# ON TRANSITIVE SIMPLE GROUPS OF DEGREE $3p$*)

To RICHARD BRAUER on his sixtieth birthday

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Let $\Omega$ be the set of symbols $1, 2, \ldots, 3p$, where $p$ is an prime number greater than 3. Let $\mathfrak{G}$ be a transitive permutation group on $\Omega$, which is simple and in which the normalizer of a Sylow $p$-subgroup has order $2p$. Our purpose is to prove the following two theorems:

THEOREM 1. *If $\mathfrak{G}$ is primitive on $\Omega$, then $p=5$ and $\mathfrak{G}$ is isomorphic to the alternating group $\mathfrak{A}_6$ of degree* 6.

THEOREM 2. *If $\mathfrak{G}$ is imprimitive on $\Omega$, then $\mathfrak{G}$ is isomorphic to the linear fractional group $LF(2, 2^m)$ with $2^m+1=p$.*

Our proof of Theorem 1 is fairly complicated. Theorem 1 implies that such a group $\mathfrak{G}$ cannot be doubly transitive. This fact will be proved in § 2. There the irreducible characters of dimension two of the symmetric group on $\Omega$ play an essential role as in our previous papers [14], [15]. We need also, however, recent result of Thompson [18] concerning groups of odd order. In § 3 we treat, roughly speaking, the almost doubly transitive case. There a result of Wielandt concerning the eigenvalues of intertwining matrices is very useful [21]. With the help of this theorem of Wielandt, some results of Brauer and Suzuki [4], [17] concerning groups whose Sylow 2-subgroups are dihedral groups of order either 4 or 8 respectively can be used. In § 4 we consider, roughly speaking, the strongly simply transitive case. For this case we need again some deep results.

Theorem II is a simple consequence of our previous result [14].

Finally, we want to emphasize that we need from beginning to end Brauer's $p$-block theory of irreducible characters.

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§ **1.** Proof of Theorem 1. Generalities.

**1.** Since $\mathfrak{G}$ is simple, the normalizer of a Sylow $p$-subgroup of $\mathfrak{G}$ is a dihedral group of order $2p$ by the splitting theorem of Burnside. Hence the principal $p$-block $B_1(p)$ of irreducible characters of $\mathfrak{G}$ consists of two non-exceptional characters, the principal character $\mathbf{A}$ and the other character $\mathbf{X}$, whose degree is congruent to $\pm 1$ modulo $p$, and a family of $\frac{1}{2}(p-1)$ $p$-conjugate exceptional characters $\mathbf{C}_i$ $(i=1, \ldots, \frac{1}{2}(p-1))$. The equation

$$\mathbf{A}(X)+\varepsilon\mathbf{X}(X)-\varepsilon\mathbf{C}_i(X)=0 \tag{1}$$

holds for every $p$-regular element $X$ of $\mathfrak{G}$ and for every $i=1, \ldots, \frac{1}{2}(p-1)$, where $\varepsilon=\pm 1$ according as the degree of $\mathbf{X}$ is congruent to $\pm 1$ modulo $p$. Let $P$ be an element of order $p$. Then we have

$$\mathbf{X}(P)=\varepsilon \tag{2}$$

and

$$\sum_{i=1}^{\frac{1}{2}(p-1)} \mathbf{C}_i(P)=-\varepsilon. \tag{3}$$

All the other irreducible characters $\mathbf{D}_j$ $(j=1, 2, \ldots)$ of $\mathfrak{G}$ belong to $p$-blocks of defect 0([3], §1).

We consider $\mathfrak{G}$ as usual as a linear group consisting of permutation matrices. Let $\alpha$ be the character of $\mathfrak{G}$ in this sense. Then for every element $X$ of $\mathfrak{G}$ $\alpha(X)$ denotes the number of symbols of $\Omega$ fixed by $X$. Since $\mathfrak{G}$ is transitive on $\Omega$, the decomposition of $\alpha$ into its irreducible components is as follows:

$$\alpha(X)=\mathbf{A}(X)+x\mathbf{X}(X)+c\sum\mathbf{C}_i(X)+\mathbf{Y}(X), \tag{4}$$

where $x$ and $c$ are non-negative integers and $\mathbf{Y}$ is a linear combination of $\mathbf{D}_j$'s with non-negative integers. All the $\mathbf{C}_i$'s have the same coefficient $c$, because they are algebraically conjugate to one another $\left(i=1, \ldots, \frac{1}{2}(p-1)\right)$.

**2.** Now we want to show that

$$\varepsilon=-1,\ x=1 \text{ and } c=0 \text{ in (4)}. \tag{5}$$

In order to show this, let us assume at first that $p>5$. Put $X=P$ in (4). Then from (2), (3) and (4) we have

(6) $$c = x + \varepsilon,$$

because $\mathbf{Y}$ vanishes at $P$ by a theorem of Brauer-Nesbitt ([8], Theorem 1). Put $X=1$ in (4). Then from (1) and (6) we have

(7) $$3p = 1 + x\mathbf{X}(1) + (x+\varepsilon)\frac{1}{2}(p-1)(\mathbf{X}(1)+\varepsilon) + \mathbf{Y}(1).$$

Now assume that $\varepsilon = 1$. Then since $\mathfrak{G}$ is simple and hence $\mathbf{X}(1) \geqq p+1$, we obtain from (7)

$$3p \geqq 1 + \frac{1}{2}(p-1)(p+2),$$

which implies the contradiction $p \leqq 5$. Hence $\varepsilon = -1$. Next assume that $x \geqq 2$. Then since $\mathfrak{G}$ is simple and hence $\mathbf{X}(1) \geqq p-1$, we obtain from (7)

$$3p \geqq 1 + 2(p-1) + \frac{1}{2}(p-1)(p-2),$$

which implies the contradiction $p \leqq 5$. Hence $x = 1$ and $c = 0$ by (6).

Now let us assume that $p = 5$. Though it is a little troublesome to handle with this case from the beginning, all the primitive groups of degree 15 are known. There are 6 types of such groups. Among them only the group, which is isomorphic to $\mathfrak{A}_6$, appears here. Therefore it is easy to check the validity of (5) in this case.

Put $\mathbf{X} = \mathbf{B}$. Then (1), (2), (3) and (4) can be rewritten as follows:

(1.1) $$\mathbf{A}(X) + \mathbf{C}_i(X) = \mathbf{B}(X)\left(i = 1, 2, \ldots, \frac{1}{2}(p-1)\right).$$

(2.1) $$\mathbf{B}(P) = -1.$$

(3.1) $$\sum_{i=1}^{\frac{1}{2}(p-1)} \mathbf{C}_i(P) = 1.$$

(4.1) $$\alpha(X) = \mathbf{A}(X) + \mathbf{B}(X) + \mathbf{Y}(X).$$

**3.** Let $J$ be an involution in the normalizer of the Sylow $p$-subgroup $\langle P\rangle$ of $\mathfrak{G}$. Let $g$ and $z$ denote the orders of $\mathfrak{G}$ and the centralizer of $J$. Then applying the method of Brauer-Fowler ([7], (23)) we have

(8) $$p = \frac{g}{z^2}\sum_{\mathbf{Z}} \frac{\mathbf{Z}(J)^2\mathbf{Z}(P)}{\mathbf{Z}(1)},$$

where $\mathbf{Z}$ ranges over all the irreducible characters of $\mathfrak{G}$. Since all the characters of defect 0 for $p$ vanish at $P$ by a theorem of Brauer-Nesbitt ([8], Theorem 1), (8) can be written as follows:

$$p = \frac{z^2}{g} \sum_{\mathbf{Z} \in B_1(p)} \frac{\mathbf{Z}(J)^2 \mathbf{Z}(P)}{\mathbf{Z}(1)} . \tag{9}$$

Let $vp-1$ be the degree of $\mathbf{B}$. Then the following equation can be obtained from (9) using (1.1), (2.1) and (3.1):

$$(vp-1)(vp-2)pz^2 = g(vp-1-\mathbf{B}(J))^2. \tag{10}$$

There is just one class of conjugate involutions in $\mathfrak{G}$. In fact let $K$ be an involution which is not conjugate to $J$. Then the method of Brauer-Fowler yields us $\mathbf{B}(K) = vp-1$, which contradicts the simplicity of $\mathfrak{G}$.

Now since the centralizer of $J$ contains a Sylow 2-subgroup of $\mathfrak{G}$, the equation (10) tells us something about the order of a Sylow 2-subgroup of $\mathfrak{G}$.

According to the degree of $\mathbf{B}$ we distinguish three cases, each of which is handled separately, since we see from (4.1) that $v$ equals either 3 or 2 or 1.

## § 2. The case in which the degree of $\mathbf{B}$ is $3p-1$.

**4.** Let us assume that the degree of $\mathbf{B}$ equals $3p-1$. Then the equations (4.1) and (10) take the following forms:

$$\alpha(X) = \mathbf{A}(X) + \mathbf{B}(X). \tag{4.2}$$

$$(3p-1)(3p-2)pz^2 = g(3p-1-\mathbf{B}(J))^2. \tag{10.1}$$

The equation (4.2) tells us in particular that $\mathfrak{G}$ is doubly transitive on $\Omega$.

By a theorem of Brauer ([3], Lemma 3) we have

$$\mathbf{B}(J) = -2 \text{ or } 0 \text{ or } 2.$$

Since $\alpha(J) \geqq 0$ the case $\mathbf{B}(J) = -2$ does not occur by (4.2). Now assume that $\mathbf{B}(J) = 2$. Then by (4.2) we have

$$\alpha(J) = 3, \tag{11}$$

and (10.1) can be read as follows:

$$(3p-1)(3p-2)pz^2 = 9(p-1)^2 g. \tag{10.2}$$

Since $\mathfrak{G}$ is doubly transitive, $\mathfrak{G}$ contains an involution $I$ with the cycle structure (12) . . . . Let $\mathfrak{R}$ denote the subgroup of $\mathfrak{G}$ consisting of all the permutations of $\mathfrak{G}$ each of which fixes each of the symbols 1 and 2. Then $I$ is contained in the normalizer of $\mathfrak{R}$. Hence there exists a Sylow 2-subgroup $\mathfrak{T}$ of $\mathfrak{R}$, whose normalizer contains $I$. $\mathfrak{S}=\mathfrak{T}\langle I\rangle$ is a Sylow 2-subgroup of $\mathfrak{G}$. In fact otherwise we must have $3p\equiv 1$ (mod. 4). Then the equality (10.2) shows that $g$ must be odd, which is a contradiction. Since $I$ and $J$ are conjugate with each other, $I$ fixes by (11) just three symbols different from 1 and 2, say 3, 4 and 5 of $\Omega$. Let $X$ be an element of $\mathfrak{T}$, which is commutative with $I$. Then since $\alpha(X)\leqq 3$ and is odd, $X$ must fix just one symbol, for instance 5, of the symbols 3, 4 and 5, and the cycle structure of $X$ is of the form (34)(5) . . . . Since every involution fixes just three symbols of $\Omega$, $X$ must be an involution. Let $Y\neq X$, $Y$ be an element of $\mathfrak{T}$, which is commutative with $I$. Then $Y$ must fix, like $X$, just one symbol of 3, 4 and 5. If it is 3, $Y$ has the cycle structure (3)(45) . . . . Then $XY$ belongs to $\mathfrak{T}$ and has the cycle structure (354) . . . , which is a contradicition. The same holds for 4, too. Hence $Y$ must fix 5, and has the cycle structure (34)(5) . . . . Then $XY$ belongs to $\mathfrak{T}$ and fixes the symbols 1, 2, 3, 4 and 5. This implies that $XY=1$, and since $X$ is an involution, $X=Y$, which contradicts our assumption on $Y$. Therefore the centralizer of $I$ in $\mathfrak{S}$ has order 4. Thus by a theorem of Suzuki ([18], Lemma 4) $\mathfrak{S}$ contains an element $L$ such that $\mathfrak{S}=\langle I, L\rangle$ and $ILI=L^{-1+2^{a-2}\varepsilon}$, where $2^a$ is the order of $\mathfrak{S}$ and $\varepsilon$ equals either 1 or 0. Let $f$ be the exact exponent of 2 dividing $p-1$. Then we obtain from (10.2) the following equality:

$$a=2f-1. \tag{12}$$

The simplicity of $\mathfrak{G}$ implies that $a$ is greater than 1. This implies by (12) that the order of $L$ is greater than 2. Now it is easy to see that the cycle structure of $L$ is of the form either $L=(1)(2)(i)R$ or $L=(12)(i)R$, where $i\neq 1, 2$ is a symbol of $\Omega$ and $R$ consists of cycles of order $2^{a-1}$. In any case this shows that $p-1$ is divisible by $2^{a-1}$, that is, $f\geqq a-1$. Hence we obtain from (12) that $a=3$ and $\mathfrak{S}$ is a dihedral group of order 8.

