STRONGLY REGULAR EXTENSIONS OF RINGS

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As defined by Arens and Kaplansky [2] a ring A is *strongly regular* (s.r.) in case to each $a \in A$ there corresponds $x = x_a \in A$ depending on a such that $a^2x = a$. In the present article a ring A is defined to be a *s.r. extension* of a subring B in case each $a \in A$ satisfies $a^2x - a \in B$ with $x = x_a \in A$. S.r. rings are, then, s.r. extensions of each subring. A ring A which is a s.r. extension of the center has been called a ξ-ring (see Utumi [13], Drazin [3], Martindale [11], and their bibliographies).

Arens and Kaplansky showed that a s.r. ring is a subdirect sum of divison rings. Since any s.r. ring is semisimple, a later result stating that any semisimple ξ-ring is a subdirect sum of division rings (see [11]) contains this result.[2)] In §2 of the present article, a further generalization is obtained: (1) *If a semisimple ring A is a s.r. extension of a commutative subring B, then A is a subdirect sum of division rings.* For the proof, the reduction to the case A is primitive is immediate, but at this stage an innovation is made. Instead of specializing B, as has been done in the previous work along these lines, a structure theorem (Theorem 2.1) for a primitive s.r. extension A of an arbitrary ring B is obtained first of all: (2) *If A is a primitive ring, not a division ring, and if A/B is s.r., then B is dense in the finite topology on A.* Of course, (1) is an immediate consequence, but more can be squeezed out of (2). For example, (2) shows that in order that a primitive ring A be a s.r. extension of a subring B, it is necessary that B be a primitive ring, or an integral domain. (A bit of duality can be introduced here, since in § 1 it is shown that a directly

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2) A generalization in another direction has been given by Utumi [13].

irreducible s.r. extension of an integral domain is necessarily an integral domain.) A fairly easy consequence of this is that in order that a semisimple ring A be a s.r. extension of a subring B, it is necessary that B be a subdirect sum of primitive rings and integral domains (Corollary 2.2 ff.).

Another consequence of (2) is that any s.r. extension of a division ring is a subdirect sum of division rings. This fact is also implied by the theorem of Arens and Kaplansky inasmuch as a s.r. extension of a s.r. ring is a s.r. ring. However, this and the above results are all obtained in a new way independently of the previous results for ξ-rings and s.r. rings.

The structure of A is not known in the general case when A is a s.r. extension of a commutative subring B. However the centralizer of B in A is a ξ-ring, so that some information on the structure of A is available.

In § 3 the results on s.r. extensions are applied in extending the results of Nakayama [12] on the commutativity of rings, continuing a program which I began in [4]. Any future improvements in the theory of s.r. extensions will net corresponding improvements in this direction also.

A simple computation shows that a ring is regular ($axa = a$) in the sense of von Neumann if and only if every principal one-sided ideal has an idempotent generator. Arens and Kaplansky introduced the notion of strong regularity ($a^2x = a$), whereby not only are these idempotent generators demanded but also nilpotent elements are banished. Here, and more generally in ξ-rings, the emphasis has shifted from the manufacture of idempotents to the disposition of the nilpotent elements of index two: they must all lie in the center. In § 5 the position in a primitive ring A of the subring $T(A)$ generated by the nilpotent elements of index two is investigated. One finds in important special cases (*e.g.*, if A is an algebraic algebra, or if A has a minimal left ideal) that the subring $T(A)$, and also the subring $E(A)$ generated by the idempotents of A, is dense in A, if A is not division. This clearly illustrates my allusion above to the extent to which the structure of an s.r. extension A/B is influenced by the fact that B contains the subring $T(A)$.

1. Directly irreducible strongly regular extensious. If A is a ξ-ring with center Z, and if $a, x \in A$ satisfy $a^2x - a \in Z$ then [11, Theorem 1] states that $ax = xa$. The verbatim proof (accredited to Herstein) given there establishes this implication for CN-rings[3], a fact which is stated in the proposition below. By reproducing its proof here, I have been able to make this section, and the section following, relatively self-contained.

