. We have omitted the details of several proofs. In general the calculations are similar to those found in differential geometry.

Section 5 contains the basic notion of connections in projective modules. In Section 6 we define the Chern class using the curvature form of a connection, and prove the product formula, which is the characteristic property of Chern classes. In our definition, the Chern class depends on a connection, but the independence of such a connection is proved in Section 7, thereby reducing the problem to the case where the module is free.

We would like to remark here: if the ring contains rational numbers, then the independence of characteristic classes on connections can be proved in a way analogous to the differential geometric proof given by Weil (Chern [2], Kobayashi and Nomizu [8]).

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Section 1. Projective modules and endomorphisms

Let R be a commutative ring with a unit. An R-module v is called *projective* if every diagram

$$\begin{tikzcd} & v \arrow[d] & \\ w \arrow[r] & u \arrow[r] & 0 \end{tikzcd}$$

of R-modules, in which the row is exact, can be imbedded in a commutative diagram

$$\begin{tikzcd} & v \arrow[d] \arrow[dl] & \\ w \arrow[r] & u \arrow[r] & 0. \end{tikzcd}$$

It is well known that an R-module v is a finitely generated projective module if and only if it is a direct summand of a finitely generated R-free module (cf. Cartan-Eilenberg [1]).

From this we see,

LEMMA 1. *Let v, v_1, and v_2 be f.g.-projective modules over R. Then* Hom (v, R), $\Lambda^n v$, v_1+v_2, $v_1\otimes v_2$ *and* Hom (v_1, v_2) *are f.g.-projective.*

An R-module M together with an R-homomorphism; $M\otimes M$ $x\otimes y\to[x,y]\varepsilon M$ is called a *Lie algebra* over R if it satisfies i) $[x, x]=0$, ii) $[[x, y], z]+[[y, z], x]+[[z, x], y]=0$ for any $x, y, z\in M$. ii) is called Jacobi's ideutity.

For any R-module v, set

$$L(v)=\mathrm{End}_R(v).$$

$L(v)$ is a Lie algebra over R by $[f, g]=f\cdot g-g\cdot f$ for $f, g\varepsilon L(v)$.

Let v be a f.g.-projective module over R. We shall define the trace for any element f in $L(v)$. Let v' be another f.g.-projective module such that $v+v'$ is free with a base $e_1, \ldots, e_n$. Denote by π the projection of $v+v'$ onto v. For $f\in L(v)$, we have $f\cdot\pi\in L(v+v')$. $f\cdot\pi$ will be expressed in a matrix form (a_{ij}) by

$$(f\cdot\pi)(e_i)=\sum a_{ij}e_j.$$

We set

$$\mathrm{Tr}(f)=\mathrm{Tr}(f : v)=\sum a_{ii}.$$

The following is clear: 1) $\mathrm{Tr}(f)$ does not depend on the choice of a base $e_1, \ldots, e_n$ in $v+v'$, 2) $\mathrm{Tr}(f)$ does not depend on the choice of such v'. Thus we have an R-homomorphism of $L(v)$ into R.

LEMMA 2. *Tr satisfies*

i) $\mathrm{Tr}(1 : R)=1$. *where the first* 1 *denotes the identity mapping of* R.

ii) $\mathrm{Tr}([f, g]: v)=0$ *for auy* $f, g\in L(v)$.

iii) *Given an exact sequence* $0\to v_1\to v\to v_2\to 0$ *of f.g.-projective modules over* R *and* $f_1\in L(v_1)$, $f\in L(v)$, $f_2\in L(v_2)$ *such that*

$$\begin{tikzcd} v_1 \arrow[r] \arrow[d, "f_1"] & v \arrow[r] \arrow[d, "f"] & v_2 \arrow[d, "f_2"] \\ v_1 \arrow[r] & v \arrow[r] & v_2 \end{tikzcd}$$

commutes, then

$$\mathrm{Tr}(f : v)=\mathrm{Tr}(f_1 : v_1)+\mathrm{Tr}(f_2 : v_2).$$

The properties ii) and iii) can be reduced to the case where every module is free. Then they can be seen easily. On the other hand, it can be shown that the three properties in Lemma 2 characterize the trace completely.

LEMMA 3. *Let v_1, and v_2 be f.g.-projective modules over R. For $f \in L(v_1)$ and $g \in L(v_2)$, we have*

$$\mathrm{Tr}(f \otimes g;\ v_1 \otimes v_2) = \mathrm{Tr}(f : v_1) \cdot \mathrm{Tr}(g : v_2).$$

The problem can be reduced to the case where v_1 and v_2 are free. Then it is well known.

We shall define a symmetric n-form P_n on $L(v)$ for any f.g.-projective module v over R. For $f_1, \ldots, f_n$ in $L(v)$, consider the mapping

$$\textstyle\sum_\sigma f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}$$

where σ runs through all permutations of n-letters. One can see that this induces an R-homomorphism of $\Lambda^n v$ into $\Lambda^n v$.

By definition,

$$P_n(f_1, \ldots, f_n : v) = \text{trace of } \textstyle\sum f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)} \text{ on } \Lambda^n v.$$

Obviously $P_n(f_1, \ldots, f_n : v)$ is symmetric with respect to $f_1, \ldots, f_n$.

LEMMA 4. *Let v_1 and v_2 be f.g.-projective modules. For f_i in $L(v_1)$ $(i = 1, 2, \ldots, k)$ and g_i in $L(v_2)$ $(i = 1, \ldots, l)$ we have*

$$P_{k+l}(f_1, \ldots, f_k, g_1, \ldots, g_l;\ v_1 + v_2) = P_k(f_1, \ldots, f_k;\ v_1) \cdot P_l(g_1, \ldots, g_l;\ v).$$

This follows from the fact that $\Lambda^n(v_1 + v_2)$ is isomorphic with $\sum \Lambda^i(v_1) \otimes \Lambda^{n-i}(v_2)$ and from iii) of Lemma 2.

LBMMA 5. *Let v be a f.g.-projective module over R. For any g in $L(v)$ and for any f_i in $L(v)$ $(i = 1, 2, \ldots, n)$, we have*

$$\textstyle\sum_i P_n(f_1, \ldots, [g, f_i], \ldots, f_n) = 0.$$

Now suppose v is free with a base $e_1, \ldots, e_n$. Then $L(v)$ is also free with the base (E_{ij}) where $E_{ij} \cdot e_k = \delta_{jk} e_i$.

LEMMA 6. *P_k has the following expression on a free module:*

$$P_k(f_1, \ldots, f_k) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \cdot \sum_{\binom{i}{j}} \mathrm{sign}\binom{i}{j} a^1_{i_1 j_1} \cdots a^k_{i_k j_k}$$

for $f_i = \sum a^i_{jk} E_{jk}$.

Lemma 6 can be verified directly from definitions.

Lemma 5 can be reduced to the case where v is free. Then it will be checked for each base E_{ij}.

Remark 1. Lemma 5 is a classical fact when v is a vector space over the field of reals or complexes. Since P_n and the bracket operation are defined over Z, one sees Lemma 5 holds on any Z-free module. Then it is easy to see that it is true for any ring.

Remark 2. On the free module v, we have the notion of the characteristic polynomial $\sum a_k t^k$ for $f \in L(v)$. One has

$$P_k(f, \ldots, f) = k!\ a_k \times (-1)^k.$$

For a free module, the cardinality of a base is called its rank. We know that every f.g.-projective module over a local ring (not necessarily Noetherian) is free (cf. Northcott [12]). A f.g.-projective module over R is said to have a constant rank if the rank of a localized module $(v)p = v \otimes R_p$ does not depend on the prime ideal p of R.

Remark 3. If v, v_1 and v_2 are f.g.-projective modules with constant ranks, then Hom (v, R), $\Lambda^n v$, $v_1 + v_2$, $v_1 \otimes v_2$ and Hom (v_1, v_2) have constant ranks.

LEMMA 7. *If v is a f.g.-projective module with constant rank, then the map*

$$j : R \to L(v)$$

defined by $j(r)p = r \cdot p$ for $r \in R$, $p \in v$, is injective unless $v = 0$.

Proof. Let I be the kernel of j. One can see that the ideal I satisfies: $I \otimes R_p = 0$ for any prime ideal p of R. Thus we have $I = 0$ (see Northcott [12]).

Hereafter we shall identify R with the image of j in $L(v)$ if v has a constant rank.

Let k and R be commutative rings. R is called a *k-algebra* if a ring homomorphism of k into R is given which maps the unit in k to the unit in R. Thus any ring is a Z-algebra.

Let R be a k-algebra. A k-endomorphism X of R as a k-module is called a *k-derivation* of R if

$$X(r \cdot s) = X(r) \cdot s + r \cdot X(s) \text{ for any } r, s \text{ in } R.$$

The set of all k-derivations of R will be denoted by $D_k(R)$, or simply D. $D_k(R)$ forms a Lie algebra over k by $[X, Y]=X\cdot Y-Y\cdot X$ for $X, Y\in D_k(R)$. $D_k(R)$ forms also an R-module by

$$(sX)(r)=s\cdot X(r)$$

for $s, r\in R, X\in D_k(R)$.