Let us consider the principal 2-block $B_1(2)$ of irreducible characters of $\mathfrak{G}$. By a theorem of Brauer-Tuan ([10], Corollary of Lemma 3) $B_1$ (2) contains at least either B or all of the $\mathrm{C}_i$'s $(i=1, \ldots, \frac{1}{2}(p-1))$, because there is no element of order $2p$ from our assumptions. Assume that $B_1(2)$ does not contain any $\mathbf{C}_i$. Then by a theorem of Brauer-Tuan ([10], Lemma 3) we have the congruence

$$\sum \mathbf{Z}(1)\mathbf{Z}(P) \equiv 0 \pmod{2^a}, \tag{13}$$

where $\mathbf{Z}$ ranges over all the irreducible characters of $\mathfrak{G}$ belonging simultaneously to $B_1(p)$ and $B_1(2)$. But the left hand side of (13) equals $1+(3p-1)(-1) = -(3p-2)$, which is a contradicition. Hence $B_1(2)$ contains all the $\mathbf{C}_i$'s. On the other hand $B_1(2)$ consists of five characters ([5], [17] and for a detailed presentation see [13]). Thus we have obtained the inequality $\frac{1}{2}(p+1) \leqq 5$, which implies that $p=5$. Now again we have only to check six primitive groups of degree 15 and we see that there is no group with required properties. Therefore we must have that $\mathbf{B}(J)=0$ and by (4.2) that

$$\alpha(J) = 1. \tag{14}$$

Furthermore (10.1) becomes the following form:

$$(3p-2)pz^2 = (3p-1)g. \tag{10.3}$$

(10.3) tells us in particular that the order of a Sylow 2-subgroup of $\mathfrak{G}$ equals the power of 2 dividing $3p-1$. Hence $\mathbf{B}$ is a character of defect 0 for 2. In particular by a theorem of Brauer-Nesbitt ([8], Theorem 1) we have

$$\alpha(X) = 1 \tag{15}$$

for every 2-singular element $X$ of $\mathfrak{G}$.

**5.** Let $\mathfrak{S}$ denote the symmetric group on $\Omega$. Let $\mathbf{X}_{:}$ and $\mathbf{X}_{..}$ be irreducible characters of $\mathfrak{S}$ corresponding to the diagrams

$$\begin{array}{l} \circ\ \circ \cdots \circ \\ \circ \\ \circ \end{array} \quad \text{and} \quad \begin{array}{l} \circ\ \circ \cdots \circ. \\ \circ\ \circ \\ \phantom{\circ} \end{array}$$

By a theorem of Frobenius (12) we have the formulae

$$\mathbf{X}_{:}(X) = \binom{\alpha(X)-1}{2} - \beta(X) \tag{16}$$

and

$$\mathbf{X}_{..}(X) = \frac{\alpha(X)(\alpha(X)-3)}{2} + \beta(X), \tag{17}$$

where $X$ ia an element of $\mathfrak{S}$ and $\beta(X)$ denotes the number of transpositions in the cycle structure of $X$.

Now since $\mathfrak{G}$ is doubly transitive, we have ((11), p. 164)

$$\sum_{X\in\mathfrak{G}}\alpha(X)=g,\quad \sum_{X\in\mathfrak{G}}\alpha(X)^2=2g \text{ and } \sum_{X\in\mathfrak{G}}\beta(X)=\frac{1}{2}g. \tag{18}$$

Using (18) we obtain from (16) and (17)

$$\sum_{X\in\mathfrak{G}}\mathbf{X}_{:}(X)=\sum_{X\in\mathfrak{G}}\mathbf{X}_{..}(X)=0.$$

Hence by the reciprocity theorem of Frobenius $\mathbf{A}$ does not appear as an irreducible component of $\mathbf{X}_{:}$ and $\mathbf{X}_{..}$ restricted to $\mathfrak{G}$. Let

$$\mathbf{X}_{:}=b\mathbf{B}+c\sum\mathbf{C}_i+\sum a_j\mathbf{D}_j \tag{19}$$

and

$$\mathbf{X}_{..}=b'\mathbf{B}+c'\sum\mathbf{C}_i+\sum b_j\mathbf{D}_j \tag{20}$$

be the decompositions of $\mathbf{X}_{:}$ and $\mathbf{X}_{..}$ into irreducible characters of $\mathfrak{G}$.

We want to show that

$$b=b'=c'=c-1\leqq 1. \tag{21}$$

To this end, we first compare the values of both sides of (19) and (20) at $P$. Then using (2.1), (3.1) and a theorem of Brauer-Nesbitt ([8], Theorem 1) we obtain from (16) and (17) the equalities $1=-b+c$ and $0=-b'+c'$.

Next let us observe the generalized character $(\mathbf{X}_{:}-\mathbf{X}_{..})\mathbf{B}$. Then we have

$$\begin{aligned}
&\sum_{X\in\mathfrak{G}}(\mathbf{X}_{:}(X)-\mathbf{X}_{..}(X))\mathbf{B}(X)\\
&=\sum_{X\in\mathfrak{G}}(1-2\beta(X))(\alpha(X)-1)\qquad \text{(by (4.2), (16), (17))}\\
&=\sum_{X\in\mathfrak{G}}(-1+\alpha(X)2\beta(X)-2\alpha(X)\beta(X))\\
&=\sum_{X\in\mathfrak{G}}(-1+\alpha(X))=0\qquad \text{(by (15))}.
\end{aligned}$$

This implies $b=b'$.

Let us assume that $b>1$. Then we have that $b\geqq 2$ and $c\geqq 3$. Comparing the degrees of the characters on both sides of (19) we have that

$$\frac{1}{2}(3p-1)(3p-2)\geqq 2(3p-1)+3\cdot\frac{1}{2}(p-1)(3p-2),$$

which implies the contradiction $0\geqq p$. Therefore we must have that $1\geqq b$.

Now we distinguish two subcases $b=0$ and $b=1$, though they can be treated rather similarly. In any case, we can use, roughly speaking, the same routine as in the previous paper [15].

**6.** At first we handle the subcase $b=0$. Then the equations (19) and (20) are read as follows:

$$\mathbf{X}_{:}=\sum \mathbf{C}_i+\sum a_j\mathbf{D}_j \tag{19.1}$$

and

$$\mathbf{X}_{..}=\sum b_j\mathbf{D}_j. \tag{20.1}$$

Since $\mathbf{B}$ is orthogonal to $\mathbf{X}_{:}+\mathbf{X}_{..}$ in this case, using (18) we obtain

$$\sum_{X\in\mathfrak{G}}\alpha(X)^3=5g. \tag{22}$$

In particular $\mathfrak{G}$ is triply transitive on $\Omega$ [21].

Using (15), (18) and (22) we can calculate the norm of $\mathbf{X}_{:}$ and $\mathbf{X}_{..}$ from (16), (17) and (19.1), (20.1) as follows:

$$\begin{aligned}(23)\qquad &\sum_{X\in\mathfrak{G}}\left(\frac{1}{2}(\alpha(X)-1)(\alpha(X)-2)-\beta(X)\right)^2\\ &=\sum_{X\in\mathfrak{G}}\frac{1}{4}\alpha(X)^4+\sum_{X\in\mathfrak{G}}\beta(X)^2-3\\ &=\frac{1}{2}(p-1)+\sum a_j^2\\ (24)\qquad &\sum_{X\in\mathfrak{G}}\left(\frac{1}{2}\alpha(X)(\alpha(X)-3)+\beta(X)\right)^2\\ &=\sum_{X\in\mathfrak{G}}\frac{1}{4}\alpha(X)^4+\sum_{X\in\mathfrak{G}}\beta(X)^2-4\\ &=\sum b_j^2.\end{aligned}$$

Eliminating the expression $\sum_{X\in\mathfrak{G}}\frac{1}{4}\alpha(X)^4+\sum_{X\in\mathfrak{G}}\beta(X)^2$ from (23) and (24) we have

$$\sum b_j^2=\frac{1}{2}(p-3)+\sum a_j^2. \tag{25}$$

**7.** Let $\mathbf{e}$ be the principal character of $\mathfrak{K}$ and $\mathbf{e}^*$ be the character of $\mathfrak{G}$ induced by $\mathbf{e}$. Since $\mathfrak{G}$ is doubly transitive, by a theorem of Frobenius [12] we have the following equation

$$\mathbf{e}^*=\mathbf{A}+2\mathbf{B}+\mathbf{X}_{:}+\mathbf{X}_{..}.$$

Substituting (19.1) and (20.1) into this equation, we have

$$\mathbf{e}^* = \mathbf{A} + 2\mathbf{B} + \sum \mathbf{C}_i + \sum (a_j + b_j)\mathbf{D}_j. \tag{26}$$

Let $\Omega_2$ denote the set of vectors $(x, y)$, where $x \neq y$ and $x, y$ belong to $\Omega$. The basis of our proof rests on the following theorem ([22], 28.4, 29.2): the norm of $\mathbf{e}^*$ equals the number of domains of transitivity of $\mathfrak{K}$ on $\Omega_2$.

By (26) the norm of $\mathbf{e}^*$ equals

$$1 + 4 + \frac{1}{2}(p-1) + \sum (a_j + b_j)^2.$$

Put $T = \Omega - \{1, 2\}$. $T_2$ is the set of vectors $(x, y)$, where $x \neq y$, and $x, y \in T$. The vectors $(1, 2)$ and $(2, 1)$ themselves constitute domains of transitivity of $\mathfrak{K}$ and furthermore the vectors of forms $(i, T)$ and $(T, i)$ $(i = 1, 2)$ each constitute domains of transitivitity of $\mathfrak{K}$. Therefore we see that the vectors of $T_2$ are divided into

$$\frac{1}{2}(p-3) + \sum (a_j + b_j)^2$$

domains of transitivity of $\mathfrak{K}$. By (25) this number will be transformed into

$$p - 3 + 2\sum a_j^2 + 2\sum a_j b_j. \tag{27}$$

Since $\mathfrak{G}$ is triply transitive on $\Omega$ and hence $\mathfrak{K}$ is transitive on $T$, every domain of transitivity of $\mathfrak{K}$ from $T_2$ contains a vector of the form $(3, x)$ with $x$ $(\neq 3) \in T$.

8. Let $\mathfrak{L}$ denote the subgroup of $\mathfrak{G}$ consisting of all the permutations of $\mathfrak{G}$ each of which fixes each of the symbols 1, 2, 3. At first assume that $\mathfrak{L}$ fixes no symbol from $\Omega$ other than 1, 2 and 3. Then since the order of $\mathfrak{L}$ is by (15) odd, every domain of transitivity of $\mathfrak{K}$ from $T_2$ contains at least three different vectors of the form $(3, x)$ with $x \in T$. Then we see at once that there exist at most $p-1$ domains of transitivity of $\mathfrak{K}$ from $T_2$. Then from (27) we have the following inequality

$$1 \geqq \sum a_j^2 + \sum a_j b_j. \tag{28}$$

If all the $a_j$'s are zero, comparing the values at the identity element of both sides of (19.1) we have the contradiction

$$\frac{1}{2}(3p-1)(3p-2)=\frac{1}{2}(p-1)(3p-2).$$

Hence (28) turns out to be an equality. This means that there exist just $p-1$ domains of transitivity of $\mathfrak{K}$ from $T_2$ and every domain of transitivity of $\mathfrak{L}$ from $T-\{3\}$ has length 3. The latter fact implies that $\mathfrak{L}$ is an elementary abelian 3-group. It is easy to check that the normalizer of $\mathfrak{L}$ in $\mathfrak{K}$ coincides with $\mathfrak{L}$. Therefore by the splitting theorem of Burnside $\mathfrak{K}$ contains the normal 3-complement $\mathfrak{M}$ of order $3p-2$. Every element $\neq 1$ of $\mathfrak{M}$ fixes just two symbols of $\Omega$, 1 and 2. Now let $I$ be an involution of $\mathfrak{G}$ with the cycle structure $(12)(3)\ldots.$ Then $I$ normalizes $\mathfrak{K}$ and therefore $\mathfrak{M}$. By (15) $I$ fixes only the symbol 3 from $\Omega$. Hence $I$ centralizes only the identity element of $\mathfrak{M}$. Therefore $\mathfrak{M}$ must be abelian. Under this circumstances we want to show that the order of $\mathfrak{L}$ is smaller than $3p-2$.