PROPOSITION 1.1. *In any CN-ring, any two elements a and x which satisfy* $a^2x-a\in Z$ *are commutative, that is, then* $ax=xa$.

Proof. Since $a^2x-a\in Z$, $(a^2x-a)a=a(a^2x-a)$, and so (1) $a^2(ax-xa)=0$. Using (1), it follows that $[a(ax-xa)a]^2=0$, so that, since A is CN, (2) $a(ax-xa)a\in Z$. Commuting this with a and using (1), there results (3) $a(ax-xa)a^2=0$. Since $a^2x-a\in Z$, one easily verifies that

$$(4)\quad ax-xa=[a(ax-xa)+(ax-xa)a]x.$$

If (4) is multiplied on the left by a, using (1), the result can be simplified to $a(ax-xa)=a(ax-xa)ax$, so that, by (2), one has

$$(5)\quad a(ax-xa)=a(ax-xa)ax=xa(ax-xa)a.$$

Multiplying (5) on the right by a produces (6) $a(ax-xa)a=xa(ax-xa)a^2$, which is $=0$ by (3). Reapplying this latter fact to (5) yields (7) $a(ax-xa)=0$. Thus, (4) can be simplified to (8) $ax-xa=(ax-xa)ax$. From (7) $[(ax-xa)a]^2=0$, so that $(ax-xa)a\in Z$. Commuting this with a, and using (7), one obtains (9) $(ax-xa)a^2=0$. Since $(ax-xa)a\in Z$, (8) becomes $(ax-xa)=x(ax-xa)a$, and so, by (9), one has (10) $(ax-xa)ax=x(ax-xa)a^2x=0$. Then (8) reduces to $ax=xa$, which is the desired result.

COROLLARY 1.2. *If a and x are elements of a ring A, and if the element* a^2x-a *and all nilpotent elements of A commute with both a and x, then a and x commute.*

Proof. Let Q denote the subring of A generated by a and x, and let $\mathfrak{z}$ denote the center of Q. Then the condition of the corollary implies that Q is a CN-ring, and that $a^2x-a\in\mathfrak{z}$, so that the corollary follows from the proposition.

An element a of a ring A is (von Neumann) *regular* if $axa=a$, and *strongly*

[3] A CN-ring is a ring in which every nilpotent element belongs to the center. It seems that in each case where I assume that a ring is CN, I actually require only that the center contains all nilpotent elements of index two. I do not know whether this latter condition is equivalent to the CN hypothesis.

regular (Arens and Kaplansky) if $a^2x = a$, for suitable $x \in A$. The next corollary in the case $n = 1$ shows that in *CN*-rings strong regularity of an element a implies its regularity. The corollary follows from the proposition by observing that $a^{n+1}x = a^n$ implies $(a^n)^2x^n = a^n$.

COROLLARY 1.3. *If A is a CN-ring, then the equation $a^{n+1}x = a^n$ for two elements a and x in A, and a natural number n, implies the equation $a^n x^n = x^n a^n$.*

LEMMA 1.4. *A directly irreducible CN-ring A has an identity element $1 \neq 0$ if and only if there exist a, $x \in A$ such that $a^2x = a \neq 0$. Then $ax = xa = 1$.*

Proof. By the corollary $a^2x = a$ implies $ax = xa$, so that $e = ax$ is a nonzero idempotent when $a \neq 0$. Since the sets $eA(1-e)$, $(1-e)Ae$ are central, they commute with e, whereby they are $= 0$. By the direct irreducibility of A, $e = 1$. The converse is trivial.

In a s.r. extension of a division ring, to each element a there corresponds an element x such that $a^2x - a$ has certain regularity properties. This situation for directly irreducible rings is slightly generalized directly below, and following this a similar generalization of s.r. extensions of integral domains is considered.

