A NOTE ON CONFORMAL MAPPINGS OF CERTAIN RIEMANNIAN MANIFOLDS

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The contents of this note were reported at a meeting of the Japan Mathematical Society five years ago, but it was not printed. Prof. K. Yano advised me to do so and it was as follows.

1. We take n-dimensional compact orientable Riemannian manifolds V and $\bar{V}$, and denote their line elements by ds^2 and $d\bar{s}^2$ and their scalar curvatures by R and $\bar{R}$ respectively (Signs of the curvatures are taken in such a way that they are positive for the spheres). We consider a conformal homeomorphism f from V to $\bar{V}$ and put

$$f^*(d\bar{s}^2) = a^2 ds^2 \qquad (a>0),$$

where f^* means a mapping of differential forms dual to f. We take a neighborhood of any point of V and orthogonal frames on it. Then ds^2 can be written as $ds^2 = \sum_i \omega_i^2$ with 1-forms ω_i $(i=1, \ldots, n)$. We put as usual

$$d(\log a) = \sum_i b_i \omega_i, \qquad b^2 = \sum b_i^2,$$

$$b_{ij} = \nabla_j b_i - b_i b_j + \frac{1}{2} b^2 \delta_{ij},$$

where ∇ means a covariant differentiation with respect to the Riemannian metric on V. Then we get a wellknown formula

$$R - \bar{R}a^2 = 2(n-1)\sum_i b_{ii}, \tag{1}$$

where we write $\bar{R}$ briefly instead of $f^*\bar{R}$. We take a number s which shall be determined later and put

$$a^s d(\log a) = dc = \sum_i c_i \omega_i. \tag{2}$$

Then we have $c_i = b_i a^s$ and

Received October 19, 1961.

$$\nabla_j c_i - (s+1)a^s b_i b_j + \frac{1}{2} a^s b^2 \delta_{ij} = a^s b_{ij}.$$

For Laplacian $\Delta c = \sum_i \nabla_i c_i$ of c we have

$$\Delta c + \left(\frac{n}{2} - 1 - s\right) a^s b^2 = a^s \sum_i b_{ii}.$$

If we choose such s that

$$s = \frac{n}{2} - 1, \tag{3}$$

we have

$$\Delta c = a^s \sum_i b_{ii}. \tag{4}$$

By (1) and (4) we get

$$(R - \bar{R}a^2)a^s = 2(n-1)\Delta c. \tag{5}$$

Thus for s determined by (3) the relations (2) holds good when we take

$$c = \frac{2}{n-2} a^{(n/2)-1} \text{ for } n > 2, \quad \text{and} \quad c = \log a \text{ for } n = 2. \tag{6}$$

2. We denote a volume element of V and $\bar{V}$ by dv and $d\bar{v}$ respectively, each corresponding to the orientations of V and $\bar{V}$. Then we have

$$f^*(d\bar{v}) = a^n dv. \tag{7}$$

Integrating (5) on the whole manifold V we obtain

$$\int_V R a^s dv - \int_V \bar{R} a^{s+2} dv = 2(n-1) \int_V \Delta c \, dv = 0.$$

Hence we have by (7) and (3)

$$\int_V R a^{(n/2)-1} dv = \int_{\bar{V}} \bar{R} a^{-((n/2)-1)} d\bar{v}. \tag{8}$$

Thus we get

THEOREM 1. *We assume that V and $\bar{V}$ are compact orientable Riemannian manifolds whose scalar curvatures are R and $\bar{R}$ respectively, and $\bar{V}$ is conformally homeomorphic to V with a magnification function a. Then a formula* (8) *holds good.*

3. Next we assume that scalar curvatures R and $\bar{R}$ are constant and will prove theorem 2. We take any differentiable function $u = \varphi(c)$ with a function

c on V. Then $du = \varphi'(c)dc$ and when we put $du = \sum_i u_i \omega_i$, $dc = \sum_i c_i \omega_i$, we get

$$u_i = \varphi'(c)c_i.$$

By taking a covariant derivative with respect to the Riemannian metric on V we obtain

$$\nabla_j u_i = \varphi'(c)\nabla_j c_i + \varphi''(c)c_i c_j.$$

Contracting with respect to i and j

$$\Delta u = \varphi'(c)\Delta c + \varphi''(c)\sum_i c_i^2.$$

We denote by dv a volume element on V. By virtue of the relation $\int_V \Delta u\, dv = 0$ we have

$$\int_V \varphi'(c)\Delta c\, dv + \int_V \varphi''(c)\sum_i c_i^2\, dv = 0. \tag{9}$$

If we can find such a function $\varphi(c)$ that

$$\varphi'(c)\Delta c \geq 0, \qquad \varphi''(c) > 0, \tag{10}$$

we have by (9) $\sum_i c_i^2 = 0$ and so $c_i = 0$, and c is constant. We take such c that (6) holds good. Then for $n > 2$

$$a = \left(\frac{n-2}{2}c\right)^{2/(n-2)}.$$

We take $\varphi(c)$ in such a way that

$$\varphi'(c) = R - \bar{R}a^2 \tag{11}$$

holds good, which is always possible as R and $\bar{R}$ are constant. Then we have

$$\varphi''(c) = -\bar{R}\frac{da^2}{dc} = -2\bar{R}\left(\frac{n-2}{2}c\right)^{-(n-6)/(n-2)}$$

For $n = 2$ we have $a = e^c$ by (6) and for $\varphi(c)$ determined by (11) we get

$$\varphi''(c) = -\bar{R}\frac{da^2}{dc} = -2\bar{R}e^{2c}.$$

In both cases we have by (11) and (5) $\varphi'(c)\Delta c \geq 0$. If $\bar{R} < 0$, we have $\varphi''(c) > 0$, and (10) is satisfied, and so c and a are constant. If $\bar{R} = 0$, we can deduce $R = 0$ from (8), and we get $\Delta c = 0$ and hence a is constant. Thus we get

THEOREM 2. *We assume that V and $\bar{V}$ are compact orientable Riemannian* *manifolds whose scalar curvatures R and $\bar{R}$ are both constant and non-positive. Then a conformal homeomorphism between V and $\bar{V}$ is homothetic, namely a magnification function is constant.*

Next we consider the case $V = \bar{V}$. Then we have

THEOREM 3. *We assume that V is a compact orientable Riemannian manifold whose scalar curvature is a non-positive constant. Then a conformal homeomorphism of V onto itself is an isometry.*

In fact a magnification function a is constant by theorem 2 and hence by the integration of (7) on the whole manifold V we get $a = 1$.

Theorem 3 is an answer to a question raised by T. Sumitomo in [2] p. 118, the case of vanishing curvature being solved by himself.

REFERENCES

[1] Bochner, S. and Yano, K.: Curvature and Betti-numbers, 1953.

[2] Sumitomo, T.: Projective and conformal transformations in compact Riemannian manifolds. Tensor, vol. **9** (1959), pp. 113-135.

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