Let $\mathfrak{Q}$ be a Sylow $q$-subgroup of $\mathfrak{M}$ and let $\mathfrak{L}_{\mathfrak{Q}}$ be the centralizer of $\mathfrak{Q}$ in $\mathfrak{L}$. Then the factor group $\mathfrak{L}/\mathfrak{L}_{\mathfrak{Q}}$ is isomorphic to an automorphism group of $\mathfrak{Q}$. Let $q$ vary over all the prime divisors of $3p-2$. Then obviously $\mathfrak{L}$ is isomorphic to a subgroup of the direct product of all the $\mathfrak{L}/\mathfrak{L}_{\mathfrak{Q}}$'s. Therefore we have only to show that for every prime divisor $q$ of $3p-2$ the order of $\mathfrak{L}/\mathfrak{L}_{\mathfrak{Q}}$ is smaller than that of $\mathfrak{Q}$. Then the ordinary Frattini argument allows us to assume that $\mathfrak{Q}$ is elementary abelian (of order $q^n$). So we can assume that $\mathfrak{L}$ is a subgroup of the general linear group $GL(u, q)$. Moreover we can assume that $\mathfrak{L}$ is irreducible in the prime field of characteristic $q$. This implies that $\mathfrak{L}$ is cyclic (of order 3). There remains nothing to prove.

Let $l$ be the order of $\mathfrak{L}$. Then there holds

$$g=3p(3p-1)(3p-2)l.$$

Substituting this value of $g$ into (10.3) we have

$$z^2=3(3p-1)^2 l.$$

Hence we can put

$$3l=m^2. \tag{29}$$

On the other hand by the theorem of Sylow (for $p$) we have that $m^2\equiv 1 \pmod{p}$, which implies $m\equiv \pm 1 \pmod{p}$. Since $m$ is odd $>1$ by (29), we obtain that $m \geqq 2p-1$. So we have the following inequality

$$(2p-1)^2<3(3p-2),$$

which implies the contradiction $p\leqq 2$.

**9.** Therefore $\mathfrak{L}$ must fix at least one symbol from $\Omega$, say 4 different from 1, 2 and 3. Now we can assume, without loss of generality, that $\mathfrak{L}$ fixes just $i$ symbols, $1, 2, \ldots, i\ (i\geqq 4)$ of $\Omega$. Let $Ns\mathfrak{L}$ denote the normalizer of $\mathfrak{L}$ in $\mathfrak{G}$. Put $\Phi=\{1, 2, \ldots, i\}$. Then the factor group $Ns\mathfrak{L}/\mathfrak{L}$ is a triply transitive permutation group on $\Phi$ ([22], 9.4). Clearly every permutation $\neq 1$ of $Ns\mathfrak{L}/\mathfrak{L}$ fixes at most two symbols of $\Phi$. Hence the order of $Ns\mathfrak{L}/\mathfrak{L}$ equals $i(i-1)(i-2)$. The degree $i$ must be odd by (15). Therefore using a theorem of Zassenhaus [24] we obtain that $Ns\mathfrak{L}/\mathfrak{L}$ is isomorphic to $LF(2, 2^m)$ with $2^m+1=i$.

In these circumstances let us assume at first that $\mathfrak{L}$ has at least one domain of transitivity from $T$ whose length is greater than 3. Now we can show that

$$i<\sqrt{p}. \tag{30}$$

To this end let $\Psi$ be a domain of transitivity of $\mathfrak{L}$ from $T$ with length $f>3$. Let $\mathfrak{N}/\mathfrak{L}$ be a Sylow 2-subgroup of $Ns\mathfrak{L}/\mathfrak{L}$. Then for any involution $X$ of $\mathfrak{N}$ we have $\Psi\cap\Psi^X=\emptyset$. In fact $\Psi^X$ is again a domain of transitivity of $\mathfrak{L}$ from $T$. If $\Psi\cap\Psi^X\neq\emptyset$, then we have $\Psi=\Psi^X$. But this means that $X$ fixes at least one symbol in $\Psi$, because the length of $\Psi$ is odd. This contradicts (15). Let $\Psi^*$ be the set of all the different $\Psi^X$ with any element $X$ from $Ns\mathfrak{L}$. Then we can consider $Ns\mathfrak{L}/\mathfrak{L}$ as a transitive permutation group on $\Psi^*$. Let $\mathfrak{F}/\mathfrak{L}$ be the subgroup of $Ns\mathfrak{L}/\mathfrak{L}$ consisting of all the elements of $Ns\mathfrak{L}/\mathfrak{L}$ each of which fixes $\Psi$. Then the order of $\mathfrak{F}/\mathfrak{L}$ is, as is shown above, odd. Then we see from a property of $LF(2, 2^m)$ that $\mathfrak{F}/\mathfrak{L}$ is cyclic of order at most $2^m+1$. Therefore $\Psi^*$ contains at least $f2^m(2^m-1)$ symbols of $T$. Thus we have obtained the following inequality

$$2^m+1+5.2^m(2^m-1)\leqq 2^m+1+f2^m(2^m-1)\leqq 3p.$$

Let assume that $i\geqq\sqrt{p}$. Then we obtain from above the following inequality:

$$\sqrt{p}+5(\sqrt{p}-1)(\sqrt{p}-2)\leqq 3p,$$

which implies that

$$p+5\leqq 7\sqrt{p}.$$

So we obtain that $p \leqq 37$. Since $p \equiv -1 \pmod 4$ by (15) we have only the following possibilities $p=7$; 11; 19; 31. Furthermore $3p-1$ must be divisible by 32, because $m$ is odd and bigger than 3. The last fact follows from the fact that any Sylow 3-subgroup of $\mathfrak{L}$ has index 3 in a Sylow 3-subgroup of $\mathfrak{G}$. Then we see that only the case $p=11$ is possible. But if $p=11$, we must have that $\mathfrak{L}=1$, which contradicts our assumption on $\mathfrak{L}$.

Let $j$ be the number of domains of transitivity of $\mathfrak{L}$ with length 3 from $T$. Then by a theorem of Bochert [1] we have that

$$i+3j \leqq 2p. \tag{31}$$

Now there exist at most

$$i+j+\frac{3p-i-3j}{5}$$

domains of transitivity of $\mathfrak{K}$ from $T_2$. Here we notice that the number in (27) is not smaller than $p-1$, because it is shown to be impossible in 8 that all the $a_j$'s are zero. Then we have the following inequality

$$4i+2j+5 \geqq 2p,$$

which implies

$$10i+2(i+3j)+15 \geqq 6p.$$

So by (30) and (31) we obtain the following inequality

$$10\sqrt{p}+15 \geqq 2p,$$

which implies that $p \leqq 37$. This has already been shown above to be impossible.

Thus we can assume that all the domains of transitivity of $\mathfrak{L}$ from $T-\Phi$ have length 3. Then we want to show that we are essentially in the same situation as in 8. At any rate $\mathfrak{L}$ is an elementary abelian 3-group. Let $I$ be an involution with the cycle structure (12) .... Let $q$ be a prime divisor of $3p-2$ and $\mathfrak{Q}$ be a Sylow $q$-subgroup of $\mathfrak{K}$ such that the normalizer of $\mathfrak{Q}$ contains $I$. Then we see as in 8 that $\mathfrak{Q}$ is abelian. Hence $\mathfrak{K}$ is an $A$-group of odd order. Therefore by a theorem of Thompson [18] $\mathfrak{K}$ is soluble. Let $\mathfrak{M}$ be a Sylow 3-complement of $\mathfrak{K}$ such that the normalizer of $\mathfrak{M}$ contains $I$. Then we see again that $\mathfrak{M}$ is abelian. Let $\underline{\mathfrak{M}}$ be the largest normal subgroup of $\mathfrak{K}$ contained in $\mathfrak{M}$. We want to see that $\mathfrak{M}=\underline{\mathfrak{M}}$. Assume that $\mathfrak{M} \neq \underline{\mathfrak{M}}$. Then let us consider the centralizer of $\underline{\mathfrak{M}}$ in $\mathfrak{K}$. Since $\mathfrak{M}$ is abelian, this has the form $\mathfrak{M}\mathfrak{L}'$ with $\mathfrak{L}' \leqq \mathfrak{L}$. If $\mathfrak{L}' \neq 1$, then $\mathfrak{L}'$ becomes a normal 3-subgroup $\neq 1$ of $\mathfrak{K}$. This is a contradiction. So we have that $\mathfrak{M} = \underline{\mathfrak{M}}$. The rest is just the same as in 8. Therefore the subcase $b=0$ cannot occur.

**10.** Next we consider the subcase $b=1$. In this case the equations (19) and (20) take the following forms:

(19.2) $$\mathbf{X}_{:} = \mathbf{B} + 2\sum \mathbf{C}_i + \sum a_j \mathbf{D}_j$$

and

(20.2) $$\mathbf{X}_{..} = \mathbf{B} + \sum \mathbf{C}_i + \sum b_j \mathbf{D}_j.$$

Corresponding to (22), (23), (24) and (25) we have now

(22.1) $$\sum_{X \in \mathfrak{G}} \alpha(X)^3 = 7g.$$

(23.1) $$\begin{aligned}&\sum_{X \in \mathfrak{G}} \left(\frac{1}{2}(\alpha(X)-1)(\alpha(X)-2) - \beta(X)\right)^2 \\ &= \sum_{X \in \mathfrak{G}} \frac{1}{4}\alpha(X)^4 + \sum_{X \in \mathfrak{G}} \beta(X)^2 - 6 \\ &= 1 + 4 \cdot \frac{1}{2}(p-1) + \sum a_j^2.\end{aligned}$$

(24.1) $$\begin{aligned}&\sum_{X \in \mathfrak{G}} \left(\frac{1}{2}\alpha(X)(\alpha(X)-3) + \beta(X)\right)^2 \\ &= \sum_{X \in \mathfrak{G}} \frac{1}{4}\alpha(X)^4 + \sum_{X \in \mathfrak{G}} \beta(X)^2 - 7 \\ &= 1 + \frac{1}{2}(p-1) + \sum b_j^2.\end{aligned}$$

(25.1) $$\sum b_j^2 = \frac{1}{2}(3p-5) + \sum a_j^2.$$

Furthermore corresponding to (26) we have now

(26.1) $$\mathbf{e}^* = \mathbf{A} + 4\mathbf{B} + 3\sum \mathbf{C}_i + \sum (a_j + b_j)\mathbf{D}_j.$$

Hence the norm of $\mathbf{e}^*$ equals

$$1 + 16 + 9 \cdot \frac{1}{2}(p-1) + \sum (a_j + b_j)^2.$$

Let $\mathfrak{H}$ denote the subgroup of $\mathfrak{G}$ consisting of all the permutations of $\mathfrak{G}$ each of which fixes the symbol 1, and let $h$ be the order of $\mathfrak{H}$. Let us consider the norm of **B** restricted to $\mathfrak{H}$ and put

$$\sum_{X\in\mathfrak{H}} \mathbf{B}(X)^2 = \sum_{X\in\mathfrak{H}} (\alpha(X)-1)^2 = ah. \tag{32}$$