THEOREM 1.5. *Let A be a directly irreducible CN-ring containing a left identity $1 \neq 0$, and such that to each $a \in A$ there correspond $b \in A$ and a natural number $n = n_a$ such that either $a^{n+1}b - a^n = 0$, or else $a^{n+1}b - a^n$ has a right inverse in A. Then the totality N of nilpotent elements of A is an ideal, and $A - N$ is a division ring.*

Proof. By Lemma 1.4, 1 is a two-sided identity. If $a^{n+1}b - a^n$ has the right inverse x, then a has the right inverse $a^n bx - a^{n-1}x$. If $a^{n+1}b = a^n$, then, by Corollary 1.3, $a^n b^n = b^n a^n$, so that $e = a^n b^n$ satisfies $e^2 = e$. Then, by Lemma 1.4, either $e = 1$, whence a has right inverse $a^{n-1}b^n$, or else $0 = e = ea^n = a^n$. Thus, every nonnilpotent element has a right inverse. It is easy to see that this means that every nonnilpotent element has a two-sided inverse. Then (e.g., [9, p. 21]), since N is a (central) ideal of A, $A - N$ is division.

Remark. One can show in general that in a ring A with identity such that every nonnilpotent element has an inverse, that N is a two-sided ideal such that $A-N$ is division (cf. the proof of Lemma 1.9 below.)

COROLLARY 1.6. *If A is directly irreducible with a left identity $1 \neq 0$ such that to each $a \in A$ there corresponds $b \in A$ such that either $a^2b-a=0$, or else a^2b-a has a right inverse, then A is division.*

Proof. It is trivial to show that $N=\mathbf{0}$, so that A is CN, and the theorem applies.

The corollary shows that a nonzero directly irreducible s.r. ring is division if only if there exists a left identity. In such a ring a two-sided identity exists, according to Lemma 1.4.

COROLLARY 1.7. *A nonzero directly irreducible ring is s.r. if and only if it is division.*

THEOREM 1.8. *If A is a directly irreducible CN-ring, and if to each $a \in A$ there correspond $b \in A$ and a natural number $n=n_a$ such that $a^{n+1}b-a^n$ is not a proper right divisor of zero in A, then the set N of nilpotent elements of A is a nil ideal, and $A-N$ is an integral domain.*

Proof. In a way completely analogous to the proof of the last theorem, one sees that N coincides with the set D of all right divisors of 0 in A. Thus, the theorem is a consequence of the following lemma. The lemma no doubt is known, but I have not been able to find a published proof. For this reason, I include one here.

LEMMA 1.9. *If $N=D$ in a ring A, then N is an ideal of A, and $A-N$ is an integral domain.*

Proof. If $N=\mathbf{0}$, there is nothing to prove. Now let $0 \neq x \in N$ have index of nilpotency $=m$. Then, since $(ax)x^{m-1}=0$, $ax \in D=N$, for all $a \in A$, that is, $Ax \subseteqq N$, for all $x \in N$. Since $(ax)^n=0$ implies that $(xa)^{n+1}=0$, this shows that $Ax \subseteqq N$ implies that $xA \subseteqq N$, so that $AxA \subseteqq N$, for all $x \in N$. In order to show that N is an ideal, it renains to show that N is closed under addition. If x, $y \in N$, then, since $(x+y)^3 \in AxA+AyA$, it follows that $(x+y) \in N$. It remains to show that $A-N$ is integral. It suffices to show that $a \notin N$, $b \notin N$, $ab=q \in N$ leads to a contradiction. Clearly, $q \neq 0$, so q is nilpotent of index $m \geqq 2$. Since $(ab)^m=[(ab)^{m-1}a]b=0$, $b \notin D$ implies that $(ab)^{m-1}a=0$. But $a \notin D$ implies that $(ab)^{m-1}=0$, which is the desired contradiction.