For any R-module v, $\mathrm{End}_k(v)$ forms a Lie algebra over k by $[\alpha, \beta]=\alpha\cdot\beta-\beta\cdot\alpha$. Set

$$N_k(v)=\{\alpha \text{ in } \mathrm{End}_k(v): \text{For any } r\in R, \text{ there is some } s\in R \text{ such that } [\alpha, r]=s\}$$

where r and s denote the mappings of v.

Consider the case where $v=R$. By the definition, we see $D_k(R)\subset N_k(R)$. For any $\alpha\in N_k(R)$, set

$$X=\alpha-\alpha(1),$$

where $\alpha(1)$ stands for the mapping of R multiplying the element $\alpha(1)$ in R. One can check easily $X\in D_k(R)$. Thus we see $N_k(R)=R+D_k(R)$.

Example 1. Let v be a f.g.-free module with a base $e_1, \dots, e_n$. For each $X\in D_k(R)$, consider the endomorphism X of v defined by

$$X(\sum r_i e_i)=\sum(Xr_i)\cdot e_i.$$

We have $X\in N_k(v)$, and for any $r\in R$,

$$[X, r]=X(r).$$

Suppose that v is a f.g.-projective module with constant rank. By Lemma 6, the map $R\to L(v)$ is injective. For any $\alpha\in N_k(v)$, consider the k-endomorphism $\theta(\alpha)$ of R defined by

$$\theta(\alpha)(r)=[\alpha, r] \qquad r\in R$$

where the right hand side is considered an element in R.

LEMMA 8. *Let v be a f.g.-projective module over R with constant rank, and R be a k-algebra. We have*

i) $[N_k(v), N_k(v)]\subset N_k(v)$, *thus $N_k(v)$ is a Lie algebra over k.*

ii) $[N_k(v), L(v)]\subset L(v)$, *thus $L(v)$ is an ideal of $N_k(v)$.*

iii) *The mapping θ maps $N_k(v)$ into $D_k(R)$ and gives a homomorphism of k-Lie algebras.*

iv) $[r\alpha, \beta] = r[a, \beta] + (\theta(\alpha)r)\beta$, *for* $r \in R$, $\alpha, \beta \in N_k(v)$.

They can be verified from definitions.

PROPOSITION 1. *Let v be a f.g.-projective module over R with constant rank, and R be a k-algebra. We have an exact sequence*

$$0 \longrightarrow L(v) \longrightarrow N_k(v) \xrightarrow{\theta} D_k(R) \longrightarrow 0$$

of R-modules, and of k-Lie algebras.

Proof. It is clear that they are homomorphisms of R-modules and, at the same time, of k-Lie algebras. We see that

$$0 \to L(v) \to N(v) \to D$$

is exact from definitions. We shall show the exactness of $N(v) \to D \to 0$. Take another f.g.-projective module v' such that $v + v'$ is free. Let π be the projection of $v + v'$ onto v. As we have shown in Example 1, $N_k(v + v') \to D$ is onto. Let X be any element in $D_k(R)$. We have a β in $N_k(v + v')$ such that $[\beta, r] = X(r)$ for any $r \in R$. Consider the endomorphism α of v defined by

$$\alpha = \pi \cdot \beta.$$

Let $r \in R$. We have

$$\begin{aligned}[\alpha, r] &= \pi \cdot \beta \cdot r - r \cdot \pi \cdot \beta \\ &= \pi[\beta, r] \\ &= \pi \cdot X(r)\end{aligned}$$

since $[\pi, r] = 0$. This shows $[\alpha, r] = X(r)$ on v. Thus $N(v) \to D$ is onto. Q.E.D.

Section 2. Multiplications of alternating forms

Let M and L be arbitrary R-modules. We denote by $A^n(M, L)$ the set of all alternating n-forms on M with values in L. $A^n(M, L)$ forms an R-module in the usual way. One sees also that $A^n(M, \mathrm{L})$ is nothing but $\mathrm{Hom}(\Lambda^n M, L)$ where Λ^n is taken over the ring R. Set

$$A(M, L) = \sum A^n(M, L).$$

We shall define several types of multiplications on the graded module $A(M, L)$

in the case where L is just R or where L is a Lie algebra over R.

Let Q be a finite set. We denote the group of all permutations of Q by $G(Q)$, or simply by $G(n)$ if the number of elements in Q is n. The multiplication in the group will be viewed as composition of mappings. Let $(Q_j^i)\, i=1, \ldots, l;\ j=1, 2, \ldots, n_i$. By definition,

$$H(Q_1^1, \ldots, Q_{n_1}^1;\ Q_1^2, \ldots, Q_{n_2}^2;\ \ldots)$$

is the subgroup of $G(Q)$ consisting of those elements which, for each i map Q_j^i onto some Q_k^i. For example

$$H((1);\ (2);\ \ldots;\ (n)) = \text{the identity}$$
$$H((1, 2, \ldots, n)) = H((1), (2), \ldots, (n)) = G(n).$$

Let M be an R-module. To each element p in $G = G(1, 2, \ldots, n)$, we associate a mapping, denoted by the same letter $p : (M, \ldots, M) \to (M, \ldots, M)$ by

$$p(a_1, \ldots, a_n) = (a_{p(1)}, \ldots, a_{p(n)}).$$

LEMMA 1. *Let A be an n-linear mapping of M into another R-module. Suppose that, for a certain subgroup H of $G(n)$, we have*

$$\operatorname{sign}(p)\cdot A\cdot p = A$$

for any p in H. Then, the map $[A]$, defined by

$$[A](a_1, \ldots, a_n) = \sum \operatorname{sign}(p)\cdot A(p(a_1, \ldots, a_n))$$

where p runs through the right cosets of G mod. H, is skew-symmetric.

Proof. First note that the value $\operatorname{sign}(p)\cdot A(p(a_1, \ldots, a_n))$ does not depend on the choice of representative p of a coset because of the assumption. Let q be in $G(n)$. We have

$$\begin{aligned}
&[A](q(a_1, \ldots, a_n)) \\
&\quad = \sum \operatorname{sign}(p)\cdot A(p\cdot q(a_1, \ldots, a_n)) \\
&\quad = \operatorname{sign}(q)\cdot \sum \operatorname{sign}(p\cdot q)\cdot A(p\cdot q(a_1, \ldots, a_n))
\end{aligned}$$

where p runs through the cosets, $p\cdot q$ runs through the cosets. Thus $[A]$ is skew-symmetric.

a) Interior product. Let M and L be arbitrary R-modules. For each a in M, we define a map

$$i_a : A^n(M, L) \to A^{n-1}(M, L)$$

by

(a.1) $$(i_a A)(b_1, \ldots, b_{n-1}) = A(a, b_1, \ldots, b_{n-1})$$

and

$$i_a A = 0 \text{ if } n = 0.$$

Remark. For A in $A^n(M, L)$, we have

$$i_a A = 0 \text{ for all } a \text{ in } M \text{ if and only if } A = 0.$$

b) Exterior product $\wedge$. Let M be an arbitrary R-module. Consider $A(M, R) = \sum A^n(M, R)$. We define, for A in $A^k(M, R)$ and B in $A^l(M, R)$, a $(k+l)$-form $A \wedge B$ by

(b.1) $$(A \wedge B)(a_1, \ldots, a_{k+l}) = \sum \mathrm{sign}(p) \cdot A(q_{p(1)}, \ldots, a_{p(k)}) \cdot B(a_{p(k+1)}, \ldots, a_{p(k+l)})$$

where p runs through the cosets of $G(1, \ldots, k+l)$ modulo $H((1, \ldots, k); (k+1, \ldots, k+l))$.

From Lemma 1, we see $A \wedge B \in A^{k-l}(M, R)$. Clearly $(A, B) \to A \wedge B$ is R-bilinear. Moreover, from the definition, we have

(b.2) $$A \wedge B = (-1)^{kl} B \wedge A.$$

We have, for any A, B, C,

(b.3) $$(A \wedge B) \wedge C = A \wedge (B \wedge C).$$

To show this, we shall show first

(b.4) $$((A \wedge B) \wedge C)(a_1, \ldots, a_n) = \sum \mathrm{sign}(p) A(p(1), \ldots, p(k)) B(p(k+1), \ldots, \ ) C(\ , \ldots, p(n))$$

where p runs through the cosets of $G(n)$ $H((1, \ldots, k);$ $(k+1, \ldots, k+l);$ $(\ , \ldots, n))$ and where $n = \deg. A + \deg. B + \deg. C$, and i stands for a_i.

Proof.

$$((A \wedge B) \wedge C)(1, \ldots, n) = \sum \mathrm{sign}(p)(A \wedge B)(p(1), \ldots, p(k+l) \cdot C(p(k+l+1), \ldots, p(n)),$$

where p runs through the cosets of $G(n)$ modulo $H((1, \ldots, k+l);$ $(\ , \ldots, n))$. We have

$$(A\wedge B)(p(1), \ldots, p(k+l))$$
$$= \sum \operatorname{sign}(q)\cdot A(qp(1), \ldots, q\cdot p(k))B(q\cdot p(k+1)), \ldots, qp(n))$$

where q runs through the cosets of $G(p(1), \ldots, p(k+l))$ modulo $H((p(1), \ldots, p(k)); (\ , \ldots, p(n)))$. One sees that if q' runs through $(k+1, \ldots, k+l); (\ , \ldots, n))$, then $q=pq'p^{-1}$ runs through the required cosets. Now it is easy to see the rest of the proof.

