R-SEQUENCES AND HOMOLOGICAL DIMENSION[1)]

IRVING KAPLANSKY

To RICHARD BRAUER on his 60th birthday

1. Introduction. The motivation for the results in this note comes from a theorem of Macaulay. Let $f_1, \ldots, f_n$ be elements of a polynomial ring R over a field, and let I be the ideal they generate. Assume $I \neq R$ and rank $(I) = n$. Then the theorem of Lasker and Macaulay asserts that I is unmixed (all prime ideals belonging to I have rank n). Macaulay [1, p. 51] proved further that *any power of I is unmixed.*

In the modern formulation of the problem we operate in any commutative ring R with unit, and let $I = (a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ is an R-sequence. We seek to prove that for any k the homological dimension of R/I^k is n. For details on how this implies unmixedness in case R is Noetherian, see [2].

In 1959 I noticed that the methods used by Rees in [2] could be adapted to prove the above theorem. Recently I discovered a still simpler proof that yields information not just on the powers of I, but on ideals generated by monomials in the a's. Since there are as yet not too many examples where homological dimensions can be computed explicitly, the details are perhaps worthy of public scrutiny.

2. Formulation of results. R will always denote a commutative ring with unit. Let A be an R-module. The *homological dimension* of A is the smallest integer m such that there exists an exact sequence

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to A \to 0$$

with P_i projective; if no such sequence exists, the homological dimension of A is ∞. We write $d(A)$ for the homological dimension, or $d_R(A)$ when it is necessary to call attention to the ring.

The elements $a_1, \ldots, a_n$ in R form an R-sequence if $(a_1, \ldots, a_n) \neq R$ and

Received November 12, 1961.

1) Research supported in part by the office of ordnance Research, U. S. Army.

for $i=1, \ldots, n$, a_i maps into a non-zero-divisor in the ring $R/(a_1, \ldots, a_{i-1})$. If R is a local ring, it is known that any permutation of an R-sequence is an R-sequence, so that Theorem 1 below is applicable if R is a local ring.

We shall be concerned with an ideal I which is generated by monomials in the a's. It is easily seen that (even if we allow an infinite number) a finite number of monomials will suffice to generate I.

The simplest result to state and prove is that $d(R/I) \leqq n$ if every permutation of the given R-sequence is an R-sequence.

THEOREM 1. *Let R be a commutative ring with unit, and $a_1, \ldots, a_n$ elements of R constituting an R-sequence in any order. Let I be an ideal generated by monomials in the a's. Then $d(R/I) \leqq n$.*

If we assume only that $a_1, \ldots, a_n$ is an R-sequence in the given order, then some extra hypothesis is needed even to get $d(R/I) < \infty$. For instance, it is easily possible to arrange that a_2 is a divisor of 0 and $d(R/I) = \infty$ with $I = (a_2)$. If it is assumed that I contains a power of each a_i, then $d(R/I)$ can be proved equal to n. The argument that proves this also yields the extra information recorded in Theorem 2.

THEOREM 2. *Let R be a commutative ring with unit and $a_1, \ldots, a_n$ an R-sequence in R. Let I be an ideal generated by monomials in the a's. Assume that I contains a power of a_i for $i=1, \ldots, n-1$. Then $d(R/I) \leqq n$. If further I contains a power of a_n then $d(R/I) = n$.*

3. Proof of Theorm 1. In the proofs we will use two basic facts on homological dimension which are given as Lemmas 1 and 2. Both already rank as "folk theorems" in this young subject. Lemma 2 is valid for any ring R, and so is Lemma 1 provided x is central.

LEMMA 1. *Let x be a non-zero-divisor in R, and write $S = R/(x)$. Let A be a non-zero S-module with $d_S(A) < \infty$. Then $d_R(A) = 1 + d_S(A)$.*

LEMMA 2. *Let A be an R-module, B a submodule, $C = A/B$.*

(a) *If $d(C) < 1 + d(B)$, then $d(A) = d(B)$.*

(b) *If $d(C) > 1 + d(B)$, then $d(A) = d(C)$.*

(c) *If $d(C) = 1 + d(B)$, then $d(A) \leqq d(C)$.*

In any case $d(A) \leqq \max(d(B), d(C))$.

