A THEOREM ON FACTORIZABLE GROUPS OF ODD ORDER

OSAMU NAGAI

To RICHARD BRAUER on his 60th birthday

Recently, W. Feit [2] obtained some results on factorizable groups of odd order. By using his procedure and applying the theory of R. Brauer [1], we can prove the following theorem similar to that of W. Feit [2]:

THEOREM. *Let G be a factorizable group of odd order such that*

$$G = HM$$

where H is a subgroup of order $3p$, p being a prime greater than 3, and M is a maximal subgroup of G. Then G contains a proper normal subgroup which is contained either in H or in M.

Proof. It is sufficient to prove the theorem in the case in which H is non-abelian. In fact, if H is abelian, then, as $p \neq 3$, the theorem follows immediately from the theorem of W. Feit [2].

Now, assume that no proper normal subgroup of G is contained in M. Suppose that $D = H \cap xMx^{-1} \neq 1$ for some element x in G. If $D = H$, then $H \subseteq xMx^{-1}$. Since every subgroup of G conjugate to M is of the form yMy^{-1} for some element y in H, it follows that H is contained in every subgroup conjugate to M. Hence the intersection of all subgroups conjugate to M is a normal subgroup of G, contained in M. This contradicts our assumption. Thus $D \neq H$. In this case H is represented as the form $H = AD$, where A is a subgroup of prime order which is either p or 3. Since the conjugate subgroup xMx^{-1} is the form yMy^{-1} for an element y in H, $G = A \cdot yMy^{-1}$. By a theorem of T. Ikuta [3], either A is normal in G or yMy^{-1} contains a proper normal subgroup of G. Thus we can assume that $H \cap xMx^{-1} = 1$ for every element x in G.

Let π be the permutation representation of G induced by the subgroup M.

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Since the kernel of π is contained in M, π is faithful. Therefore we can assume that G itself is a transitive permutation group of degree $3p$. Since M is a maximal subgroup, G is a primitive permutation group. Since $H \cap xMx^{-1} = 1$ for every element x in G, H is a regular subgroup of G. Since the order of G is odd, G cannot be doubly transitive. Therefore, by the results in [4], G has the following properties:

(a) The order of G contains the prime p to the first power only.

(b) The centralizer of a Sylow p-subgroup P is contained in P.

(c) G^*, considered as matrix-representation of G, contains no irreducible constituent of degree 1 except the unit representation. Furthermore,

(d) G^* contains no irreducible constituents of the exceptional type (in Brauer's sense). In fact, if G^* contains an irreducible constituent of exceptional type, then by Theorem 3 of H. Tuan [5], either $G \cong A_7$ or $G \cong LF(2, p)$. Since the order of G is odd, this is a contradiction.

Under these circumstances, the degrees of the irreducible constituents of G^* can be determined completely (see [1], or [4], p. 204). They are 1, p and $2p-1$. Corresponding to this decomposition, the subgroup G_1 leaving fixed one letter has just three transitive sets whose lengths are 1, v and w (see [6], p. 77). Of course $1+v+w=3p$. If $v=w$, then $3p=1+2v$. Since $p-1 \equiv 0 \pmod 3$, we can put $p-1=6l$ where l is a rational integer. Then $q = 3pvw/p(2p-1) = 3(9l+1)^2/(12l+1)$ is not a rational integer. By a theorem of J. S. Frame (see [6], p. 83), this is a contradiction. Hence $v \neq w$.

Now, assume that $1<v<w$. By the methods of H. Wielandt (see [6], in particular p. 92), we obtain the following two equations:

(1) $v+sp+t(2p-1)=0$,

(2) $v^2+s^2p+t^2(2p-1)=3pv$,

where s and t are rational integers. Since $1+v+w=3p>1+2v$, $(3p-1)/2>v$. From (2), $t^2<3pv/(2p-1)<p^2$. This means $|t|<p$. From (1), $t \equiv v \pmod p$. If we put $t=v+xp$, then, since $v>0$ and $|t|<p$, $x \leqq 0$. If $x \leqq -3$, then $2p \leqq p(-x-1)<v$. This is impossible, since $(3p-1)/2>v>0$. Hence $x=0$, or -1, or -2, that is, $t=v$ or $t=v-p$ or $t=v-2p$. If $t=v$, then, from (1), $s=-2t$. Substitute this in (2), $2t=1$. This is a contradiction. If $t=v-p$, then from (1), $s=-2t-1$. From (2) $2p=6t^2+3t+1$. On the other hand, since H is non-abelian, $p-1\equiv 0 \pmod 3$. This is a contradicton. If $t=v-2p$, then as above, we have $2p=6t^2+9t+4$. This is also a contradiction.

Thus the proof is completed.

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Department of Mathematics
Yamaguchi University