Now we can see that the same formula holds for $A\wedge(B\wedge C)$, and the results are the same. Thus we have (b.3).

Remark. The same formulas as (b.4) can be seen for $A_1\cdots A_n$. Now by (b.3), we see that $A(M, R)=\sum A^n(M, R)$ forms a graded ring over R by the multiplication Λ.

(b.5) For any a in M, and for A in $A^n(M, R)$, B in $A^k(M, R)$, we have
$$i_a(A\wedge B)=i_a(A)\wedge B+(-1)^n A\wedge i_a(B).,$$

i.e. i_a is a derivation of order -1.

The proof is just a direct verification from definitions.

c) n-th power A^n. Let M be an R-module, and A be in $A^l(M, R)$. When deg. $A=l$ is even and positive we can define an n-th power of A for any n. By definition, A^n is an nl-form defined by

(c.1)
$$A^n(a_1, \ldots, a_{nl})$$
$$= \sum \operatorname{sign}(p)\cdot A(a_{p(1)}, \ldots, a_{p(l)})A(a_{p(l+1)}, \ldots)\cdots A(\cdots a_{p(nl)})$$

where p runs through the cosets of $G(1, \ldots, nl)$ modulo $H((1, \ldots, k), (l+1, \ldots, 2l), \ldots, (\ , \ldots, nl))$.

By Lemma 1, we see easily that A^n is skew-symmetric.

Suppose A and B are in $A^l(M, R)$ where l is positive and even. We have:

(c.2) $$n!\ A^n=A\wedge\cdots\wedge A \quad (n\text{-times})$$

(c.3) $$(A+B)^n=\sum A^k\wedge B^{n-k}$$

(c.4) $$i_a(A^n)=i_a(A)\wedge A^{n-1} \text{ for any } a\in M.$$

They can be verified easily.

d) [,]. Let M be an R-module, and L be a Lie algebra over R. For A in $A^k(M, L)$, B in $A^l(M, L)$, we define a $(k+l)$-form $[A, B]$ in $A^{k+l}(M, L)$ by (d.1) $$[A, B](a_1, \dots, a_{k+l}) = \sum \operatorname{sign}(p)[A(a_{p(1)}, \dots, a_{p(k)}), B(, \dots, a_{p(k+l)})]$$

where p runs through the cosets of $G(1, \dots, k+l)$ modulo $H((1, \dots, k); (k+1, \dots, k+l))$.

We have:

(d.2) $$[A, B] = (-1)^n[B, A]$$

(d.3) $$[A, [B, C]] = [[A, B], C] + (-1)^n[B, [A, C]].$$

where $n = \deg. A \times \deg. B$.

(d.4) $$i_a[A, B] = [i_aA, B] + (-1)^k[A, i_aB]$$

for any $a \in M$, where k is deg. A.

(d.2) and (d.4) follows from definitions. (d.3) is a consequence of Jacobi's identity in L.

e) $\tilde{A}$. Let M be an R-module and L be a Lie algebra over R. For A in $A^n(M, L)$ with odd degree n, we define a $2n$-form $\tilde{A}$ in $A^{2n}(M, L)$ by

(e.1) $$\tilde{A}(a_1, \dots, a_{2n}) = \sum \operatorname{sign}(p)[A(a_{p(1)}, \dots, a_{p(n)}), A(a_{p(n+1)}, \dots, a_{p(2n)})]$$

where p runs through the cosets of $G(1, \dots, 2n)$ modulo $H((1, \dots, n), (n+1, \dots, 2n))$.

For example, if A is a 1-form in $A^1(M, L)$, then

$$\tilde{A}(a, b) = [A(a), A(b)].$$

Later we shall need $\tilde{A}$ just for 1-form A.

Let A be in $A^n(M, L)$ with odd degree.

We have:

(e.2) $$2 \cdot \tilde{A} = [A, A]$$

(e.3) $$i_a\tilde{A} = [i_aA, A] \text{ for any a in } M.$$

(e.4) $$[\tilde{A}, A] = 0$$

(e.5) $$[A, [A, B]] = [\tilde{A}, B].$$

(e.2) and (e.3) follow from definitions. (e.4) and (e.5) are consequences of Jacobi's identity in L.

Remark. If A is an element in $A^n(M, L)$ with a positive even degree, then we have always $[A, A]=0$.

f) Let M and L be arbitrary R-modules, To any n-linear mapping P of $L\otimes\cdots\otimes L$ into R, we associate a mapping of $A(M, L)\otimes\cdots\otimes A(M, L)$ into $A(M, R)$. Let A_i be in $A^{k_i}(M, L)$ and m be $\sum k_i$. Define an n-form $P(A_1, \ldots, A_n)$ in $A^m(M, R)$ by

$$\text{(f.1)}\qquad \begin{aligned} &P(A_1, \ldots, A_n)(a_1, \ldots, a_m) \\ &\quad =\sum \operatorname{sign}(p)\, P(A_1(a_{p(1)}, \ldots), A_2(\ , \ldots), \ldots,) \end{aligned}$$

where p runs through the cosets of $G(1, \ldots, m)$ modulo $H((1, \ldots, k_1); (k_1+1, \ldots, k_1+k_2); \ldots)$.

Now suppose that P is symmetric. We define a new form $P(A^n)$ in $A^{nk}(M, R)$ for A in $A^k(M, L)$ with even k, which is corresponding to A^n in c). Set

$$\text{(f.2)}\qquad P(A^n)(a_1, \ldots)=\sum \operatorname{sign}(p) P(A(a_{p(1)}, \ldots), \ldots)$$

where p runs through the cosets of $G(1, \ldots, nk)$ modulo $H((1, \ldots, k), (k+1, \ldots, 2k), \ldots)$.

Since P is symmetric and A is of even degree, $P(A^n)$ is well defined by Lemma 1.

Taking (f. 1) and (f. 2) together, we can define $P(A_1^{n_1}, \ldots, A_k^{n_k})$ where $\sum n_i=n$ and deg. A_i is even if $n_i\geq 2$. The element p of the permutation group runs through the cosets of $G(\sum l_i n_i)$ modulo $H((1, \ldots, l_1), (l_1+1, \ldots, 2l_1), \ldots, (\ldots, n_1 l_1); (n_1 l_1+1, \ldots, n_1 l_1+l_2), \ldots, (\ldots, n_1 l_1+n_2 l_2); \ldots)$.

We have

$$\text{(f.3)}\qquad P((A+B)^n)=\sum P(A^k, B^{n-k})$$

$$\text{(f.4)}\qquad i_a\cdot P(A_1^{n_1}, \ldots, A_k^{n_k})=\sum_i P(A_1^{n_1}, \ldots, i_a A_i, A_i^{n_i-1}, \ldots, A_k^{n_k}).$$

Remark. We have defined $A^n(M, R)$ to be the set of all alternating n-forms on M. Set $M^*=\operatorname{Hom}(M, R)$. We have a canonical mapping J of $\Lambda^n M^*$ into $A^n(M, R)$ defined by

$$J(f_1, \ldots, f_n)(a_1, \ldots, a_n)=\det.(f_i(a_j)).$$

J is well defined and gives and gives an R-homomorphism. It is known that if M is a f.g.-projective module with constant rank, then J is an isomorphism. One can see that the multiplication $\wedge$ in $A(M, R)$ is compatible with that in ΛM^*. Thus if M is a f.g.-projective module with constant rank, then our graded ring $A(M, R)$ is nothing but ΛM^*.

Section 3. De Rham cohomology

Let k be a commutative ring and R a commutative k-algebra (see Section 1) throughout this section.

Definition. A k-Lie algebra M together with an R-module structure compatible with the k-module structure and with a mapping θ of M into $D_k(R)$ is called a *Lie d-algebra* if

i) θ is a homomorphism of k-Lie algebras and a homomorphism of R-modules,

ii) $[ra, b]=\theta(b)(r)\cdot a+r\cdot[a, b]$ for any $r\in R$, $a, b\in M$ (cf. Palais [13]).

Example 1. For any f.g.-projective R-module v with constant rank, $N_k(v)$ is a Lie d-algebra over R with $\theta : N_k(v)\rightarrow D$, defined by $\theta(a)r=[a, r]$ (see Lemma 8 and Proposition 1 in Section 1).

Example 2. For any k-algebra R, the set $D_k(R)$ of derivations is itself a Lie d-algebra over R with $\theta=$ identity map on $D_k(R)$.

Definition. Let M be a Lie d-algebra over R iwht $\theta : M\rightarrow D_k(R)$, and let v be an R-module. A mapping $t : M\rightarrow N_k(v)$ is called a *representation* of M on v if

i) t is a homomorphism of k-Lie algebras and a homomorphism of R-modules,

ii) $[t(a), r]=\theta(a)\cdot r$ in $\mathrm{End}_k(v)$, for any a in M, r in R.

Leter, we shall consider a representation of M only on f.g.-projective modules with constant rank.