A consequence of Corollary 1.6 is that a directly irreducible s.r. extension of a division ring is a division ring. In analogy with this fact one has

Corollary 1.10. *A directly irreducible s.r. extension of an integral domain is an integral domain.*

Proof. If A is the extension, and B the integral domain, then A contains no nilpotent elements $\neq 0$. Thus, if $a, y \in A$, then $ay = 0$ if and only if $ya = 0$. Since A is a CN-ring, by the theorem it suffices to show that $a - a^2 b \in B$ is not a proper right divisor of zero in A. Hence assume that $0 \neq y \in A$ is such that $y(a - a^2 b) = 0$, and $a - a^2 b \neq 0$. Then $(a - a^2 b)y = 0$, so that $(a - a^2 b)(y - y^2 c) = 0$, where c can be chosen such that $y - y^2 c \in B$. Since B is integral, $y = y^2 c$, so that, by Lemma 1.4, $yc = cy$ is the ring identity, which contradicts the choice of y as a proper left divisor of 0.

Since a s.r. ring is a s.r. extension of every subring, it would seem that the hypothesis "A is a s.r. extension of B" would have more force if one assumes at the outset that A is not an s.r. ring. (Then B is not s.r.!) For these rings the structure theory can be reduced in some cases to that of directly irreducible s.r. extensions.

Proposition 1.11. *Let B be a simple ring with identity e, and let A be a s.r. extension of B, A not a s.r. ring. Then $A = Q \oplus P$, where Q is a directly irreducible s.r. extension of B having the identity e, and P is a s.r. ring. Conversely, $Q \oplus R$ is a s.r. extension of B, if Q is any s.r. extension of B, and P is any s.r. ring.*

Proof. The sufficiency is clear. The necessity requires the following lemma which is also of interest in more general situations.

Lemma. *If B is a ring with a central idempotent e, and if A/B is a s.r. extension, then e is a central element of A.*

Proof of the Lemma. B contains all nilpotent elements of index two, so that B contains the sets $eA(1-e)$, $(1-e)Ae$. Since e is central in B, these sets $= 0$, so $A = eAe \oplus (1-e)A(1-e)$, and e is central.

Going back to the proof of the proposition, since $(1-e)A(1-e) \cap B = \mathbf{0}$, $P = (1-e)A(1-e)$ is s.r. as required. It remains to show that $Q = eAe$ is directly irreducible. To this end assume that $Q = M \oplus N$, where M and N are ideals. $M \cap B = 0$ implies that M is s.r. If both M and N were s.r., then so would Q, whence A, be s.r., contrary to assumption. On the other hand, if $M \cap B \neq 0$, then $M \supseteqq B$, so that $M = Q$, $N = 0$, and Q is directly irreducible.

The existence of directly irreducible s.r. proper extensions, not division rings, of simple rings is guaranteed by the example in § 4 of [4].

§ 2. **Semisimple strongly regular extensions.** The next theorem shows that in order that a ring B possess a primitive s.r. extension, it is necessary that B be a primitive ring, or an integral domain.

2.1. STRUCTURE THEOREM. *Let A be a primitive ring, not a division ring, which is represented as a dense ring of l.t.'s in a vector space V over a division ring D. Then: if B is any subring of A such that A/B is s.r., then B is isomorphic to a dense ring of l.t.'s in V.*

Proof. Let V_n be a vector subspace of V of finite dimension n, let $U = \{a \in A \mid V_n a \subseteqq V_n\}$, and let $K = \{a \in A \mid V_n a = 0\}$. Then, as is well known [9], the difference ring $\bar{U} = U - K$ is isomorphic to D_n, the complete ring of $n \times n$ matrices over D. First assume that $n > 1$, and let $u \in U$ be such that $u^2 \in K$. Then, if $c \in A$ is such that $u - u^2 c \in B$, then, since $u^2 c \in K$, it follows that $u - u^2 c \in Q = B \cap U$. Thus, the subring $\bar{Q}$ determined by Q under the canonical homomorphism $U \to \bar{U}$ contains every $\bar{u} \in \bar{U}$ satisfying $\bar{u}^2 = 0$. By [7, p. 602, Proposition 1], $\bar{Q} = \bar{U}$, that is, $U = Q + K$, and, consequently, every l.t. of V_n is induced by an element of B, in case $n > 1$. Now V_1 is contained in a subspace V_2, and if $\bar{a}_1$ is any l.t. in V_1, there exists a l.t. $\bar{a}_2$ in V_2 such that a_2 induces $\bar{a}_1$. Then, if $b \in B$ induces $\bar{a}_2$, then b also induces $\bar{a}_1$. Thus, in all cases, the l.t.'s in V_n can be induced by elements of B. This establishes that B is isomorphic to a dense ring of l.t.'s in V.