The spirit of the next lemma is that the "relative primeness" of the a's that is built into the definition of an R-sequence can be extended to more complicated objects.

LEMMA 3. *Let $a_1, \ldots, a_n$ be elements constituting an R-sequence in any order. Let J be an ideal generated by monomials in $a_2, \ldots, a_n$. Then $ta_1 \in J$ implies $t \in J$.*

Proof. We may suppose that a_2 actually occurs in one of the monomials generating J. Write $J = (a_2 K, L)$ where K is generated by monomials in $a_2, \ldots, a_n$ and L just by monomials in $a_3, \ldots, a_n$. We have $ta_1 = ua_2 + v$, $u \in K$, $v \in L$. We pass to the ring $R/(a_1)$, noting that the homomorphic images of $a_2, \ldots, a_n$ constitute an R-sequence. Writing $*$ for homomorphic image, we have $u^* a_2^* \in L^*$. By induction on n, $u^* \in L^*$, whence $u \in (a_1, L)$, say $u = wa_1 + x$ with $x \in L$. Since $u \in K$, this implies $wa_1 \in (K, L)$. We make an induction on the sum of the degrees of the monomials generating J, and deduce $w \in (K, L)$. Next we substitute for u in the equation $ta_1 = ua_2 + v$, and find $(t - wa_2)a_1 \in L$. Since $a_1, a_3, a_4, \ldots, a_n$ is also an R-sequence we have, again by induction on n, $t - wa_2 \in L$. Hence $t \in (a_2 K, L) = J$.

Proof of Theorem 1. We may suppose that a_1 actually occurs in one of the monomials generating I. Let $I_0 = (a_1, I)$. We study the module R/I in the two steps R/I_0, I_0/I.

(1) R/I_0 is annihilated by a_1 and so may be regarded as an S-module where $S = R/(a_1)$. As such, it has the same form relative to a sequence of length $n-1$ (the images of $a_2, \ldots, a_n$) which is an R-sequence in any order, as R/I does relative to $a_1, \ldots, a_n$. By induction on n, $d_S(R/I_0) \leqq n-1$. By Lemma 1, $d_R(R/I_0) \leqq n$.

(2) I_0/I is a cyclic module, generated by a_1. The annihilator is the set of all s with $sa_1 \in I$. Write $I = (a_1 I', J)$ where J is generated by monomials in $a_2, \ldots, a_n$. Now if $sa_1 \in I$, then $sa_1 = ya_1 + z$, $y \in I'$, $Z \in J$. Thus $(s-y)a_1 \in J$. By Lemma 3, $s - y \in J$, whence $s \in (I', J)$. Hence I_0/I is isomorphic to $R/(I', J)$. By induction on the sum of the degrees of the monomials generating I, $d(R/(I', J)) \leqq n$. Hence $d(I_0/I) \leqq n$.

To complete the proof of Theorem 1 it remains only to put these two pieces together with the aid of Lemma 2.

4. Proof of Theorem 2. The plan of proof is the same as soon as we have the appropriate analogue of Lemma 3.

LEMMA 4. *Suppose* $a_1, \ldots, a_n$ *is an R-sequence and* $ta_1 \in J$ *where* J *is generated by monomials in* $a_2, \ldots, a_n$ *and contains a power of* a_i *for* $i = 2, \ldots, n-1$. *Then* $t \in J$.

It turns out that to give a smooth inductive proof of Lemma 4 it is advisable to prove simultaneously a companion lemma.

LEMMA 5. *Suppose* $a_1, \ldots, a_n$ *is an R-sequence and* $ta_n \in J$ *where* J *is generated by monomials in* $a_1, \ldots, a_{n-1}$ *and contains a power of each. Then* $t \in J$.