Example 3. Let v be a f.g.-projective R-module with constant rank, and $M=N_k(v)$. We shall give a representation t of $N_k(v)$ on $L(v)$. For a in $N_k(v)$, consider a mapping of $L(v)$ into itself defined by

$$t(a) : f\rightarrow[a, f].$$

As $[a, f]$ is in $L(v)$ (see Lemma 8, Section 1), $t(a)$ is in $N_k(L(v))$. In fact,

$$\begin{aligned}[t(a), r](f)&=t(a)\cdot r(f)-r\cdot t(a)(f)\\&=[a, r\cdot f]-r\cdot[a, f]=[a, r]\cdot f\end{aligned}$$

One can see easily that t gives a representation of $N_k(v)$ on $L(v)$.

Example 4. Let M be $D_k(R)$ itself. Each X in $D_k(R)$ gives a mapping $X: R \to R$, which belongs to $N_k(R)$. Setting $\theta = id.$, the identity mapping of $D_k(R)$ gives a representation of $D_k(R)$ on R.

Example 5. Suppose that an R-module v is free with a base $e_1, \ldots, e_n$. Then, for each X in $D_k(R)$, consider the k-endomorphism X' of v, defined by

$$X'(\sum r_i e_i) = \sum (X r_i) e_i.$$

The mapping $X \to X'$ gives a representation of $D_k(R)$ on v with $\theta = id.$

Let M be a Lie d-algebra over R with map $\theta: M \to D_k(R)$. Suppose a representation t of M on v is given. We shall derive the cohomology group of the graded module $A(M, v) = \sum A^n(M, v)$.

Lie derivative ∂_a. For any a in M, we define the mapping $\partial_a: A^n(M, v) \to A^n(M, v)$ as follows: for $A \in A^n(M, v)$ and $b_1, \ldots, b_n \in M$,

$$(\partial_a A)(b_1, \ldots, b_n) = t(a)(A(b_1, \ldots, b_n)) - \sum_i A(b_1, \ldots, [a, b_i], \ldots, b_n).$$

One can verify easily that $(\partial_a A)(b_1, \ldots, b_n)$ is skew-symmetric with respect to $b_1, \ldots, b_n$ and that $\partial_a A$ is R-linear on each b_i. The latter follows from the property $[t(a), r] = \theta(a) r$ (see the complete proof in Palais [13]). ∂_a is called the Lie derivative of A with respect to a.

We have:

LEMMA 1. i) $[\partial_a, \partial_b] = \partial_{[a,b]}$,
ii) $[\partial_a, i_b] = i_{[a,b]}$

for any a, b in M.

The proofs are not so hard and can be found in Palais [13], Nijenhuis-Froelicher [4].

Coboundary operator d. Let A be an n-form in $A^n(M, v)$. We define

$$(dA)(a_1, \ldots, a_{n+1}) = \sum (-1)^{i+1} t(a_i)(A(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1})) + \sum_{i<j} (-1)^{i+j} A([a_i, a_j], a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{n+1})$$

for $a_1, \ldots, a_{n+1}$ in M.

One sees that $dA(a_1, \ldots, a_{n+1})$ is skew-symmetric with respect to $a_1, \ldots, a_{n+1}$ and that dA is R-linear on each a_i (see Palais [13]). d is called the coboundary operator on $A(M, v)$.

LEMMA 2. *For any a in M, we have*

$$\partial_a = i_a \cdot d + d \cdot i_a.$$

LEMMA 3. *For any a in M, we have*

$$\partial_a \cdot d = d \cdot \partial_a.$$

LEMMA 4. $d^2 = 0$.

Lemma 2 follows from the definition of d using Lemma 1.

Proof of Lemma 3. By induction on degree of forms. Suppose A is in $A^0(M, v) = v$. We have

$$\begin{aligned}
&(d\partial_a A)(b) = t(b)(\partial_a A) = t(b)(t(a)(A)),\\
&(\partial_a dA)(b) = t(a)(dA(b)) - dA([a, b])\\
&\quad = t(a)(t(b)(A)) - t([a, b])(A) = t(b)(t(a)(A)),
\end{aligned}$$

since t is a representation.

Suppose it is true on forms with less degree than n. It suffices to show that on $A^n(M, v)$ we have $i_b d\partial_a = i_b \partial_a d$ for any b in M, where deg. $A = n$. We have, on $A^n(M, v)$,

$$i_b \partial_a d = [i_b, \partial_a] d + \partial_a i_b d = -i_{[a,b]} d + \partial_a \cdot \partial_b - \partial_a d i_b,$$

and since $i_b A$ is an n-1-form for $A \in A^n(M, v)$ this is equal to, by our assumption of induction,

$$\begin{aligned}
&= -i_{[a,b]} d + \partial_a \cdot \partial_b - d\partial_a i_b = -i_{[a,b]} d + [\partial_a, \partial_b] + \partial_b \cdot \partial_a - d\partial_a i_b\\
&= \partial_{[a,b]} - i_{[a,b]} d + \partial_b \cdot \partial_a - d[\partial_a, i_b] - d i_b \partial_a\\
&= d \cdot i_{[a,b]} + (\partial_b - d i_b) \cdot \partial_a - d i_{[a,b]} = i_b \cdot d \cdot \partial_a, \qquad \text{Q.E.D.}
\end{aligned}$$

Proof of Lemma 4. Again by induction on degrees. On $A^0(M, v)$, it is easy. Suppose $d^2 = 0$ on $A^{n-1}(M, v)$. It suffices to show

$$i_a d^2 = 0$$

on $A^n(M, v)$.

$$\begin{aligned}
i_a d^2 &= i_a dd = \partial_a \cdot d - d i_a d\\
&= d\partial_a - d\partial_a + d d i_a = d d i_a
\end{aligned}$$

Since $i_a A \in A^{n-1}(M, v)$, for $A \in A^n(M, v)$, we have $d^2 i_a A = 0$, Q.E.D.

Thus we can derive the cohomology group of the representation of M on v, which will be denoted by

$$H(M,\ v)=\sum H^n(M,\ v).$$

For any Lie d-algebra M over R, we have the canonical representation of M on R, by $\theta : M\to D_k(R)\subset N_k(R)$.

Definition. For a k-algebra R, the cohomology group $H(D_k(R), R)$ of $D_k(R)$ with respect to the canonical representation on R will be called the ***de Rham cohomology*** of the ring R.

Example 6. If k is the field of complex numbers, and R is the ring of complex-valued differentiable functions on a paracompact finitedimensional manifold, then the de Rham cohomology of R is nothing but the topological cohomology group of the manifold with coefficients in complexes (see Weil [16], de Rham [14]).

Let M be a Lie d-algebra over R with $\theta : M\to D_k(R)$. We consider $A(M, R)$ with respect to the canonical representation of M on R by θ. As we have defined in Section 2, $A(M, R)$ has a multiplicative structure Λ.

LEMMA 5. *For any A, B in $A(M, R)$, we have*

$$\partial_a(AB)=\partial_a A\wedge B+A\wedge\partial_a B,\ d(AB)=dA\wedge B=dA\wedge B+(-1)^n A\wedge dB$$

where $n=\deg. A$.

We have defined A^n for A in $A^m(M, R)$ with positive even degree m in Section 2. We have

LEMMA 6. $\partial_a(A^n)=\partial_a(A)\wedge A^{n-1}$, $d(A^n)=d(A)\wedge A^{n-1}$.

Lemma 5 can be proved as follows: the formula for ∂_a can be proved by the induction of deg. $A+$deg. B in the same way as Lemma 3. Then using $\partial_a=di_a+i_a d$ and (b. 5), one has the formula for d.

To prove Lemma 6, one must first show the formula for ∂_a, which follows from the definitions. Then, using $\partial_a=di_a+i_a d$ and (c. 4), we get the formula for d.

Section 4. Invariant forms

Throughout this section, M is a Lie d-algebra over R with map $\theta : M\to D_k(R)$, L is a Lie algebra over R and t is a representation of M on L (see Example 3 in Section 3). d and ∂_a are defined on $A(M, L)\sum A^n(M, L)$. In Section 2, we have defined a form $[A, B]$ for A, B in $A(M, L)$, and $\tilde{A}$ for A in $A(M, L)$ with odd degree.

LEMMA 1. *For A, B in $A(M, L)$, we have*

$$\partial_a[A, B]=[\partial_a A, B]+[A, \partial_a B],$$
$$d[A, B]=[dA, B]+(-1)^n[A, dB]$$

where $n=\deg. A$.

LEMMA 2. *For A in $A(M, L)$ with odd degree, we have*

$$\partial_a\tilde{A}=[\partial_a A, A],\ d\tilde{A}[dA, A].$$

In both cases, the formulae for ∂_a can be verified from definitions. Then using $\partial_a=di_a+i_a d$ and formulae for i_a of (d. 3) and (e. 3) in Section 2, one has the formulae for d.

We say an R-linear form $P: L\otimes\cdots\otimes L\to R$ is *L-inariant* if

$$\textstyle\sum_i P(f_1, \ldots, f_{i-1}, [g, f_i], \ldots, f_n)=0$$

for any $f_1, \ldots, f_n$ and g in L.