(1) of the next corollary is immediate.

COROLLARY 2.2. *Let A be a s.r. extension of a ring B.* (1) *If A is a primitive ring, not a divison ring, then B is a primitive ring, and so is any intermediate ring of A/B.* (2) *If A is semisimple, then B is a subdirect sum of primitive rings, and integral domains.*

Proof. (2) Let $\{P\}$ denote the collection of primitive ideals in A. Since

$\cap P = 0$, necessarily $\cap (P \cap B) = 0$, so that B is a subdirect sum of the rings $\{B-(P\cap B)\}$. Now $A-P$ is a s.r. extension of $(P+B)-P$, so that by the theorem: if $A-P$ is not a division ring, then $(P+B)-P$ is primitive; $(P+B)-P$ is an integral domain, otherwise. (2) is completed by observing that $(P+B)-P$ is isomorphic to $B-(P\cap B)$, for each $P \in \{P\}$.

COROLLARY 2.3. *Let A be a semisimple ring which is a s.r. extension of a commutative subring B. Then A is a subdirect sum of division rings, and B is a subdirect sum of (commutative) integral domains. (If in addition A is subdirectly irreducible, then A is a division ring).*

Proof. Let A' be any primitive homomorph of A, and let B' denote the corresponding map of B. Since A'/B' is s.r., and B' is commutative, density of B' in A' would imply commutativity of A', which in turn would imply that A' is a field. Thus, by the theorem, A' is a division ring, so that A is a subdirect sum of division rings. By the corollary, B must be a subdirect sum of (commutative) primitive rings and integral domains. Since a commutative primitive ring is a field, B has the desired struture. (The parenthetical remark is obvious).

3. Commutativity theorems. If S is a nonempty subset of a ring A, then $[S]$ denotes the subring generated by S. If R is a subring, then $R[S]$ denotes the subring generated by R and S. If A is a division ring, and if R is a division subring, $R(S)$ is the division subring generated by R and S.

Let Φ be a commutative ring with identity. A ring A is a Φ-ring (in the sense of Jacobson [8, p. 55]) if A is a unitary left Φ-module satisfying $c(xy) = (cx)y = x(cy)$ for all $c \in \Phi$, and all $x, y \in A$.

DEFINITION. Let Φ be a commutative ring with identity which contains a (possibly 0) subring K with the property that (1) a nonzero homomorph K' of K is an integral domain if and only if K' is an algebraically closed field, and (2) there exist finitely many $c_1, \ldots, c_r \in \Phi$ such that $\Phi = K[c_1, \ldots, c_r]$. Let A be a Φ-ring, and B a Φ-subring of A such that to each $a \in A$ there corresponds a polynomial $P_a(x)$ in the polynomial ring $\Phi[x]$ such that

$$a^n - a^{n+1}p_a(a) \in B$$

for some natural number n depending on a. Then A/B is an *N-extension.* If

A/B is an *N-extension*, then it is an *N_1-extension* if $n=1$ for all $a \in A$, and it is an *N_2-extension* if B contains all idempotents of A.