The same equality holds for any conjugate subgroup of $\mathfrak{H}$ in $\mathfrak{G}$. Adding up (32) over all the conjugate subgroups of $\mathfrak{H}$ in $\mathfrak{G}$, we have

$$\sum_{X\in\mathfrak{G}} \alpha(X)(\alpha(X)-1)^2 = ag. \tag{33}$$

By (18) and (22.1) we see that the left hand side of (33) equals $4g$. Thus we have proved that $a=4$. Therefore by ([22], 28.4, 29.2) $\Omega-\{1, 2\}$ is divided into three domains of transitivity of $\mathfrak{K}$, say $T(i)$ $(i=1, 2, 3)$. Let $t_i$ be the length of $T(i)$. Then we have

$$t_1+t_2+t_3=3p-2. \tag{34}$$

By $T(i)_2$ is meant the set of vectors $(x, y)$, with $x \neq y$, $x, y \in T(i)$. Now the vectors (1, 2) and (2.1) themselves constitute domains of transitivity of $\mathfrak{K}$ and furthermore the vectors of $(i, T(j))$ and $(T(j), i)$ $(i=1, 2;\ j=1, 2, 3)$ each constitute domains of transitivity of $\mathfrak{K}$ from $\Omega_2$. Therefore we see that the vectors of $T(i)_2$ and $(T(i), T(j))$ $(i, j=1, 2, 3;\ i\neq j)$ are divided into

$$\frac{1}{2}\cdot 3(3p-1)+\sum(a_j+b_j)^2$$

domains of transitivity of $\mathfrak{K}$ from $\Omega_2$. By (25.1) this number will be transformed into

$$6p-4+2\sum a_j^2+2\sum a_jb_j. \tag{27.1}$$

Let $n_k$ be a symbol of $T(k)$ and $\mathfrak{L}_k$ be the subgroup of $\mathfrak{K}$ consisting of all the permutations of $\mathfrak{K}$ each of which fixes the symbol $n_k$ $(k=1, 2, 3)$. Let $i_h$ and $j_k$ denote the numbers of domains of transitivity of $\mathfrak{L}_k$ from $T(1)+T(2)+T(3)$ having lengths 1 and 3, respetively $(k=1, 2, 3)$. Let us assume at first that for every $k=1, 2, 3$, $\mathfrak{L}_k$ has a domain of transitivity of length greater than 3 from $\Omega$. Then since $\mathfrak{G}$ is doubly transitve, we have, by a theorem of Bochert [2], the following inequalities:

$$2p+\frac{2\sqrt{3p}}{3} \geqq 2+i_k+3j_k \qquad (k=1, 2, 3) \tag{35}$$

Every domain of transitivity of $\mathfrak{K}$ from $T(1)_2$, $(T(1), T(2))$ and $(T(1), T(3))$

contains a vector of the form $(n_1, *)$. Hence there exist at most

$$i_1 - 1 + j_1 + \frac{3p-2-i_1-3j_1}{5} \tag{36}$$

domains of transitivity from $T(1)_2$, $(T(1), T(2))$ and $(T(1), T(3))$. The same holds also for $T(2)_2$, $(T(2), T(1))$, $(T(2), T(3))$ and $T(3)_2$, $(T(3), T(1))$, $(T(3), T(2))$. Adding up three numbers of type (36) we see that there exist at most

$$\frac{9p-21}{5} + \frac{4}{5}(i_1+i_2+i_3) + \frac{2}{5}(j_1+j_2+j_3) \tag{37}$$

domains of transitivity of $\mathfrak{K}$ from $T(k)_2$ and $(T(k), T(1))$ $(k, 1 = 1, 2, 3;\ k \neq 1)$.

Let $J$ be an involution whose cycle structure has the form (12) .... By (14) $J$ fixes just one symbol, say $\alpha_J$, of $\Omega$. Without loss of generality we can assume that $\alpha_J$ belongs to $T(3)$ and $\alpha_J = \alpha_3$. Since $J$ belongs to the normalizer of $K$, $J$ transfers $T(1)$ into one of $T(i)$'s. $(i = 1, 2, 3)$. If it is $T(1)$, then since $J$ does not fix any symbol of $T_1$ the length of $T(1)$ must be even, which is a contradiction. Moreover since $J$ fixes the symbol $\alpha_3$, $J$ fixes $T(3)$. Hence $J$ interchanges $T(1)$ with $T(2)$. In particular we see that $L_1$ and $L_2$ are conjugate in the normalizer of $K$. and that $i_1 = i_2$, $j_1 = j_2$ and $t_1 = t_2$.

Let $\Phi_3$ be the set of all the symbols of $T(1) + T(2) + T(3)$, each of which is fixed by all the permutations of $\mathfrak{L}_3$.

In the first place, let us assume that $\Phi_3$ is contained in $T(3)$. We consider the normalizer $Ns\,\mathfrak{L}_3$ of $\mathfrak{L}_3$ in $\mathfrak{G}$. Then by a theorem of Witt ([22], 9.4) $Ns\,\mathfrak{L}_3/\mathfrak{L}_3$ is doubly transitive on $\Phi_3 \cup \{1, 2\}$. Furthermore since $\mathfrak{K}$ is transitive on $T(3)$, we see by a theorem of Jordan ([22], 3.6) that $Ns\,\mathfrak{L}_3 \cap \mathfrak{K}$ is transitive on $\Phi_3$. Hence $Ns\,\mathfrak{L}_3/\mathfrak{L}_3$ is triply transitive on $\Phi_3 \cup \{1, 2\}$ and has the order $(i_3+2)(i_3+1)i_3$. Since $i_3$ is odd, we obtain by a theorem of Zassenhaus ([24]) that $Ns\,\mathfrak{L}_3/\mathfrak{L}_3 \simeq LF(2, 2^m)$, where $2^m = i_3 + 1$.

Now if $i_3 \geqq \sqrt{p}$, then we obtain as in **9.** that $p \leqq 37$. Hence again by (14) we have only the following possibilities $p = 7$; 11; 19; 23; 31. Here $3p-2$ cannot be a prime number. In fact, otherwise, since the degree of $C_i$ equals $3p-2$, the order of $\mathfrak{D}$ must be divisible by $3p-2$ by a well known theorem and this implies the triple transitivity of $\mathfrak{G}$ contradicting our assumption $b = 1$. So it remains only the following two possibilities $p = 19$; 31. Furthermore if $\mathfrak{L}_3$ has the domain of transitivity of length $>5$, the same method as in 9 assures us that $p < 19$. Hence we can assume that $\mathfrak{L}_3$ does not possess any domain of transitivity of length $>5$. The order of $\mathfrak{L}_3$ is therefore of the form $3^{\mu}5^{\nu}$. If $p=31$, then since the order of $\mathfrak{K}$ is, as is noticed above, divisible by 91, we have that $t_3\equiv 0 \pmod{91}$. This contradicts (34), because $t_1=t_2\geq 1$. So we must have that $p=19$. Let $k_3$ denote the number of domains of transitivity of $\mathfrak{L}_3$ with length 5. Then we have the following equality: $2+i_3+3j_3+5k_3=57$. The same method as in **9** shows us that $k_3\geqq i_3(i_3+1)$. Hence we have that $i_3+5i_3(i_3+1)\leqq 55$, whence follows that $i_3\leqq 3$. This contradicts our assumption that $i_3\geqq 19>4$.

Therefore we can assume that $i_3<\sqrt{p}$. Then using this inequality we have from (27.1), (35) and (37) that

$$\frac{9p-21}{5}+\frac{4}{5}\sqrt{p}+\frac{4}{5}\left(4p+\frac{4\sqrt{3p}}{3}\right)+\frac{2}{5}\left(\frac{2}{3}p+\frac{2\sqrt{3p}}{9}\right)>6p-4.$$

Then we have easily that $p<19$. This is, as is already shown above, a contradiction.

Next let us assume that $\Phi_3$ is not contained in $T_3$. Then without loss of generality we can assume that $\Phi_3$ contains a symbol of $T(1)$ and namely $\alpha_1$. Then $\mathfrak{L}_3$ is contained in $\mathfrak{L}_1$. Since we can choose the symbol $\alpha_2$ in such a way that the cycle structure of $J$ has the form $J=(12)(\alpha_3)(\alpha_1\alpha_2)\ldots$, we can assume that $\mathfrak{L}_3$ is also contained in $\mathfrak{L}_2$. In particular we have that $t_3\equiv 0$ (mod $t_1(=t_2)$). In this case $\Phi_1$, $(\Phi_2)$ the sets of all the symbols of $T(1)+T(2)+T(3)$, each of which is fixed by all the permutations of $\mathfrak{L}_1(\mathfrak{L}_2)$, must be contained in $T(1)+T(2)$. Otherwise, for instance, if $\Phi_1$ is not contained in $T(1)+T(2)$, we obtain that $\mathfrak{L}_1\subseteq\mathfrak{L}_3$ and $t_1=t_2=t_3$. The latter fact contradicts (34). In particular we have that $t_1>t_3$. If $t_3:t_1>3$, then we have from (34) that $t_1<\frac{3}{7}p-\frac{2}{7}$. Now using the fact $\Phi_1\cup\Phi_2\subseteq T(1)+T(2)$ we obtain from (27.1), (35) and (37) the following inequality

$$\frac{9p-2}{5}+\frac{4}{5}\left(\frac{12}{7}p-\frac{8}{7}\right)+\frac{4}{5}\left(2p+\frac{2\sqrt{3p}}{3}\right)+\frac{2}{5}\left(\frac{2p}{3}+\frac{2\sqrt{2p}}{9}-2\right)\geqq 6p-4.$$

This implies a contradiction that $p<5$. Hence we must have that $t_3=3t_1$. Then we have from (34) that $t_1=\frac{3}{5}p-\frac{2}{5}$. Finally using again $\Phi_1\cup\Phi_2\leq T(1)+T(2)$ we obtain from (27.1), (35) and (37) the following inequality

$$\frac{9p-21}{5}+\frac{4}{5}\left(\frac{12}{5}p-\frac{8}{5}\right)+\frac{4}{5}\left(2p+\frac{2\sqrt{3p}}{3}\right)+\frac{2}{5}\left(\frac{2p}{3}+\frac{2\sqrt{3p}}{9}-2\right)\geqq 6p-4.$$

This implies a contrudiction that $p<7$.

Hence we can assume that at least one of $\mathfrak{L}_k$ $(k=1, 2, 3)$, say $\mathfrak{L}_1$, has only domains of transitivity with length either 1 or 3 from $\Omega$. Then $\mathfrak{L}_1$ must be an elementary abelian 3-group. On the other hand, $\mathfrak{G}$ qossesses an irreducible character of degree $3p-2$, for instance, $\mathbf{C}_1$. Therefore by a famous theorem $g$ and hence the order of $\mathfrak{R}$ must be divisible by $3p-2$. Hence finally $t_1$ must be divisible by $3p-2$. By (34) this is a contradiction.

Therefore the case in which the degree of $\mathbf{B}$ is $3p-1$ cannot occur.

§ **3.** The case in which the degree of $\mathbf{B}$ is $2p-1$.

**11.** Now let us assume that the degree of $\mathbf{B}$ equals $2p-1$. Then the equations (4.1) and (10) read as follows:

$$\alpha(X)=\mathbf{A}(X)+\mathbf{B}(X)+\mathbf{D}_1(X), \tag{4.3}$$

where $X$ is any element of $\mathfrak{G}$ and the degree of $\mathbf{D}_1$ equals $p$;

$$2(p-1)(2p-1)pz^2=g(2p-1-\mathbf{B}(J))^2. \tag{10.4}$$

By a theorem of Brauer ([3], Lemma 3) we have

$$\mathbf{B}(J)=1 \text{ or } -1.$$

If $\mathbf{B}(J)=-1$, then from (10.4) we obtain the following equality

$$(p-1)(2p-1)z^2=2pg,$$

which shows that $z$ is divisible by $p$. This is a contradiction. Hence we must have

$$\mathbf{B}(J)=1, \tag{38}$$

and (10.4) takes the following form:

$$p(2p-1)z^2=2(p-1)g. \tag{10.5}$$

(10.5) tells us in particular that the order of a Sylow 2-subgroup of $\mathfrak{G}$ equals the power of 2 dividing $2(p-1)$, say $2^{a+1}$. Therefore every character $\mathbf{C}_i$ becomes by (1.1) a character of 2-defect 0 $\left(i=1, \ldots, \frac{1}{2}(p-1)\right)$.

We consider the representation $\mathfrak{D}_1$ corresponding to $\mathbf{D}_1$ and the matrix $\mathfrak{D}_1(J)$ corresponding to $J$. Let us assume that $\mathfrak{D}_1(J)$ possesses the eigenvalues 1 and $-1$ in the multiplicities $m$ and $n$ respectively. Then we have that

$$(39) \qquad m+n=p.$$

On the other hand, again by a theorem of Brauer ([3], Lemma 3) we have

$$(40) \qquad \mathrm{D}_1(J)=m-n=\varepsilon,$$

where $\varepsilon$ is either 1 or $-1$. From (39) and (40) we obtain that

$$(41) \qquad n=\frac{1}{2}(p-\varepsilon).$$

Now since $\mathfrak{G}$ is simple, the determinant of $\mathfrak{D}_1(J)$, $(-1)^n$, must be the unity, and hence $n$ must be even. Here it may be convenient to distinguish two subcases, (I) $p\equiv 1 \pmod 4$ and (II) $p\equiv -1 \pmod 4$, though the second subcase will be eliminated rather promtly later. Then in the subcase (I) (41) and (40) imply that $\varepsilon=1$ and $\mathrm{D}_1(J)=1$. Hence by (38) and (4.3) we have that

$$(42) \qquad \alpha(J)=3.$$

In the subcase (II) (41) and (40) imply that $\varepsilon=-1$ and $\mathrm{D}_1(J)=-1$. Hence by (38) and (4.3) we have that

$$(43) \qquad \alpha(J)=1.$$

**12.** Now we are in a position to apply a method of Wielandt [21]. By (4.3), $\Omega-\{1\}$ is divided into two domains of transitivity of $\mathfrak{H}$, say $T(i)$ $(i=1, 2)$ ([22], 28.4, 29.2). Let $t_i$ be the length of $T(i)$ and assume that $t_1\leqq t_2$. Then we have

$$(44) \qquad t_1+t_2=3p-1$$

and

$$(45) \qquad t_1\leqq\frac{1}{2}(3p-1)\leqq t_2.$$

We define matrices $V(T(i))$ as follows: put $V(T(i))=(v_{k,l})$. Then $v_{k,l}=1$, if there exist an element $X$ of $\mathfrak{G}$ and a symbol $n$ of $T(i)$ such that $X(1)=1$ and $X(n)=k$ hold, and $v_{k,l}=0$ otherwise. $V(T(i))$ is commutative with every matrix of $G$, which is as usual considered as a linear group consisting of permutation matrices. By the definition of $V(T(i))$ we have

$$(46) \qquad E+V(T(1))+V(T(2))=W=\begin{pmatrix} 1 & \cdots & 1 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ 1 & \cdots & 1 \end{pmatrix}.$$

where $E$ is the unit matrix of degree $3p$. Let us bring $\mathfrak{G}$ into the completely reduced form. Then by the lemma of Schur $V(T(i))$ and $W$ become diagonal matrices. Without loss of generality we can assume that the diagonal form of $V(T(i))$ is

$$\begin{bmatrix} z(i,1) & & & & & & \\ & \underbrace{z(i,2) & & \\ & \ddots & \\ & & z(i,2)}_{p} & & & \\ & & & \underbrace{z(i,3) & & \\ & \ddots & \\ & & z(i,3)}_{2p-1} \end{bmatrix}.$$

Now as in [21] we obtain the following:

(47) (i) $z(i, j)$ is a rational integer $(i=1, 2; j=1, 2, 3)$,

and $z(i, 1)=t_i$, $z(i, 2)\neq t_i$ and $z(i, 3)\neq t_i$ $(i=1, 2)$.