As we have shown in Section 1, the symmetric form P_n defined in Section 1 is $L(v)$-invariant.

LEMMA 3. *Suppose P is an L-inariant symmetric n-form on L. Then, for any 1-form B in $A^1(M, L)$ we have*

$$\textstyle\sum_i P(A_1^{n_1}, \ldots, A_{i-1}^{n_{i-1}}, [A_i, B], A_i^{n_i-1}, \ldots, A_k^{n_k})=0$$

where $\sum n_i=n$, and $\deg. A_i$ *is even.*

LEMMA 4. *Suppose P is an L-inariant symmetric n-form on L. Then, for any 1-form B in $A^1(M, L)$ and for any 2-form A in $A^2(M, L)$, we have*

$$P(A^k, [A, B], (\tilde{B})^l)=0$$

where $k+l+1=n$. In particular, we have

$$P((\tilde{B})^n)=0,\ P(A^{n-1}, [A, B])=0.$$

Lemma 4 is a direct consequence of Lemma 3 together with the fact the $[\tilde{B}, B]=0$ in Section 2.

Proof of Lemma 3. We shall give here a proof of

$$P([A, B], A^{n-1})=0$$

for 2-form A. The geueral case can be proved in the same way. $P([A, B], A^{n-1})$ is a $2n+1$-form in $A(M, L)$. Take $a_0, \ldots, a_{2n+1}$ in M. We use i instead of a_i. $P([A, B], A^{n-1})$ has a form:

$$P([A, B], A^{n-1})(0, \ldots, 2n+1)$$
$$= \Sigma_k \cdot \Sigma \pm P([A(i_1, i_2), B(k)], A(i_3, i_4), \ldots)$$

We shall show that for each k, the sum of terms contianing $B(k)$ is zero Suppose $k=0$. The terms containing $B(0)$ is given by

$$-\sum \mathrm{sign}(p) P([B(0), A(p(1), p(2))], A(p(3), p(4)), \ldots)$$

where p runs through the cosets of $G(1, 2, \ldots, 2n)$ modulo $H((1, 2); (3, 4), \ldots, (2n-1, 2n))$. This shows that the above is the sum of the following terms

$$\Sigma_i P(([B(0), A(j_{2i-1}, j_{2i})], A(j_1, j_2), \ldots, A_{(j_{2i-1}, j_{2i})}, \ldots),$$

which is zero since P is symmetric and L-invariant.

Definition. An R-linear map $P: L \otimes \cdots \otimes L \to R$ is called *M-inariant* with respect to the representation t if

$$\theta(a)(P(f_1, \ldots, f_n)) = \Sigma_i P(f_1, \ldots, (t(a) f_i), \ldots, f_n)$$

for any $a \varepsilon M$, and for any $f_1, \ldots, f_n$ in L.

Let be a f.g.-projective module with constant rank. Then $N_k(v)$ has a canonical representation on $L(v)$ (see Example 3 in Section 3).

PROPOSITION 1. *The symmetric form P_n on $L(v)$ defined in Section 1 is $N(v)$-inariant.*

Proof. First assume that v is free with a base $e_1, \ldots, e_n$. We have

$$N_k(v) = L(v) + D'$$

where D' consists of those elements X' such that

$$X'(\Sigma r_i e_i) = \Sigma (X r_i) e_i$$

for X in D. (see Example 5 in Section 3 and Proposition 1 in Section 1.)

Suppose a is in $L(v)$. Then $\theta(a)R = 0$, and $t(a)f = [a, f]$ for f in $L(v)$. Thus in this case, we have

$$\Sigma_i P_n(f_1, \ldots, [a, f_i], \ldots, f_n) = 0$$

by Lemma 5 in Section 1. Now for the element X' in D'

$$t(X')(f) = [X', f] = \Sigma(Xr_{ij}) E_{ij}$$

where $f = \Sigma r_{ij} E_{ij}$. Thus $t(X')$ operates on $L(v)$ by taking the derivative of each coefficient. Since X is a derivation, we see

$$\theta(X')\cdot P_n(f_1, \ldots, f_n) = \Sigma_i P_n(f_1, \ldots, t(X')f_i, \ldots, f_n)$$

by Lemma 6 in Section 1.

Now for an arbitrary v, take another v' such that $v+v'$ is free and finite. Denote by π the projection of $v+v'$ onto v. By Lemma 5, for any a in $N_k(v)$ we can find b in $N_k(v+v')$ such that

$$a = \pi\cdot b\cdot\pi \text{ and } \theta(a) = \theta(b).$$

We have

$$\theta(b)\cdot P_n(g_1, \ldots, g_n;\ v+v') = \Sigma_i P_n(g_1, \ldots, t(b)g_i, \ldots, g_n;\ v+v')$$

For any $f_1, \ldots, f_n$ in $L(v)$, consider

$$g_i = \pi\cdot f_i\cdot\pi.$$

g_i is in $L(v+v')$. We have

$$\theta(a) P_n(f_1, \ldots, f_n;\ v) = \theta(b)\cdot P_n(g_1, \ldots, g_n;\ v+v').$$

Since $[b, g_i] = \pi[a, f_i]$, we have also

$$P_n(f_1, \ldots, [a, f_i], \ldots, f_n;\ v) = P_n(g_1, \ldots, [b, g_i], \ldots, g_n;\ v+v') \quad \text{Q.E.D.}$$

LEMMA 5. *For any a in $N_k(v)$, there exists b in $N_k(v+v')$ such that*

$$a = \pi\cdot b\cdot\pi,\ \theta(a) = \theta(b)$$

where π denotes the projection of $v+v'$ to v.

Proof. Let c be an element in $N(v+v')$ such that $\theta(c) = \theta(d)$. Set

$$b' = \pi\cdot c\cdot\pi + (1-\pi)\cdot c\cdot(1-\pi).$$

We see easily $b' \varepsilon N(v+v')$ and $\theta(b') = \theta(c)$. Set, $a' = \pi b'\pi - a$. It is just a direct verification to see $a' \varepsilon L(v)$. Set

$$b = b' - \pi a'\pi$$

b satisfies the required properties.

Suppose that P is an M-invariant symmetric n-form on L with respect to t.

LEMMA 6. *Let $A_1, \ldots, A_k$ be in $A(M, L)$ with even degree and $n = \Sigma n_i$, where $n_i \geq 2$. We have*

$$\partial_a P(A_1^{n_1}, \ldots, A_k^{n_k}) = \Sigma_i P(A_1^{n_1}, \ldots, \partial_a A_i, A_i^{n_i-1}, \ldots, A_k^{n_k}),\ dP(A_1^{n_1}, \ldots, A_k^{n_k})$$
$$= \Sigma_i P(A_1^{n_1}, \ldots, dA_i, A_i^{n_i-1}, \ldots, A_k^{n_k})$$

The proofs are similar to those of Lemma 2.

Section 5. Connections in projective modules

Let R be a k-algebra, v a f.g.-projective module over R with constant rank. By Proposition 1 in Section 1, we have the exact sequence

$$0 \longrightarrow L(v) \longrightarrow N_k(v) \xrightarrow{\theta} D_k(R) \longrightarrow 0.$$

Definition. A *connection* in v is a mapping Δ of $D_k(R)$ into $N_k(v)$ such that

i) Δ is a homomorphism of R-modules.

ii) $\theta \cdot \Delta$ is the identity mapping on $D_k(R)$.

Defining the connection in this way originates with Nomizu [11] in differential geometry for tangent bundles.

Set

$$\omega = 1 - \Delta \cdot \theta$$

on $N_k(v)$, where 1 stands for the identity mapping on $N_k(v)$. ω has the following properties:

i) ω is an R-homomorphism of $N_k(v)$ into $L(v)$,

ii) $\omega(f) = f$ for any f in $L(v)$.

ω is called the connection form of Δ.

Conversely, suppose that such an ω satisfying i) and ii) is given. We see that the kernel of ω is isomorphic with $D_k(R)$ by the map θ. For any X in $D_k(R)$, we have a unique a in $N_k(v)$ such that $\theta(a) = X$, and $\omega(a) = 0$. Defining $\Delta(X) = a$, we get a connection, whose connection form is just ω. Thus giving such an ω is equivalent to defining a connection.

The existence of a connection. First let v be a free module with a base $e_1, \ldots, e_n$. Define a mapping Δ of $D_k(R)$ into $N_k(v)$ by

$$\Delta(X)(\Sigma r_i e_i) = \Sigma(X r_i) e_i$$

This gives a connection in, which will be called the trivial connection in v with respect to the base $e_1, \dots, e_n$.

Now let v be an arbitrary f.g.-projective module with constant rank, and v' another module such that $v+v'$ is free with a base $e_1, \dots, e_n$. Denote by π the projection of $v+v'$ onto v. We have the connection Δ in $v+v'$ with respect to the base $e_1, \dots, e_n$, where $\Delta : D_k(R) \to N_k(v+v')$. Set

$$\Delta'(X) = \pi \cdot \Delta(X)$$

for X in $D_k(R)$. We have

$$[\Delta'(X), r] = \pi \cdot [\Delta(X), r] = \pi \cdot (Xr).$$

Thus $\Delta'(X)$ is in $N_k(v)$ and $\theta\Delta'(X) = X$, i.e. Δ' gives a connection in v.