Proof of Lemmas 4 *and* 5. We assume both to be true for $n-1$. Furthermore for the given n we make an induction on the sum of the degrees of the monomials generating J.

We first treat Lemma 4. If a_n does not occur in a generating monomial, induction applies at once. Otherwise write $J = (K, a_nL)$; here K is generated by monomials in $a_2, \ldots, a_{n-1}$ and contains a power of each. Say $ta_1 = u + a_n v$, $u \in K$, $v \in L$. We pass to the ring $R/(a_1)$, using $*$ for homomorphic image. Then $v^* a_n^* \in K^*$, whence $v^* \in K^*$ by our inductive assumption of Lemma 5 for $n-1$. Thus $v \in (a_1, K)$, say $v = wa_1 + x (x \in K)$. This implies $wa_1 \in (K, L)$ whence $w \in (K, L)$ by our second induction. Now $ta_1 = u + a_n(wa_1 + x)$, $(t - wa_n)a_1 = u + xa_n \in K$, $t - wa_n \in K$ by Lemma 4 for $n-1$, $t \in (K, a_nL) = J$.

We proceed to the proof of Lemma 5. This time we write $J = (a_1K, L)$, where L is generated by monomials in $a_2, \ldots, a_{n-1}$ and contains a power of each. Say $ta_n = ua_1 + v$, $u \in K$, $v \in L$. We look at this equation mod (a_1), and apply Lemma 5 for $n-1$. The result is $t \in (a_1)$, $t = wa_1$. Then $(wa_n - u)a_1 = v \in L$. By the case $n-1$ of Lemma 4, $wa_n - u \in L$, so $wa_n \in (K, L)$, and $w \in (K, L)$ by the induction on the sum of the degrees of the monomials. Finally $t = a_1 w \in (a_1K, L) = J$.

Proof of Theorem 2. That $d(R/I) \leqq$ is proved verbatim as in Theorem 1 (except for citing Lemma 4 in place of Lemma 3), and we shall not repeat the proof.

If I contains a power of a_n, then by induction we get both $d(R/I_0)$ and

$d(I_0/I)$ to be n, whence $d(R/I)=n$ by Lemma 2. (To be absolutely accurate we should distinguish the case $a_1 \in I$; but then $I_0 = I$ and we are finished when we show $d(R/I_0)=n$).

5. Further remarks. We append three concluding remarks.

1. If R is Noetherian, it is possible to sharpen Theorem 2 by showing that $d(R/I)=n-1$ or n and that $d(R/I)=n$ if a_n "actually occurs" in I (in a sense easily made precise). Whether this holds in the non-Noetherian case I have been unable to determine.

2. Let us say that a module A has a *finite free resolution* if there exists an exact sequence

$$0 \to F_n \to \cdots \to F_0 \to A \to 0$$

with the modules F_i free and finitely generated. It is known that the analogues of Lemmas 1 and 2 are valid in the context of finite free resolutions. By tracing through the proofs we then see that under the hypothesis of either Theorem 1 or Theorem 2, R/I has a finite free resolution.

3. Lemma 4 has a corollary of some interest. Let $m_1, m_2, \ldots$ be monomials in the a's and suppose we have a relation $t_1 m_1 + t_2 m_2 + \cdots = 0$. Suppose further that m_1 is not a formal multiple of any other of the m's. Then: $t_1 \in (a_1, \ldots, a_n)$. The deduction of this form Lemma 4 is simple and is left to the reader.

Here is a further consequence which shows that the resemblance between R-sequences and independent indeterminates is more than a resemblance. Let R be a commutative ring with unit containing a field F (with the same unit). Let $a_1, \ldots, a_n$ be an R-sequence in R. Then $F[a_1, \ldots, a_n]$ is a polynomial ring, i.e. the a's are independent indeterminates over F.

References

[1] F. S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge, 1916.
[2] D. Rees, The grade of an ideal or module, Proc. Camb. Phil. Soc. **53** (1957), 28–42.

University of Chicago