(ii) $z(i, 1)+pz(i, 2)+(2p-1)z(i, 3)=0$.

(iii) $z(i, 1)^2+pz(i, 2)^2+(2p-1)z(i, 3)^2=3pt_i$.

Furthermore since $W$ possesses the eigenvalues $3p$ and $0$ in the multiplicities $1$ and $3p-1$ respectively, by (46) we have the following equalities:

(48) $$z(1, i)+z(2, i)=-1 \qquad (i=2, 3).$$

From (i) and (ii) we derive at once that

(49) $$z(i, 3)\equiv t_i \pmod{p}.$$

Moreover we obtain from (iii) that

$$z(i, 3)^2 \leqq \frac{3pt_i}{2p-1}<p^2.$$

In fact assume that

$$t_i \geqq \frac{(2p-1)p}{3}$$

But we have that $\frac{p(2p-1)}{3} \geqq 3p$ for $p\geqq 5$, which contradicts (44). Hence we have that

(50) $$-p < z(i, 3) < p.$$

From (47) (i), (49), (50) and (45) we have that

$$-p < t_1 - z(1, 3) < \frac{1}{2}(5p-1) < 3p$$

and

$$\frac{1}{2}(p-1) < t_2 - z(2, 3) < 4p.$$

Therefore we have

(51) $$t_1 - z(1, 3) = \text{either } p \text{ or } 2p,$$

and

(52) $$t_2 - z(2, 3) = \text{either } p \text{ or } 2p \text{ or } 3p.$$

Among different combinations of (51) and (52) only the following two cases are possible by (48): Case (A) $t_1 - z(1, 3) = p$ and $t_2 - z(2, 3) = 2p$; Case (B) $t_1 - z(1, 3) = 2p$ and $t_2 - z(2, 3) = p$.

At first let us consider Case (A). Then we have from (47) (ii) the following equalities:

(53) $$z(1, 2) = 2p - 1 - 2t_1 \text{ and } z(2, 2) = 2(2p-1) - 2t_2.$$

Substituting (51), (52) and (53) into (47) (iii) we obtain

(54) $$6t_1^2 - 3(4p-1)t_1 + (2p-1)(3p-1) = 0$$

and

(55) $$6t_2^2 - 3(8p-3)t_2 + 4(2p-1)(3p-1) = 0.$$

Similarly in Case (B) we have the following equations:

(56) $$6t_1^2 - 3(8p-3)t_1 + 4(2p-1)(3p-1) = 0$$

and

(57) $$6t_2^2 - 3(4p-1)t_2 + (2p-1)(3p-1) = 0.$$

Now we can show that Case (B) cannot occur. To this end let us consider the quadratic form $Q(T)$ in $T$, which is the left hand side of (57). $Q(T)$ takes its minimum value at $\frac{1}{4}(4p-1)$. By (45) we have that $Q(t_2)$

$\geqq Q\left(\frac{1}{2}(3p-1)\right)$. But a simple calculation shows that $Q\left(\frac{1}{2}(3p-1)\right)$ $=\frac{1}{2}(3p-1)(p-2)>0$. This contradicts (57).

The equation (55) tells us that $t_2$ is divisible by 8. Since $t_2$ is the length of a domain of transitivity of $\mathfrak{H}$, $t_2$ is a divisor of the order of $\mathfrak{H}$, and hence of $g$. Therefore $g$ must be divisible by 8.

Now let us assume that the subcase (II) in **11** does occur. Then we have from (43) that $\frac{1}{2}(3p-1)$ must be even, because $\mathfrak{G}$ is simple and contains no odd permutation. This implies, however, by (10.5) that $g$ cannot be divisible by 8. This is a contradiction.

Now by (42) we see that $3p-1 \not\equiv 0 \pmod 4$. Hence the equations (54) and (55) tells us that the exact powers of 2 dividing $t_1$ and $t_2$ are 2 and 8 respectively.

**13.** Let $\mathfrak{S}$ be a Sylow 2-subgroup of $\mathfrak{G}$, which is contained in $\mathfrak{H}$. Since $\frac{1}{2}t_1$ is odd, $T(1)$ contains a domain of transitivity $T_{\mathfrak{S}}$ of $\mathfrak{S}$ with length 2. Without loss of generality we can put $T_{\mathfrak{S}}=\{2, 3\}$. Let $\mathfrak{T}_1$ denote the subgroup of $\mathfrak{S}$ consisting of all the permutations of $\mathfrak{S}$ each of which fixes each of the symbols 2 and 3. Then $\mathfrak{T}_1$ has index 2 with respect to $\mathfrak{S}$. Let us consider $\mathfrak{T}_1$ as a permutation group on $T(2)$. Then by (42) $\mathfrak{T}_1$ must be semi-regular on $T(2)$. In particular we have that $t_2 \equiv 0 \pmod{2^a}$. This implies, together with the fact remarked at the end of **12**, that $8 \equiv 0 \pmod{2^a}$. Therefore the order of $\mathfrak{S}$ equals either 8 or 16.

Now we want to show that $\mathfrak{S}$ contains a cyclic normal subgroup of index 2. At any rate $\mathfrak{S}$ contains an element $X$ with the cycle structure $(1)(23)\ldots$. Assume that there exists such an element $X$ with order greater than 2, say $2^b$ $(b \geqq 2)$. Let $(1)(23)Y$ be the cycle structure of $X$. Then by (42) $Y$ consists of cycles of order $2^b$. Since $\mathfrak{G}$ contains no odd permutation, the number $3(p-1)/2^b$ must be odd. This implies that $b=a$. So we can assume that every element $X$ with the cycle structure $(1)(23)\ldots$ is an involution. At any rate we have the decomposition $\mathfrak{S}=\mathfrak{T}_1\langle X\rangle$ with $\mathfrak{T}_1 \cap \langle X\rangle=1$. By (42) $X$ fixes just two symbols of $\Omega$, which are different from 1, 2 and 3, say 4 and 5. Let us consider the centralizer $Zs_{\mathfrak{S}}X$ of $X$ in $\mathfrak{S}$. Then since by (42) every element $Y \neq 1$ of $\mathfrak{T}_1$ fixes no symbol of $\Omega$, which is different from 1, 2 and 3, we see that the order of $Zs_{\mathfrak{S}}X$ equals four. Hence by a theorem of Suzuki

([16], Lemma 4) $\mathfrak{S}$ contains an element of order $2^a$.

Moreover an ordinary transfer argument (see for example [19]) assures us that $\mathfrak{S}$ cannot be abelian. Therefore if $\mathfrak{S}$ is of order 8, we see, using a theorem of Brauer-Suzuki [9], that $\mathfrak{S}$ is a dihedral group.

Our next aim is to show that the order of $\mathfrak{S}$ cannot be 16. Let us assume that the order of $\mathfrak{S}$ is 16. Let us consider $\mathfrak{S}$ on $T(2)$. Then $\mathfrak{S}$ cannot be semi-regular on $T(2)$. In fact, otherwise, we have the congruence $t_2 \equiv 0 \pmod{16}$, which implies the contradiction $8 \equiv 0 \pmod{16}$. Hence there exists a symbol of $T(2)$, say 4, and an element $B \neq 1$ of $\mathfrak{S}$ such that $B$ fixes 4. Let $\mathfrak{T}_2$ denote the subgroup of $\mathfrak{S}$ consisting of all the permutations of $\mathfrak{S}$ each of which fixes the symbol 4. Then since $t_2$ is even, $\mathfrak{T}_2$ fixes at least, and by (42) just, one more symbol of $T(2)$, say 5. Moreover by (42) we have $\mathfrak{T}_1 \cap \mathfrak{T}_2 = 1$, which implies that the order of $\mathfrak{T}_2$ equals 2. Hence $B$ generates $\mathfrak{T}_2$. $B$ has the cycle structure $(1)(23)(4)(5) \ldots$. Let $A$ be an element of $\mathfrak{S}$ of order 8. Then the cycle structure of $A$ must have the form $(1)(23)A^*$, where $A^*$ consists of cycles of order 8. In fact, otherwise, it must have the form $(1)(2)(3)A^*$, which contradicts the simplicity of $\mathfrak{G}$, because $(p-1)/8$ is odd. Let us assume that $\mathfrak{S}$ is not a dihedral group. Then by a theorem of Suzuki ([16], Lemma 4) we have that $BAB = A^3$. Then $\mathfrak{H}$ contains just two classes of involutions, namely the class of $A^4$ and that of $B$. Let $z_1$ and $z_2$ denote the orders of centralizers of $A^4$ and $B$ in $\mathfrak{H}$ respectively. Let $g(2)$ and $h(2)$ denote the numbers of involutions in $\mathfrak{G}$ and in $\mathfrak{H}$ respectively. Then by (42) we have the following equality

$$g/z = g(2) = ph(2) = p(h/z_1 + h/z_2),$$

which implies the equality

$$3/z = 1/z_1 + 1/z_2. \tag{58}$$

If the centralizer $ZsA^4$ of $A^4$ in $\mathfrak{G}$ contains an element with the cycle structure $(123) \ldots$, we have $z = 3z_1$. Then (58) implies that $1/z_2 = 0$, which is a contradiction. $ZsA^4$ contains $B$. Hence if $ZsA^4$ contains no element with the cycle structure $(123) \ldots$, then we have $z = z_1$. Then (58) implies that $z_1 = 2z_2$. But the indices of the centralizers of involutions in $\mathfrak{S}$ with respect to $\mathfrak{S}$ are either 1 or 4. This contradicts that $z_1 = 2z_2$. Thus $\mathfrak{S}$ must be a dihedral group of order 16.