Suppose a connection Δ is given in a f.g.-projective module v with constant rank. The element in the kernel of ω, i.e. the element in the image of Δ is called horizontal, using the terminology in differential geometry. We set

$$h = \Delta \cdot \theta = 1 - \omega.$$

h is an R-homomorphism of $N_k(v)$ into itself. We define a mapping h' : $A^n(N_k(v), w) \to A^n(N_k(v), w)$ for any R-module w, by

$$(h'A)(a_1, \dots, a_n) = A(ha_1, \dots, ha_n)$$

for $a_1, \dots, a_n$ in $N_k(v)$.

We define a form A in $A^n(N_k(v), w)$ to be basic if A satisfies:

$$A(a_1, \dots, a_n) = 0 \text{ whenever some } a_i \text{ is in } L(v).$$

Because of the exactness of $0 \to L(v) \to N_k(v) \to D_k(R) \to 0$, one sees that a basic form in $A^n(N_k(v), w)$ comes from a certain form in $A^n(D_k(R), w)$.

LEMMA 1. *A form A in $A^n(N_k(v), w)$ is basic if and only if $h'A = A$.*

Proof. It is obvious that if $h'A = A$, then A is basic, since $h(L(v)) = 0$. Suppose A is basic. Then

$$\begin{aligned}(h'A)(a_1, \dots, a_n) - A(a_1, \dots, a_n) &= A(ha_1, \dots, ha_n) - A(a_1, \dots, a_n) \\ &= \Sigma A(a_1, \dots, a_{i-1}, ha_i - a_i, ha_{i+1}, \dots, ha_n)\end{aligned}$$

Since $ha - a$ is in $L(v)$ for any $a \varepsilon N_k(v)$, we have the result.

As we have shown in Section 3, $N_k(v)$ is a Lie d-algebra over R, and has the canonical representation on $L(v)$ (See Example 3, Section 3) and the canonical representation on R (see Examples 1 and 4, Section 3). Thus we have the coboundary operator d on $A(N_k(v), L(v)) = \Sigma A^n(N_k(v), L(v))$, and on $A(N_k(v), R) = \Sigma A^n(N_k(v), R)$.

Definition. For any form A in $A^n(N_k(v), L(v))$ (or $A^n(N_k(v), R)$), we define the covariant derivative DA with respect to a given connection by $DA = h'dA$, i.e.

$$(DA)(a_1, \ldots, a_{n+1}) = (dA)(ha_1, \ldots, ha_{n+1})$$

for $a_1, \ldots, a_{n+1}$ in $N_k(v)$.

Definition. For a given connection $\varDelta$, the curvature form K of $\varDelta$ is defined by

$$K = D\omega$$

where ω is the connection form of $\varDelta$.

Since ω belongs to $A^1(N_k(v), L(v))$, K is a 2-form in $A^2(N_k(v), L(v))$.

LEMMA 2. *For any X, Y in $D_k(R)$, we have*

$$K(\varDelta X, \varDelta Y) = \varDelta([X, Y]) - [\varDelta X, \varDelta Y]$$

Proof.

$$\begin{aligned} K(\varDelta X, \varDelta Y) &= (D\omega)(\varDelta X, \varDelta Y) = (d\omega)(h\varDelta X, h\varDelta Y) = (d\omega)(\varDelta X, \varDelta Y) \\ &= [\varDelta X, \omega(\varDelta Y)] - [\varDelta Y, \omega(\varDelta X)] - \omega([\varDelta X, \varDelta Y]) = -\omega([\varDelta X, \varDelta Y]) \\ &= -[\varDelta X, \varDelta Y] + \varDelta\cdot\theta([\varDelta X, \varDelta Y]) = \varDelta([X, Y]) - [\varDelta X, \varDelta Y], \end{aligned}$$

where we have used the fact that θ is a homomorphism of Lie algebras and $\theta\cdot\varDelta = id$.

Note that Lemma 2 explains the role of the curvature form, i.e. K measures the depature from being a homomorphism of Lie algebras.

When the curvature form K is zero, the connection is called locally trivial. Since $h'\cdot h' = h'$, the covariant derivative of any form is basic by Lemma 2. Thus K is zero if $K(\varDelta X, \varDelta Y) = 0$ for any X, Y in $D_k(R)$.

Example. When v is free the trivial connection with respect to a certain base gives $K = 0$.

If the structure group of a differentiable vector bundle is reduced to a discrete subgroup, then it has a locally trivial connection in the sense of differential geometry.

LEMMA 3. *For the curature form K, we have*

i) $K = d\omega - \tilde{\omega}$,

ii) $DK = 0$.

LEMMA 4. *For any basic form A in $A^n(N_k(v), L(v))$, we have*

i) $DA = dA - [\omega, A]$,

ii) $D^2 A = -[K, A]$.

Proof of i) in Lemma 3. Suppose a, b in $N_k(v)$ are horizontal.

$$K(a, b) = (d\omega(ha, hb) = (d\omega)(a, b),$$
$$\tilde{\omega}(a, b) = [\omega(a), \omega(b)] = 0.$$

Thus we have i) for horizontal a, b. Next suppose a is horizontal and b is in $L(v)$.

$$K(a, b) = (d\omega(ha, hb) = 0,\ \tilde{\omega}(a, b) = [\omega(a), \omega(b)] = 0,$$
$$(d\omega)(a, b) = [a, \omega(b)] - [b, \omega(a)] - \omega([a, b]) = [a, b] - [a, b] = 0,$$

since $[N_k(v), L(v)] \subset L(v)$. Suppose a and b are in $L(v)$. We have

$$K(a, b) = 0,\ \tilde{\omega}(a, b) = [\omega(a), \omega(b)] = [a, b],$$
$$(d\omega)(a, b) = [a, \omega(a)] - \omega([a, b]) = [a, b].$$

Since $N_k(v)$ is generated by $L(v)$ and horizontal elements, we have i) for any a, b.

Proof of i) in Lemma 4. Let $a_1, \ldots, a_{n+1}$ be horizontal in $N_k(v)$. Wehave

$$(DA)(a_1, \ldots, a_{n+1}) = (dA)(a_1, \ldots, a_{n+1}),$$
$$[\omega, A](a_1, \ldots, a_{n+1}) = 0.$$

Suppose a_1 is in $L(v)$ and $a_2, \ldots, a_{n+1}$ are arbitrary in $N_k(v)$. We have

$$(DA)(a_1, \ldots, a_{n+1}) = 0,$$
$$[\omega, A](a_1, \ldots, a_{n+1}) = \Sigma(-1)^{i+1}[\omega(a_i), A(a_1, \ldots, a_i, \ldots, a_{n+1})]$$
$$= [\omega(a_1), A(a_2, \ldots, A_{n+1})],$$
$$(dA)(a_1, \ldots, a_{n+1}) = \Sigma(-1)^{i+1}[a_i, A(a_1, \ldots, a_i, \ldots, a_{n+1})]$$
$$+ \Sigma(-1)^{i+j}A(a_i, a_j], a_1, \ldots, a_i, \ldots, a_j, \ldots, a_{n+1})$$
$$= [a_1, A(a_2, \ldots, a_{n+1})]$$

since $[a_1, a_j]$ is in $L(v)$ and A is basic. This proves i).

Proof of ii) in Lemma 3. Since K is basic, we can apply i) of Lemma 4. We have

$$DK = dK - [\omega, K] = d(d\omega - \tilde{\omega}) - [\omega, d\omega - \tilde{\omega}] = -d\tilde{\omega} + [\omega, \tilde{\omega}] - [\omega, d\omega].$$

Since $[\tilde{\omega}, \omega] = 0$ by (e. 4) of Section 4, and $d\tilde{\omega} = [d\omega, \omega] = -[\omega, d\omega]$ by Lemma 2 in Section 4. We have $DK = 0$.

Proof of ii) in Lemma 4. Since DA is again basic we can apply i). We have

$$\begin{aligned} D(DA) &= d(DA) - [\omega, DA] = d(dA - [\omega, A]) - [\omega, dA - [\omega, A]) \\ &= -d([\omega, A]) - [\omega, dA] + [\omega, [\omega, A]] = -[d\omega, A] + [\omega, A] = -[K, A] \end{aligned}$$

where we used Lemma 1 of Section 4 and (e. 4) of Section 2.

Reduction. Let v be a f.g.-projective module over R with constant rank. We have the exact sequence $0 \to L(v) \to N_k(v) \to D \to 0$. Suppose that the following commutative diagram is given for an R-Lie algebra L and for a Lie-d-algebra N over R:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & N & \longrightarrow & D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & L(v) & \longrightarrow & N_k(v) & \longrightarrow & D & \longrightarrow & 0 \end{array}$$

We say that the pair (L, N) gives a reduction of v if the vertical mappings are injective homomorphisms of R-modules and homomorphisms of Lie algebras and if L is an R-projective module with constant rank.

In differential geometry, the notion of reductions is very much related with the curvature forms.