Let us consider $\mathfrak{H}$ on $T(2)$. Then since $B$ (or $A$) is odd on $T(2)$, $\mathfrak{H}$ contains a normal subgroup $\mathfrak{H}^*$ of index 2, which consists of even permutations of $\mathfrak{H}$ on $T(2)$. A Sylow 2-subgroup $\mathfrak{S} \cap \mathfrak{H}^*$ of $\mathfrak{H}^*$ is generated by $A^2$ and $AB$. $A^4$ and $AB$ are not conjugate in $\mathfrak{H}^*$. Then since $\mathfrak{S} \cap \mathfrak{H}^*$ is a dihedral group of order 8, an ordinary transfer argument assures us that $\mathfrak{H}^*$ contains a normal subgroup of index 2. Then since $\mathfrak{H}$ contains a normal subgroup of index 4, $\mathfrak{H}$ contains the normal Sylow 2-complement $\mathfrak{U}$ (for instance see [13], Lemma 8). Let $\mathfrak{K}_1$ denote the subgroup of $\mathfrak{H}$ consisting of all the permutations of $\mathfrak{H}$ each of which fixes the symbol 2. Similarly let $\mathfrak{K}_2$ denote the subgroup of $\mathfrak{H}$ corresponding to 4 instead of 2. Moreover let $\mathfrak{H}'(2)$ denote the 2-commutator subgroup of $\mathfrak{H}$. Then since $\mathfrak{H}$ is 2-nilpotent, the index of $\mathfrak{H}'(2)$ in $\mathfrak{H}$ equals 4. It is easy to see that the indices of $\mathfrak{H}'(2)\mathfrak{K}_i$ with respect to $\mathfrak{H}$ are equal to 2 $(i=1, 2)$. Therefore $\Omega$ is divided into 5 domains of transitivity of $\mathfrak{H}'(2)$. Then we have the following equation: $\sum_{H\in\mathfrak{H}'(2)} \alpha(H) = 5h_2'$, where $H$ ranges over all the elements of $\mathfrak{H}'(2)$ and $h_2'$ is the order of $\mathfrak{H}'(2)$. Obviously $\sum_{H\in\mathfrak{H}'(2)} \mathbf{A}(H) = h_2'$. Furthermore since $\mathbf{C}_i$ is a character of 2-defect 0 $\left(i=1, 2, \dots, \frac{p-1}{2}\right)$, we have by (1.1) $\mathbf{B}(S)=1$ for every 2-singular element $S$ of $\mathfrak{G}$. Then since every element $H$ outside $\mathfrak{H}'(2)$ is 2-singular, we have that $\sum_{\mathfrak{H}\in\mathfrak{H}'(2)} \mathbf{B}(\mathfrak{H}) = h_2'$. Therefore using (4.3) we obtain the following equation

$$\sum_{H\in\mathfrak{H}'(2)} \mathbf{D}_1(H) = 3h_2'. \tag{59}$$

Let $e$ and $f_1$ be the principal characters of $\mathfrak{H}'(2)$ and $\mathfrak{H}$ respectively. Let $f_i$ $(i=2, 3, 4)$ be the linear characters of $\mathfrak{H}$ containing $\mathfrak{H}'(2)$ in their kernels and different from $f_1$. They can be indexed so that the following character table hold.

| | $A^4$ | $B$ | $AB$ | $A$ |
|---|---|---|---|---|
| $f_2$ | 1 | 1 | $-1$ | $-1$ |
| $f_3$ | 1 | $-1$ | 1 | $-1$ |
| $f_4$ | 1 | $-1$ | $-1$ | 1 |

Let $e^*$ and $f_i^*$ be the characters of $\mathfrak{G}$ induced by $e$ and $f_i$ $(i=1, 2, 3, 4)$. Then we have the equations:

$$e^* = f_1^* + f_2^* + f_3^* + f_4^*$$

and

$$f_1^* = \alpha = \mathbf{A} + \mathbf{B} + \mathbf{D}_1.$$

Furthermore by the reciprocity theorem of Frobenius we have from (59) the following equation:

$$e^* = \mathbf{A} + \mathbf{B} + 3\,\mathbf{D}_1 + \sum_{\lambda>1} d_\lambda \mathbf{D}_\lambda,$$

where $\mathbf{D}_\lambda$ ranges some irreducible characters of $\mathfrak{G}$ of $p$-defect 0. (We assume that $d_\lambda > 0$). From these equations we have the following equation:

$$f_2^* + f_3^* + f_4^* = 2\,\mathbf{D}_1 + \sum_{\lambda>1} d_\lambda \mathbf{D}_\lambda.$$

No $f_k^*$ ($k = 2, 3, 4$) has the form: $f_k^* = 2\,\mathbf{D}_1 + \cdots$. In fact, otherwise, we have that $f_k^* = 2\,\mathbf{D}_1 + \mathbf{D}_2$, where the degree of $\mathbf{D}_2$ equals $p$. Then we must have, as is shown in **11**, that $\mathbf{D}_2(J) = 1$ for every involution $J$ of $\mathfrak{G}$, and therefore that $f_k^*(J) = 3$. Let $X_i$ be a permutation of $\mathfrak{G}$ which transfers the symbol 1 to $i$ ($i = 1, 2, \ldots, 3p$). Then we have a decomposition of $\mathfrak{G}$ into the cosets of $\mathfrak{H}$: $\mathfrak{G} = \sum_{i=1}^{n} \mathfrak{H} X_i$. Now from the definition of induced characters we have that $f_k^*(J) = f_k^*(B) = f_k(B) + f_k(X_4^{-1}BX_4) + f_k(X_5^{-1}BX_5)$, which is less than 3 if $k = 3$ or 4, and that $f_k^*(J) = f_k^*(AB) = f_k(AB) + \cdots$, whch is less than 3 if $k = 2$. Anyway this is a contradiction.

Therefore either $f_2^*$ or $f_3^*$ takes the form: $f_l^* = \mathbf{D}_1 + \cdots$ ($l = 2$ or 3).

Since $f_l^*$ cannot be decomposed into characters of degree $p$ from the same reason as above, we have that $f_l^* = \mathbf{D}_1 + \mathbf{D}_2$, where the degree of $\mathbf{D}_2$ equals $2p$. Using again a theorem of Brauer [3], Lemma 3, we have that $D_2(J) = 2$ or $-2$ for every involution $J$ of $\mathfrak{G}$. The case $D_2(J) = 0$ can be eliminated from the simplicity of $\mathfrak{G}$. Since $f_l^*(J) < 3$ we must have here that $D_2(J) = -2$, and thetefore that $f_l^*(J) = -1$. Now from the definition of induced characters and from the fact that $A^4$, $B$ and $AB$ are conjugate with each other, we have that $f_l^*(J) = f_l^*(A^4) = f_l(A^4) + f_l(B) + \cdots$, which is not less than 1 if $l = 2$ and that $f_l^*(J) = f_l^*(A^4) = f_l(A^4) + f_l(AB) + \cdots$, which is not less than 1 if $l = 3$. This is a contradiction.

**14.** Since $\mathfrak{S}$ is a dihedral group of order 8, there exists an involution $B$ of $\mathfrak{S}$ such that the cycle structure of $B$ has the form (1), (23) .... Let $A$

be an element of $\mathfrak{S}$ with order 4. Then since $\frac{1}{4}\cdot 3(p-1)$ is odd, the cycle structure of $A$ has the form $(1), (23)A^*$, where $A^*$ consists of cycles of order 4.

Now we are in a position to use in full some excellent results of Brauer and Suzuki concerning the groups which satisfy the following two conditions: (i) Their Sylow 2-subgroups are dihedral groups of order either 8 or 4. (ii) They contain no normal subgroup of index 2 ([4], [17] and [13]).

Our group $\mathfrak{G}$ with a dihedral Sylow 2-subgroup of order 8 certainly satisfies these two conditions. Hence the principal 2-block of irreducible characters of $\mathfrak{G}$ consists of five characters $\mathbf{A}$ and $\mathbf{X}_i$ $(i=1, 2, 3, 4)$, whose degrees satisfy the following equalities:

$$\mathbf{X}_4(1)=\varepsilon+\mathbf{X}_1(1)=\mathbf{X}_2(1)+\varepsilon'\mathbf{X}_3(1), \tag{60}$$

where $\varepsilon$ and $\varepsilon'$ equal either 1 or $-1$. Since every $\mathbf{C}_j$ is a character of defect 0 for 2, we have $\mathbf{C}_j \neq \mathbf{X}_i$. Then it is easy to see from (60) that $\mathbf{X}_1=\mathbf{B}$, $\varepsilon=1$ and $\varepsilon'=1$.

Put $z=8y$. Let $ZsA$, $ZsA^2$, $ZsB$, $ZsAB$ and $Zs\mathfrak{S}$ be the centralizers of $A$, $A^2$, $B$, $AB$ and $\mathfrak{S}$ in $\mathfrak{G}$. Furthermore we denote by $2l$, $4lu$, $4lu_1$ and $4lu_2$ the orders of $Zs\Xi$, $ZsA \cap ZsA^2$, $ZsB \cap ZsA^2$ and $ZsAB \cap ZsA^2$. Then the first formula of Suzuki concerning the order of $\mathfrak{G}$ is as follows:

$$g=\frac{32yu^2(u_1+u_2)^2p(2p-1)}{(p-1)^2}. \tag{61}$$

Now we want to show by means of a contradiction that $\mathfrak{H}$ contains a normal subgroup of index 2. So let us assume that $\mathfrak{H}$ contains no normal subgroup of index 2. Then since $\mathfrak{H}$ also satisfies the above two conditions, we have the equality analogous to (61). It is clear from our choice of the elements $A$ and $B$ that $Zs\Xi$, $ZsA \cap ZsA^2$, $ZsB \cap ZsA^2$ and $ZsAB \cap ZsA^2$ are contained in $\mathfrak{H}$. Let $8y'$ be the order of $ZsA^2 \cap \mathfrak{H}$ and let $\mathbf{X}_1'$ be the irreducible character of $\mathfrak{H}$ corresponding to $\mathbf{X}_1=\mathbf{B}$ of $\mathfrak{G}$. Then the first formula of Suzuki for $\mathfrak{H}$ is as follows:

$$\frac{g}{3p}=\frac{64y'u^2(u_1+u_2)^2\mathbf{X}_1'(1)(\mathbf{X}_1'(1)+\varepsilon')}{(\mathbf{X}_1'(1)-\varepsilon')^2}, \tag{62}$$

where $\varepsilon'$ equals $\pm 1$. Furthermore all the involutions in $\mathfrak{H}$ are conjugate to one another. Hence corresponding to (58) we have here that $y=3y'$. Then we obtain from (61) and (62) the following equality:

$$\frac{X_1'(1)(X_1'(1)+\varepsilon')}{(X_1'(1)-\varepsilon')^2}=\frac{2p-1}{2(p-1)^2}. \tag{63}$$

(63) implies at once that $\varepsilon'=-1$. Furthermore it is easy to check that the right-hand side of (63) is smaller than $\frac{1}{2}$ and that the left-hand side of (63) is greater than $\frac{1}{2}$. In the latter case we use the congruence $X_1'(1)\equiv\varepsilon'$ (mod 8) due to Brauer and Suzuki. This is a required contradiction. Hence $\mathfrak{H}$ contains a normal subgroup $\mathfrak{H}^*$ of index 2.

Then we want to show that $\mathfrak{H}^*$ contains no normal subgroup of index 2. Assume that $\mathfrak{H}^*$ contains a normal subgroup of index 2. Then $\mathfrak{H}$ is 2-nilpotent. Let $\mathfrak{H}'(2)$ denote the 2-commutator subgroup of $\mathfrak{H}$. Then the index of $\mathfrak{H}'(2)$ in $\mathfrak{H}$ equals 4. It is eas to see that $\Omega$ is divided into either 5 or 7 domains of transitivity of $\mathfrak{H}'(2)$. But if $\Omega$ is divided into 5 domains of transitivity of $H'(2)$, we obtain the same contradiction as at the end of **13.** So let us assume that $\Omega$ is divided into 7 domains of transitivity of $\mathfrak{H}'(2)$. Then it follows that $\mathfrak{S}$ is semi-regular on $T(2)$. Anyway we can use the same notation as in **13.** (Instead of $A^4$ there we must consider here $A^2$). Then we have the equations:

$$e^*=\mathbf{A}+\mathbf{B}+5\,\mathbf{D}_1+\sum_{\lambda>1}d_\lambda\mathbf{D}_\lambda \tag{64}$$

and

$$f_2^*+f_3^*+f_4^*=4\,\mathbf{D}_1+\sum_{\lambda>1}d_\lambda\mathbf{D}_\lambda.$$

Then some $f_k^*$ ($k=2,3,4$) must have the form: $f_k^*=3\,\mathbf{D}_1$ or $f_k^*=2\,\mathbf{D}_1+\cdots$, which gives us a contradiction as in **13.** Thus $\mathfrak{H}^*$ contains no normal subgroup of index 2.

Now the group $\mathfrak{H}^*$ with an elementary abelian Sylow 2-subgroup of order 4 satisfies the two conditions at the beginning of this section. The principal 2-block of irreducible characters of $\mathfrak{H}^*$ consists of four characters $X_i^*$ ($i=0,1,2,3$), where $X_0^*$ is the principal character of $\mathfrak{H}^*$. Let $4l^*$ be the order of the centralizer $Zs_{\mathfrak{H}}(\mathfrak{S}\cap\mathfrak{H}^*)$ of $\mathfrak{S}\cap\mathfrak{H}^*$ in $\mathfrak{H}^*$ and let $u^*$ be the index of $Zs_{\mathfrak{H}^*}(\mathfrak{S}\cap\mathfrak{H}^*)$ in $ZsA^2\cap\mathfrak{H}^*$. Then we have the following formula of Brauer concerning the order of $\mathfrak{H}^*$:

$$\frac{g}{6p}=\frac{32\,u^{*3}l^*X_1^*(1)X_2^*(1)X_3^*(1)}{(X_1^*(1)+\delta_1)(X_2^*(1)+\delta_2)(X_3^*(1)+\delta_3)}, \tag{65}$$

where $\delta_i$ equals $\pm 1$.

Further we need the second formula of Suzuki concerning the order of $\mathfrak{G}$, which is, using the facts $\mathbf{X}_1 = \mathbf{B}$, $\varepsilon = 1$ and $\varepsilon' = 1$ in (60), stated as follows:

$$g = \frac{128\, uy^2(2p-1)p}{l(p-1)^2}. \tag{66}$$

From (61) and (66) we obtain the equality

$$y = \frac{1}{4}\, l\, u(u_1 + u_2)^2. \tag{67}$$

On the other hand, it is easy to see that $ZsA^2$ contains a normal Sylow 2-complement $\mathfrak{U}$. Let us consider $\mathfrak{S}/\langle A^2\rangle$ as usual as an operator group of $\mathfrak{U}$. Then among the orders of subgrops which consist of all the elements of $\mathfrak{U}$ each of which is fixed by $A\langle A^2\rangle$, $B\langle A^2\rangle$, $AB\langle A^2\rangle$ and $\mathfrak{S}/\langle A^2\rangle$ respectively, there holds the following identity of Brauer-Wielandt ([23]), (1.1)):

$$y = l\, uu_1u_2. \tag{68}$$

From (67) and (68) we obtain at once that

$$u_1 = u_2. \tag{69}$$

Since $\mathfrak{H}$ contains a normal subgroup of index 2, there are more than one class of involutions in $\mathfrak{H}$. Therefore the same considerations which led us to (58) yield here that $ZsA^2$ is contained in $\mathfrak{H}$. Now since every 2-regular element of $\mathfrak{H}$ is contained in $\mathfrak{H}^*$, we have together with (69) the following

$$l^* = l\, u_1, \tag{70}$$

and

$$y = l\, u_1 u^*. \tag{71}$$

Now using (68), (69), (70) and (71) we obtain from (65) and (66) the following equality:

$$\frac{2(2p-1)}{(p-1)^2} = \frac{3\,\mathbf{X}_1^*(1)\mathbf{X}_2^*(1)\mathbf{X}_3^*(1)}{(\mathbf{X}_1^*(1)+\delta_1)(\mathbf{X}_2^*(1)+\delta_2)(\mathbf{X}_3^*(1)+\delta_3)}. \tag{72}$$

Obviously the right-hand side of (72) is not smaller than 3/8. Therefore we have the following inequality

$$0 \geqq 3p^2 - 38p + 19.$$

This implies that $p \leqq 11$. Since $p \equiv 1 \pmod 4$, we can conclude that $p = 5$. Thus again we have only to check six primitive groups of degree 15 and will find that only the group isomorphic to $\mathfrak{A}_6$ satisfies our requirements. It may be convenient to refer to some data: $p = 5$; $t_1 = 6, t_2 = 8$; $z(1, 2) = -3$, $z(2, 2) = 2$, $z(1, 3) = 1$, $z(2, 3) = -2$; $y = u = u_1 = u_2 = 1 = 1$; $\mathbf{X}_1^*(1) = 3$, $\mathbf{X}_2^*(1) = \mathbf{X}_3^*(1) = 1$, $\delta_1 = -1$, $\delta_2 = \delta_3 = 1$.

§ **4.** The case in which the degree of $\mathbf{B}$ is $p-1$.

**15.** Now let us consider the case in which the degree of $\mathbf{B}$ equals $p-1$. Then (4.1) takes one of the following forms:

$$\alpha(X) = \mathbf{A}(X) + \mathbf{B}(X) + \mathbf{D}_1(X), \tag{4.4}$$

where $\mathbf{D}_1$ is an irreducible character of $\mathfrak{G}$ with degree $2p$;

$$\alpha(X) = \mathbf{A}(X) + \mathbf{B}(X) + \mathbf{D}_1(X) + \mathbf{D}_2(X), \tag{4.5}$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are different irreducible characters of $\mathfrak{G}$ with degree $p$;

$$\alpha(X) = \mathbf{A}(X) + \mathbf{B}(X) + 2\,\mathbf{D}_1(X), \tag{4.6}$$

where $\mathbf{D}_1$ is an irreducible character of $\mathfrak{G}$ with degree $p$. Moreover (10) becomes the following form:

$$(p-2)(p-1)pz^2 = g(p-1-\mathbf{B}(J))^2. \tag{10.6}$$

By a theorem of Brauer ([3], Lemma 3) we have that $\mathbf{B}(J) = 0$. Therefore we obtain from (10.6) the following

$$(p-2)pz^2 = g(p-1). \tag{10.7}$$

(10.7) tells us in particular that the order of a Sylow 2-subgroup of $\mathfrak{G}$ equals the power of 2 dividing $p-1$, say $2^a$. Therefore $\mathbf{B}$ becomes a character of defect 0 for 2. Hence as in **4** by a theorem of Brauer-Tuan ([10], Corollary of Lemma 3) we see that every $\mathbf{C}_i$ belongs to the principal 2-block $B_1(2)$ of irreducible characters of $\mathfrak{G}$ $(i = 1, \ldots, \frac{1}{2}(p-1))$.

Assume that $a = 2$. Then by a theorem of Brauer-Feit ([6], Theorem 1) $B_1(2)$ contains at most 5 characters. Therefore we have the inequality $5 \geqq \frac{1}{2}(p+1)$, which implies that $p = 5$. So we have only to consider again 6 types of primitive groups of degree 15. It is easy to check that there is no group among them with required properties. Therefore we can assume that

$a \geqq 3$.

Since $p \equiv 1 \pmod 4$, we obtain, as in (39)-(41), that $\mathrm{D}_1(J) = 2$ in Case (4.4); $\mathrm{D}_i(J) = 1$ $(i = 1, 2)$ in Case (4.5) and $\mathrm{D}_1(J) = 1$ in Case (4.6). Hence we have

$$(73) \qquad \alpha(J) = 3.$$

**16.** First of all we want to deal with Case (4.4). Then by (4.4) $\Omega - \{1\}$ is divided into two domains of transitivity of $\mathfrak{H}$, say $T(i)$ $(i = 1, 2)$ ([22], 28.4, 29.2). Let $t_i$ be the length of $T(i)$ $(i = 1, 2)$. Then we have

$$(44.1) \qquad t_1 + t_2 = 3p - 1.$$

We see at once from (44.1) that $t_1$ and $t_2$ are simultaneously even or simultaneously odd. Assume that $t_1$ and $t_2$ are odd. Let $x \neq 1$ be any symbol of $\Omega$ and let $\mathfrak{K}$ denote the subgroup of $\mathfrak{G}$ consisting of all the permutations of $\mathfrak{G}$ each of which fixes each of the symbols 1 and $x$ of $\Omega$. Then it follows from our assumption that $\mathfrak{K}$ contains a Sylow 2-subgroup of $\mathfrak{G}$. Hence $\mathfrak{G}$ cannot contain an involution whose cycle structure has the form $(1x)\dots$. Since $x \neq 1$ is an arbitrary symbol of $\Omega$, every involution must fix the symbol 1 of $\Omega$, which contradicts the simplicity of $\mathfrak{G}$. Therefore $t_1$ and $t_2$ are even.

Since $p \equiv 1 \pmod 4$, we see by (44.1) that either $t_1$ or $t_2$ is semi-odd, say $t_1$. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $\mathfrak{G}$, which is contained in $\mathfrak{H}$. Let us consider $\mathfrak{S}$ as a permutation group on $T(1)$. Then $T(1)$ contains a domain of transitivity of $\mathfrak{S}$ with length 2, say $\{2, 3\}$. Let $X$ be any element of $\mathfrak{S}$ whose cycle structure has the form $(1), (23)\dots$. Assume that the order of $X$ is $2^b$ with $b > 1$. Then we see by (73) that the cycle structure of $X$ has the form $(1)(23)Y$, where $Y$ consists of cycles of order $2^b$. Since $\mathfrak{G}$ is simple and hench $X$ must be even, $3(p-1)/2^b$ must be odd. This implies that $b = a$ and hence that $\mathfrak{S}$ is cyclic. This is a contradiction. Thus $X$ must be an involution. By (73) $X$ fixes just two symbols of $\Omega - \{1\}$, say 4 and 5. Now let $\mathfrak{T}$ denote the subgroup of $\mathfrak{S}$ consisting of all the permutations of $\mathfrak{S}$ each of which fixes the symbol 2. Then the index of $\mathfrak{T}$ in $\mathfrak{S}$ equals 2. Let us consider the centralizer of $X$ in $\mathfrak{T}$. Then since by (73) every element $\neq 1$ of $\mathfrak{T}$ does not fix the symbol 4, the centralizer of $X$ in $\mathfrak{T}$ has order 2. Therefore by a theorem of Suzuki ([16], Lemma 4) $\mathfrak{S}$ contains an element $Z$ such that $\mathfrak{S} = \langle X \rangle \langle Z \rangle$. Since $XZ$ is an involution, we have $XZX = Z^{-1}$. Therefore $\mathfrak{S}$ is a dihedral group of order $2^a$ with $a \geqq 3$.

Let $B_1(2)$ be the principal 2-block of irreducible characters of $\mathfrak{G}$. Then using a method of Suzuki ([13], (42)-(43)) we see that $B_1(2)$ contains two irreducible characters $\mathbf{X}_1$ and $\mathbf{X}_4$ whose degrees satisfy the equality

$$1+\delta_1\mathbf{X}_1(1)=\delta_1\mathbf{X}_1(1), \tag{74}$$

where $\delta_1$ equals $\pm 1$. We see at once from (74) that either $\mathbf{X}_1$ or $\mathbf{X}_4$ must be equal to some $\mathbf{C}_i$. But since $\mathbf{B}$ is a character of defect 0 for 2, (74) gives us a contradiction. This contradiction shows that Case (4.4) does not occur.

**17.** Next let us consider Case (4.6). Then by (4.6) $\Omega-\{1\}$ is divided into five domains of transitivity of $\mathfrak{H}$, say $T(i)$ $(i=1,\ldots,5)$ ([22], 28.4, 29.2). Let $t_i$ be the length of $T(i)$ $(i=1,\ldots,5)$. Then we have

$$t_1+t_2+t_3+t_4+t_5=3p-1. \tag{44.2}$$

We see from (44.2) and (73) that either every $t_i$ is even or just two of them, say $t_1$ and $t_2$, are odd. Assume that the former case occurs. Then the method in 16 can be applied and we obtain a contradiction. Therefore we can assume that the latter case occurs.

Then $\mathfrak{S}$ fixes at least one symbol, say 2, of $T(1)$ and at least one symbol, say 3, of $T(2)$. By (73) every element $\neq 1$ of $\mathfrak{S}$ fixes only the symbols 1, 2 and 3. Let $X$ be an element of $\mathfrak{G}$ whose cycle structure has the form $(21\ldots)$ $\ldots$. Then $X^{-1}\mathfrak{S}X$ fixes the symbol 1 and is contained in $\mathfrak{H}$. Therefore by Sylow's theorem there exists an element $Y$ of $\mathfrak{H}$ such that $Y^{-1}\mathfrak{S}Y=X^{-1}\mathfrak{S}X$. Then $YX^{-1}=Z$ is contained in the normalizer $Ns\mathfrak{S}$ of $\mathfrak{S}$ in $\mathfrak{G}$ and has the cycle structure $(12\ldots)\ldots$. Since $\mathfrak{S}$ fixes only the symbols 1, 2 and 3, the cycle structure of $Z$ must have the form $(123)\ldots$. Assume that there exists an involution $W$ in $\mathfrak{S}$ which is commutative with $Z$. Then since the cycle structure of $WZ$ has the form $(123)\ldots$, we have by (73) that $\alpha(WZ)=0$. Moreover since $WZ$ is 2-singular, we have by a theorem of Brauer-Nesbitt ([8], Theorem 1) that $\mathbf{B}(WZ)=0$. Therefore we obtain from (4.6) that $\mathbf{D}_1(WZ)$ $=-\frac{1}{2}$. But since $\mathbf{D}_1(WZ)$ must be an integer, this is a contradiction. Thus there is no such an involution.