Section 6. Chern classes and the product formula

Let v be a f.g.-projective module over R with constant rank, and R a k-algebra. We have the exact sequence $0 \to L(v) \to N_k(v) \to D_k(R) \to 0$. Hereafter we shall omit the symbol k. $N(v)$ is a Lie d-algebra over R and has representations on $L(v)$ and on R, thus we have the cochain groups $A(N(v), L(v))$ and $A(N(v), R)$.

On the other hand, by the canonical representation of $D(R)$ on R, we have the cochain group $A(D(R), R)$, which defines the de Rham cohomology $H^*(R)$

of the ring R. By what we have proved at the end of Section 3, the ring structure cf $A(D(R), R)$ is carried over to the cohomology of $A(D(R), R)$, thus $H^*(R)$ is a ring.

We shall start by giving a few relations among those cochain groups.

We denote by $B^n(N(v), L(v))(B^n(N(v), R))$ the set of all basic forms in $A^n(N(v), L(v))$ $(A^n(N(v), R)$ respectively$)$.

LEMMA 1. *The mapping: $R \to L(v)$ (see Lemma 7 in Section 1) gives an injective homomorphism: $A^n(N(v), R) \to A^n(N(v), L(v))$. This is compatible with d, and if a connection is given then it is compatible with D.*

For any element a in R, the representations of a in $N(v)$ on $L(v)$ and on R coincide. The rest is just a direct verification.

LEMMA 2. *The mapping $\theta : N(v) \to D(R)$ induces the mapping $\theta : A^n(D(R), R) \to A^n(N(v), R)$ by*

$$(\theta' A)(a_1, \ldots) = A(\theta a_1, \ldots).$$

θ' is compatible with d, i.e. $d\theta' = \theta' d$.

This follows from the fact that θ is a homomorphism of k-Lie algebras and a homomorphism of R-modules.

LEMMA 3. *θ' gives an isomorphism of $A^n(D(R), R)$ onto $B^n(N(v), R)$. The inverse mapping φ is also compatible with d.*

The first statement follows from the exactness of $0 \to L(v) \to N(v) \to D(R) \to 0$. Let $A \varepsilon B^n(N(v), R) \subset B^n(N(v), L(v))$. We have

$$DA = dA - [\omega, A]$$

where ω is the connection form. Since the values of A lie on R, which is in the center of the Lie algebra $L(v)$, we have $[\omega, A] = 0$. Hence dA is basic. Now the second statement of Lemma 3 follows from Lemma 2.

LEMMA 4. *For any A in $B^n(N(v), R)$, we have*

$$DA = dA.$$

This is clear from the above argument.

Definition of Chern classes. Let Δ be a connection in v, and K its curvature form. Let P be an invariant symmetric n-form on $L(v)$ with respect to $N(v)$.

We have the form $P(K^n)$ in $A^{2n}(N(v), R)$ (see section 2). Since K is basic, $P(K^n)$ is also basic. We have

$$d(P(K^n)) = D(P(K^n)) = P(DK, K^{n-1}) = 0,$$

be Lemma 6, Section 4, and by Lemma 3, Section 5. This shows, by Lemma 3, that the form $\varphi(P(K^n))$ in $A^{2n}(D(R), R)$ is a cocycle. The class determined by $\varphi(P(K^n))$ is called the *characteristic class* of v corresponding to the invariant form P.

Now let P_n be the invariant symmetric form defined in Section 1. The characteristic class of v corresponding to P_n is called the *n-th Chern class* of v, and denoted by $c_n(v)$, or by $c_n(v : \Delta)$ when we need to emphasize Δ. $c_n(v)$ is in $H^{2n}(R)$. We define $c_0(v) = 1$, where 1 denotes the class determined by 1 in $A^0(D(R), R) = R$, which is a cocycle. Set

$$c(v) = \Sigma c_n(v)$$

where the sum is just a formal sum. $c(v)$ will be called the *Chern class* of v.

Remark. If $D(R)$ is finitely generated over R, then one sees that $H^n(R) = 0$ for every sufficiently large n since $\Lambda^n D(R) = 0$ if n is greater than the number of generators of $D(R)$. Hence in this case $c(v)$ can be defined to be an element in $H^*(R)$.

Later we shall prove that the Chern class $c(v)$ does not depend on the choice of connections.

The product formula: Let v_1 and v_2 be f.g.-projective R-modules with constant rank. We have

$$c(v_1 + v_2) = c(v_1) \wedge c(v_2)$$

i.e.

$$c_n(v_1 + v_2) = \sum c_k(v_1) \wedge c_{n-k}(v_2).$$

In this section, we shall prove the product formula for a suitable connection in $v_1 + v_2$, which will be determined by connections in v_1 and in v_2. Using this result, we shall prove in the next section, that the Chern class does not depend on the choice of connection. It will complete the proof of the product formula.

Proof. Let Δ_i be any connection in v_i, ω_i the connection form and K_i the curvature form $(i = 1, 2)$. We define a connection Δ in $v_1 + v_2$ as follows. For any X in $D(R)$, consider the mapping $\Delta(X)$ of v_1+v_2 into itself, defined by

$$\Delta(X)=\Delta_1(X)\cdot\pi_1+\Delta_2(X)\cdot\pi_2$$

where π_i denotes the projection of v_1+v_2 onto v_i. $\Delta(X)$ is in $N(v_1+v_2)$. In fact, for any $r\varepsilon R$, we have

$$\begin{aligned}[\Delta(X),\ r]&=[\Delta_1(X)\cdot\pi_1,\ r]+[\Delta_2(X)\cdot\pi_2,\ r]\\&=[\Delta_1(X),\ r]\cdot\pi_1+[\Delta_2(X),\ r]\cdot\pi_2=X(r)\cdot\pi_1+X(r)\cdot\pi_2=X(r).\end{aligned}$$

This shows $\Delta(X)\varepsilon N(v_1+v_2)$, and $[\Delta(X),\ r]=X(r)$. Thus $\theta\cdot\Delta=$ the identity mapping on $D(R)$. Hence Δ defines a connection in v_1+v_2. Let K be the curvature form of Δ. By the definition of Δ, and by a lemma in Section 5, we have

$$K(\Delta X,\ \Delta Y)=K_1(\Delta_1X,\ \Delta_1Y)+K_2(\Delta_2X,\ \Delta_2Y)$$

for any X, Y in $D(R)$, where we consider $L(v_1)$ and $L(v_2)$ to be submodules of $L(v_1+v_2)$ in the natural way. Consider the forms A, A_1, A_2 in $A^{2n}(D(R),\ L(v_1+v_2'))$ defined by

$$A(X,\ Y)=K(\Delta X,\ \Delta Y),\ A_i(X,\ Y)=K_i(\Delta_iX,\ \Delta_iY).$$

The cocycle defining $c_n(v_1+v_2)$ in $A^{2n}(D(R),\ R)$ is given by $P_n(A^n)$. We have

$$A=A_1+A_2,$$

hence

$$P_n(A^n)=\Sigma P_n(A_1^k,\ A_2^{n-k}).$$

We denote by P_k', P_k'' the k-th invariant symmetric forms on $L(v_1)$, $L(v_2)$ respectively. The cocycles defining $c_k(v_1)$ and $c_k(v_2)$ are given by $P_k'(A_1^k)$ and $P_k''(A_2^k)$ respectively. Thus it suffices to show

$$P_n(A_1^k,\ A_2^{n-k})=P_k'(A_1^k)\,P_{n-k}''(A_2^{n-k}).$$

Using Lemma 4 in Section 1, we have

$$\begin{aligned}&P_n(A_1(X_1,\ X_2),\ \dots,\ A_1(X_{2k}),\ A_2(X_{2k+1}),\ \dots)\\&\quad=P_k'(A_1(X_1,\ X_2),\ \dots)\cdot P_{n-k}(A_2(X_{2k+1},\ ),\ \dots)\end{aligned}$$

for any $X_1,\ \dots,\ X_{2n}$ in $D(R)$. This together with definitions in Section 2 completes the proof.

Section 7. Invariance of Chern classes

We shall prove in this section that Chern classes of projective modules are independent of the choice of connections. The problem will be reduced step by step.

A) *It suffices to show the independence in the case where the projective module is free.*

Proof. Suppose that the Chern class of a finitely generated free module is independent of the choice of connections. Since the free module has a trivial connection, our assumption is the same as saying that the Chern class of a free module is 1 in $H^*(R)$. Let v be any f.g.projective R-module with constant rank and v' another R-module such that $v+v'$ is free. Take arbitrary connections $\varDelta_1$ and $\varDelta_2$ in v, and $\varDelta'$ in v'. By the product formula in Section 6, we have

$$c(v, \varDelta_1)\wedge c(v', \varDelta') = 1,$$
$$c(v, \varDelta_2)\wedge c(v', \varDelta') = 1$$

since $c(v+v') = 1$. We have

$$c_0(v, \varDelta_1) = c_0(v, \varDelta_2) = 1,$$
$$c_0(v, \varDelta_1)\wedge c_1(v', \varDelta') + c_1(v, \varDelta_1)\wedge c_0(v', \varDelta') = c_1(v', \varDelta') + c_1(v, \varDelta_1) = 0.$$

In the same way

$$c_1(v', \varDelta') + c_1(v, \varDelta_2) = 0.$$

Thus

$$c_1(v, \varDelta_1) = c_1(v, \varDelta_2).$$

Repeating this, we get $c_n(v, \varDelta_1) = c_n(v, \varDelta_2)$. Q.E.D.