Let $V$ be a central involution in $\mathfrak{S}$. Then the above argument implies that $V$ and $Z^{-1}VZ$ are not conjugate in $\mathfrak{H}$. Thus there exist more than one class of involutions in $\mathfrak{H}$. Assume that $t_1=1$. Then the normalizer $Ns\mathfrak{H}$ of $\mathfrak{H}$ in $\mathfrak{G}$ contains an element whose cycle structure has the form $(21\ldots)\ldots$ and is bigger than $\mathfrak{H}$. Then by the primitivity of $\mathfrak{G}$ we must have $\mathfrak{G}=Ns\mathfrak{H}$, which implies by the simplicity of $\bar{\mathfrak{G}}$ that $\mathfrak{H}=1$. Then the order of $\mathfrak{G}$ equals $3p$, which contradicts the simplicity of $\mathfrak{G}$. Thus we have that $t_1>1$. Now $T(1)$ contains at least one symbol, say 4, different from 2. Since $T(1)$ is a domain of transitivity of $\mathfrak{H}$, there exists a Sylow 2-subgroup $\mathfrak{S}^*$ of $\mathfrak{H}$ such that $\mathfrak{S}^*$ fixes the symbols 1, 4 and $x$, where $x$ is a symbol of $T(2)$. Let $U$ be an involution in $\mathfrak{S}^*$, which is not conjugate to $V$. Then by a theorem of Brauer-Fowler ([7], Lemma (3 A)) there must exist an involution $I$ of $\mathfrak{H}$ which is commutative with $U$ and $V$. Since every permutation $\neq 1$ of a Sylow 2-subgroup of $\mathfrak{H}$ fixes the same symbols, this implies that $I$ must fix at least four symbols 1, 2, 3 and 4 contradicting (73). This contradiction shows that Case (4.6) does not occur.

**18.** Finally let us consider Case (4.5). Then by (4.5) $\Omega-\{1\}$ is divided into three domains of transitivity of $\mathfrak{H}$, say $T(i)$ $(i=1, 2, 3)$ ([25], 28.4, 29.2). Let $t_i$ be the length of $T(i)$ $(i=1, 2, 3)$. Then we have

$$t_1+t_2+t_3=3p-1. \tag{44.3}$$

We see from (44.3) that either every $t_i$ is even or just two of them, say $t_1$ and $t_2$, are odd. Assume that the former case occurs. Then the method in **16** can be applied and we obtain a contradiction. Therefore we can assume that the latter case occurs.

If there exist more than one class of involutions in $\mathfrak{H}$, then the method in **17** can be applied and we obtain a contradiction. Therefore we can assume that all the involutions in $\mathfrak{H}$ are conjugate one another in $\mathfrak{H}$.

Now it follows from the argument in **17** that there exist in $\mathfrak{G}$ an involution $W$ and a 3-element $Z$, which satisfy the following two conditions: (i) $W$ and $Z$ are commutative with each other. (ii) $W$ and $Z$ have the cycle structures $(1)(2)(3)\ldots$ and $(123)\ldots$ respectively.

Next let us consider the matrices $V(T(i))$ $(i=1, 2, 3)$ as in **12.** Without loss of generality we can assume that the diagonal form of $V(T(i))$ is

$$\begin{bmatrix}
z(i,1) &&&&&&&&& \\
& z(i,2) &&&&&&&& \\
& {\scriptstyle p-1} & \ddots &&&&&&& \\
&&& z(i,2) &&&&&& \\
&&&& z(i,3) &&&&& \\
&&&& {\scriptstyle p} & \ddots &&&& \\
&&&&&& z(i,3) &&& \\
&&&&&&& z(i,4) && \\
&&&&&&& {\scriptstyle p} & \ddots & \\
&&&&&&&&& z(i,4)
\end{bmatrix}$$

Then as in [21] we obtain the following:

(47.1)

(i) $z(i, j)$ is an algebraic integer ($i=1, 2, 3$; $j=1, 2, 3, 4$). In particular, $z(i, 1)$ and $z(i, 2)$ are rational integers ($i=1, 2, 3$). Furthermore we have that $z(i, 1)=t_i$ and $z(i, j)\neq t_i$ ($i=1, 2, 3$; $j=2, 3, 4$).

(ii) $z(i, 1)+(p-1)z(i, 2)+pz(i, 3)+pz(i, 4)=0$.

(iii) $z(i, 1)^2+(p-1)z(i, 2)^2+p|z(i, 3)|^2+p|z(i, 4)|^2=3pt_i$.

Let us assume that $\mathbf{D}_1$ and $\mathbf{D}_2$ are rational characters. Then using a method of Wielandt ([22], p. 82) we see that every $z(i, j)$ is a rational integer. We consider (47.1) for $i=1$. Then since from our assumptions $t_1$ is odd, we have from (ii) that $z(1, 3)+z(1, 4)\equiv 1 \pmod 2$ and from (iii) that $z(1, 3)^2+z(1, 4)^2 \equiv 0 \pmod 2$. This is a contradiction. Now by (4.5) we see that $\mathbf{D}_2$ (and only $\mathbf{D}_2$) is an algebraically conjugate character of $\mathbf{D}_1$.

Here let us consider the element $WZ$. Assume that $\mathbf{D}_1(WZ)$ is rational. Then since $\mathbf{D}_1$ and $\mathbf{D}_2$ are algebraically conjugate, we have that $\mathbf{D}_1(WZ) =\mathbf{D}_2(WZ)$. On the other hand, since the cycle structure of $WZ$ has the form $(123)\ldots$ we have by (73) that $\alpha(WZ)=0$. Moreover since $WZ$ is 2-singular and $\mathbf{B}$ has 2-defect 0, we have by a theorem of Brauer-Nesbitt ([8], Theorem 1) that $\mathbf{B}(WZ)=0$. Therefore by (4.5) we have that $\mathbf{D}_1(WZ)=-\frac{1}{2}$. Since $\mathbf{D}_1(WZ)$ must be an integer, this is a contradiction.

Let the order of $Z$ be $3^z$. Then $\mathbf{D}_1(WZ)$ belongs to the field of the $3^z$-th roots of unity over the rational number field $\mathbf{Q}$. But this field is a cyclic field over $\mathbf{Q}$ and $\mathrm{D}_1(WZ)$ has degree two over $\mathbf{Q}$, $\mathrm{D}_1(WZ)$ belongs to the field of the cubic roots of unity over $\mathbf{Q}$: $\mathbf{Q}(\omega)$ with $\omega^3=1$, $\omega\neq 1$. Furthermore since $\mathrm{D}_1$ and $\mathrm{D}_2$ are algebraically conjugate only with each other, we see that the field of $\mathrm{D}_i$ over $\mathbf{Q}$, namely the field generated by all the numbers $\mathrm{D}_i(X)$, where $X$ ranges over all the elements of $\mathfrak{G}$, is $\mathbf{Q}(\omega)$ $(i=1, 2)$. Then again using the method of Wielandt ([25], p. 82) we see that all the $z(i, j)$'s belong to $\mathbf{Q}(\omega)$ and that $z(i, 3)$ and $z(i, 4)$ are complex-conjugate numbers $(i=1, 2, 3)$. The latter fact follows from the complex conjugacy of $\mathrm{D}_1$ and $\mathrm{D}_2$.

Now the numbers 1 and $\frac{1}{2}(1+\sqrt{3}\,\mathrm{i})$ constitute an integral basis of $\mathbf{Q}(\omega)$. Therefore we ca ı put

$$(75) \qquad z(i, 3)=\frac{1}{2}(n_i+m_i\sqrt{3}\,\mathrm{i}) \text{ and } z(i, 4)=\frac{1}{2}(n_i-m_i\sqrt{3}\,\mathrm{i}),$$

where $n_i$ and $m_i$ are rational integers $(i=1, 2, 3)$.

Choose a Sylow 2-subgroup $\mathfrak{S}$ of $\mathfrak{G}$ as in **17**. Then by (73) $\mathfrak{S}$ is semi-regular on $T(1)-\{2\}$, $T(2)-\{3\}$ and $T(3)$. Hence we have the congruences:

$$(76) \qquad t_i\equiv 1 \pmod{2^a} \ (i=1, 2) \text{ and } t_3\equiv 0 \pmod{2^a}.$$

Furthermore we see as in **17** that

$$(77) \qquad t_i>1 \ (i=1, 2, 3).$$

Now we obtain from (47.1) (ii) and (75) the following congruences:

$$n_i\equiv -1 \pmod{2^a} \ (i=1, 2) \text{ and } n_3\equiv 0 \pmod{2^a}.$$

Therefore we can put

$$(78) \qquad n_i=A_i2^a-1 \ (i=1, 2) \text{ and } n_3=A_32^a,$$

where $A_i$ is a rational integer $(i=1, 2, 3)$.

At any rate we heve by a theorem of Brauer-Feit ([6], Theorem 1) the following inequality:

$$\frac{1}{2}(p+1)\leqq 2^{2a-2},$$

which implies in particular that

$$(79) \qquad 2^{2a}>2p.$$

Now we want to show that (1) $t_i\geqq p+2$ $(i=1, 2)$ and (2) $t_3\geqq p-1$, which yield us a contradiction $t_1+t_2+t_3 \geqq 3p+3$ to (44.3). We deal only (1), because (2) can be dealt with quite similarly as (1). At first let us assume that $|A_i| \geqq 3$ or $A_i=-2$. Then we have from (78) and (79) that

$$\begin{aligned} n_i^2 &= A_i^2 2^{2a} - A_i 2^{a+1} + 1 \\ &> 8p. \end{aligned}$$

Assume that $A_i=2$. Then we have similarly that

$$\begin{aligned} n_i^2 &= 2^{2a+2} - 2^{a+2} + 1 \\ &> \frac{1}{2}\cdot 7 \cdot 2^{2a} \\ &> 7p. \end{aligned}$$

Hence if $|A_i| \geqq 2$, then we have from (47.1) (iii), (75) and (78) that

$$\begin{aligned} t_i &> (|z(i,\ 3)|^2 + |z(i,\ 4)|^2)/3 \\ &> n_i^2/6 \\ &> 7p/6 \\ &> p+2. \end{aligned}$$

Now we can assume that $|A_i| \leqq 1$. If $A_i=0$, then we have by (47.1) (ii) that

$$t_i = p-(p-1)z(i,\ 2),$$

which implies by (77) that $t_i \geqq p$. But $t_i$ cannot be equal to $p$, because $t_i$ is a divisor of the order of $\mathfrak{H}$. Since $t_i$ is odd, thus we have that $t_i \geqq p+2$. If $A_i=1$, then we have by (47.1) (ii) that

$$t_i = -(p-1)z(i,\ 2) - p(2^a-1).$$

Let us consider a linear form $L(X)=(p-1)X-p(2^a-1)$ in $X$ on the domain of rational integers. $L(X)$ attains its least positive value $p-2^a$ at $X=2^a$. The next least positive value of $L(X)$ is certainly not smaller than $p$. So let us assume that $t_i=p-2^a$ and $z(i,\ 2)=2^a$. Then we have by (76) and (77) that $p>2^{a+1}$. But since $2^a$ is an exact power of 2 dividing $p-1$, we have that $p \geqq 3.2^a$. Then we have further that $(2^a-1)^2 \geqq 4p/3$. Then finally we have by (47.1) (iii) and (79) that

$$\begin{aligned} t_i &\geqq ((t_i^2+(p-1)2^{2a}+\frac{1}{2}p(2^a-1)^2)/3p \\ &> 4p/27+2p/3-2/3+2p/9 \\ &> 28p/27-2/3 \\ &> p. \end{aligned}$$

The case of $A_i = -1$ can be handled quite similarly.

§ **5.** Proof of Theorem 2.

Let $\mathfrak{H}$ denote the subgroup of $\mathfrak{G}$ consisting of all the permutations of $\mathfrak{G}$ each of which fixes the symbol 1 of $\Omega$. Since $\mathfrak{G}$ is imprimitive on $\Omega$ and since $\mathfrak{G}$ is simple, $\bar{\mathfrak{G}}$ contains a subgroup $\mathfrak{M}$ of index $p$ containing $\mathfrak{H}$. Hence by a previous result [14] $\mathfrak{G}$ is isomorphic to a linear fractional group $LF(2, 2^m)$ with $p = 2^m + 1$ $(m \geqq 2)$, and $\mathfrak{M}$ becomes the normalizer of a Sylow 2-subgroup of $\mathfrak{G}$. Conversely let us consider any $LF(2, 2^m)$ such that $p = 2^m + 1$ is a prime number greater than 3. Let $\mathfrak{M}$ be the normslizer of a Sylow 2-subgroup of $LF(2, 2^m)$. Then since $m$ is even, the order of $\mathfrak{M}$ is divisible by 3. Hence $\mathfrak{M}$ contains a (uniquely determined) subgroup of index 3, because the factor group of $\mathfrak{M}$ by its Sylow 2-subgroup is cyclic. Therefore such an $LF(2, 2^m)$ can always be represented (uniquely) as an imprimitive permutation group of degree $3p$.

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