Hereafter we assume that v is a free module with a fixed base $e_1, \dots, e_n$. $L(v)$ is also free with the base $\{E_{ij}\}$ where $E_{ij}e_k = \delta_{jk}e_i$. We have a representation t of the Lie d-algebra $D(R)$ defined by

$$t(X)(\Sigma r_{ij}E_{ij}) = \Sigma(Xr_{ij})E_{ij}.$$

Let A be a n-form in $A^n(D(R), L(v))$ (or $A^n(N(v), L(v))$). We can write

$$A = \Sigma A_{ij}E_{ij}$$

where A_{ij} belongs to $A^n(D(R), R)$ (or $A^n(N(v), R)$). One can see easily that with respect to the representation t the above expression is compatible with the coboundary operator d, i.e., we have

$$dA = \Sigma d(A_{ij}) \cdot E_{ij}.$$

B) *It suffiees to show that for any* 1-*form* A *in* $A^1(D(R), L(v))$, *the form* $P_n((dA)^k, (A)^{n-k})$ *in* $A^{2n}(D(R), R)$ *is cohomologous to zero.*

Proof. Let Δ be the trivial connection in v with respect to the base $e_1, \ldots, e_n$ and Δ' an arbitrary connection in v. Δ is defined by

$$\Delta(X)(\Sigma r_i e_i) = \Sigma(X r_i) e_i.$$

Let ω, ω' be the connection forms of Δ, Δ' and K, K' their curvature forms. Since Δ is trivial we have $K = 0$. We shall denote by D the covariant derivative with respect to the trivial connection Δ. Set

$$B = \omega' - \omega.$$

We have

$$\begin{aligned} K' &= d\omega' - \omega' = d(\omega + B) - (\widetilde{\omega + B}) \\ &= d\omega + dB - \tilde{\omega} - \tilde{B} - [\omega, B] = dB - [\omega, B] - \tilde{B} = DB - \tilde{B}, \end{aligned}$$

where we used the fact that the form B is basic since $B(f) = \omega'(f) - \omega(f) = f - f = 0$ for any $f \varepsilon L(v)$. We have

$$P_n((K')^n) = \Sigma(-1)^{nk} P_n((DB)^k, (\tilde{B})^{n-k}).$$

Consider the form A in $A'(D(R), L(v))$, defined by

$$A(X) = B(\Delta X).$$

A is basic, and we have $\theta' A = B$, $\varphi B = A$ (see Lemmas 2 and 3 in Section 6). It can be easily shown that

$$\varphi(DB) = dA, \ \varphi(\tilde{B}) = \tilde{A}.$$

Thus we have

$$\varphi P_n((DB)^k, (\tilde{B})^{n-k}) = P_n((dA)^k, (\tilde{A})^{n-k}).$$

Hence, if this form is cohomologous to zero, any connection in a free module has the trivial Chern class. Q.E.D.

We shall show:

C) *For any* 1-*form* A *in* $A^1(D(R), R)$, *the form* $P_n((dA)^k, (\tilde{A})^{n-k})$ *is* *cohomologous to zero.*

First note:

i) For $k=0$, we have $P_n((\tilde{A})^n)=0$ by Lemma 4, Section 4,

ii) $d(P_n((dA)^k, (\tilde{A})^{n-k}))=0$.

In fact, by the last lemma in Section 4, we have

$$d(P_n((dA)^k, (A)^{n-k}))=P_n(ddA, (dA)^{k-1}, (\tilde{A})^{n-k})+P_n((dA)^k, d\tilde{A}, (\tilde{A})^{n-k-1}).$$

By Lemma 2, Section 4, $d\tilde{A}=[dA, A]$. Thus the last term is zero by Lemma 4, Section 4.

iii) For $A=\Sigma A_{ik}E_{ik}$, we have

$$A=\Sigma_j(A_{ij}\wedge A_{jk})E_{ik}.$$

This follows from a direct verification.

LEMMA 1. *Let* $A=\sum A_{ij}E_{ij}$, $B=\sum B_{ij}E_{ij}$ *be 2-forms in* $A^2(D(R), L(v))$. We have

$$P_n((A)^k, (B)^{n-k})=\sum_{1\leq i_1<\cdots<i_n-<m}\sum\operatorname{sign}\begin{pmatrix} i_1, & \ldots, & i_n \\ j_1, & \ldots, & j_n\end{pmatrix}$$
$$A_{i_1j_1}\wedge\cdots\wedge A_{i_kj_k}\wedge B_{i_{k+1}j_{k+1}}\wedge\cdots\wedge B_{i_nj_n}.$$

Proof. We have, by Lemma 6 in Section 1,

$$P_n(E_{i_1j_1}, \ldots, E_{i_nj_n})=\operatorname{sign}\begin{pmatrix} i_1, & \ldots, & i_n \\ j_1, & \ldots, & j_n\end{pmatrix}.$$

In particular,

$$P_n(E_{i_1j_1}, \ldots, E_{i_nj_n})=0$$

whenever some $E_{i_kj_k}=E_{i_lj_l}$ for $k\neq l$. Thus one sees $P_n((A_{ij}E_{ij})^l, \ldots)=0$ whenever $l\geq 2$. One can get the result expanding $P_n(A^k, B^{n-k})$. (For instance, $P_n((A_1+A_2)^2, B^{n-2})=P_n(A_1^2, B^{n-2})+P_n(A_2^2, B^{n-2})+P_n(A_1, A_2, B^{n-2})$.)

Using Lemma 1 and iii), we see that $P((dA)^k, (A)^{n-k})$ is a linear combination over integers of the following type of forms:

$$dA_{i_1j_1}\wedge\cdots\wedge dA_{i_kj_k}\wedge A_{r_1s_1}\wedge\cdots\wedge A_{r_es_e}$$

where $l=2n-2k$. Also one can verify that in the above forms, the forms of the following type are not contained

$$\cdots\wedge dA_{ij}\wedge\cdots\wedge A_{ij}\wedge\cdots.$$

Note that we have always $A_{ij} \wedge A_{ij} = 0$, $dA_{ij} \wedge dA_{ij} = 0$. Note the statement C) follows from the following Lemma 2.

For given non-negative integers k, l and n with $k+l \leq n$, we denote by $S(k, l, n)$ the set of ordered subsets $(a_1, \ldots, a_k;\ b_1, \ldots, b_l)$ of $\{1, \ldots, n\}$ such that $a_1 < \cdots < a_k$, $b_1 < \cdots < b_l$ and $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_l\} = \phi$.

LEMMA 2. *Let k, l and n be non-negatie integers such that $k+l \leq n$ and $l \geq 1$. Suppose that for each $(a_1, \ldots, a_j;\ b_1, \ldots, b_l)$* in $S(k, l, n)$, *an integer $f(a_1, \ldots, a_k;\ b_1, \ldots, b_l)$ is given with the following property:*

For any ring R, and for any 1-forms $A_1, \ldots, A_n$ in $A^1(D(R), R)$, $\Sigma f(a_1, \ldots, a_k;\ b_1, \ldots, b_l) A_{a_1} \wedge \cdots \wedge A_{a_k} \wedge dA_{b_1} \wedge \cdots \wedge dA_{b_l}$ is a cocycle.

Then, there exists a function g on $S(k+1, l-1, n)$ *with values integer satisfying:*

For any ring R, and for any 1-forms $A_1, \ldots, A_n$

$$d(\Sigma g(a_1, \ldots, a_{k+1};\ b_1, \ldots, b_{l-1}) A_{a_1} \wedge \cdots \wedge A_{a_{k+1}} \wedge dA_{b_1} \wedge \cdots \wedge dA_{b_{l-1}})$$
$$= \Sigma f(a_1, \ldots, a_k;\ b_1, \ldots, b_l) A_{a_1} \wedge \cdots \wedge A_{a_k} \wedge dA_{b_1} \wedge \cdots \wedge dA_{b_k}.$$

Proof. We may assume $k+l=n$. We omit a_i's in f, g since they are determined uniquely by b_i's. The assumption implies

$$(X) \qquad \Sigma(-1)^i f(b_1, \ldots, b_i, \ldots, b_{e+1}) = 0$$

for any $1 \leq b_1 < \cdots < b_{l+1} \leq n$. Set

$$g(b_1, \ldots, b_{l-1}) = f(1, b_1, \ldots, b_{l-1}) \text{ if } b_1 > 1$$
$$= 0 \qquad \text{if } b_1 = 1.$$

We have, for any $1 \leq b_1 < \cdots < b_l \leq n$,

$$(X, X) \qquad \Sigma(-1)^{i+1} g(b_1, \ldots, b_i, \ldots, b_l) = f(b_1, \ldots, b_l).$$

In fact, if $b_1 = 1$, then it follows from definition. If $b_1 > 1$, then it follows from (X). (X, X) gives the required result.

The condition (X) follows from the following example: $R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$, the polynomial ring over a field, $A_i = x_i dy_i$. Q.E.D.

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