N-extensions have been studied extensively by Nakayama [12] (and others, see [12, References]) where the main result states that any ring A which is an N_1-extension of its center Z is commutative (or, more generally, any CN-ring which is an N-extension of its center is commutative.) This result had been obtained earlier by Nakayama in the case $K=\mathbf{0}$. In this case it is also true that a division ring A is commutative if it is an N-extension of a division Φ-subring $\neq A$ (Faith [4, Theorem 1]), a result which is extended to the $K \neq \mathbf{0}$ case below.

THEOREM 3.1. *Let A be a division Φ-ring, Φ as in the definition, and let B be a Φ-subring such that A/B is an N-extension. Then: if B is commutative, or if B is a division subring $\neq A$, then A is a field.*

Proof. If B is commutative, so is the division subring (B) generated by B. If $(B)=A$, then A is a field as required. Hence, it suffices to consider only the case where A $(\neq Z)$ is an N-extension of the division ring $B \neq A$. Let 1 be the identity of A, and set $\varphi = \Phi 1$. Since $\varphi \neq \mathbf{0}$, A and B are algebras over the field $\bar{\varphi}$ of quotients of φ. In this case the results of [6] are applicable. The hypotheses imply that to each $a \in A$ there corresponds $p_a(x)$ with coefficients in $\varphi (\subseteq \bar{\varphi})$ such that $a^n - a^{n+1}p_a(a) \in B$, Under these conditions [6, Theorem 1.5] asserts that to each $b \in A$ there corresponds a polynomial $F_b(x)$ over $\bar{\varphi}$ such that (i) $F_b(b) \in Z$, and (ii) $F_b(x)$ is the composition of finitely many of the polynomials in the set

$$\{x^n - x^{n+1}p_a(x) \mid a \in A,\ n=1, 2, \ldots\}.$$

Clearly, then, the polynomial $F_b(x)$ has the form

$$F_b(x) = x^m - x^{m+1}g_b(x),$$

with $m=m(b)>1$, and $g_b(x) \in \varphi[x]$. (It is important to note that the $F_b(x)$ are polynomials over φ.) The effect of all of this is to show that A/Z is an N-extension, as defined above, so that $A=Z$ by the result of Nakayama.

THEOREM 3.2. *Let A be a Φ-ring, Φ as in the definition, and let B be a commutative Φ-subring such that A/B is an N_1-extension. If either A is semi-*

simple, or $B \cap J(A) = 0$*, where* $J(A)$ *denotes the Jacobson radical of* A*, then* A *is commutative.*

Proof. Since A is semisimple (if $B \cap J(A) = 0$, then J is s.r., so $J = 0$), A is a subdirect sum of division rings A' by Corollary 2.3. Each A' can be regarded as a Φ-ring, and it follows that each A' is an N_1-extension of a commutative subring, so that each A' is commutative by Theorem 3.1. Then A is commutative.

Below, if a ring is an N_1-extension of 0, then it is an N_1*-ring*. By Nakayama's result, every N_1-ring is commutative. If A is an N_1-extension of a simple subring B, and if B has an identity e, it follows from the lemma to Proposition 1.11 that $A = eAe \oplus (1-e)A(1-e)$. Since $(1-e)A(1-e)$ is an N_1-ring, it is commutative. Now suppose that $eAe = M \oplus N$, where M and N are ideals. If both M and N are disjoint from B, then both M and N are N_1-rings, whence they are commutative. Thus, if eAe is noncommutative, it can be assumed that, say, $B \cap M \neq 0$. Then, by the simplicity of B, $B \subseteq M$, and, since M now contains the identity e of eAe, $M = eAe$, $N = 0$, so that eAe is directly irreducible. This establishes the lemma.

LEMMA 3.3. *If* A *is an* N_1*-extension of a simple* Φ*-subring* B*, and if* B *contains an identity element* e*, then*

$$A = Q \oplus P,$$

where $Q = eAe$*, and* $P = (1-e)A(1-e)$ *is a (commutative)* N_1*-ring. Furthermore, either* A *is commutative, or else* $eAe = Q$ *is directly irreducible.*

Now suppose that B in the lemma is a division Φ-subring. Then, if A is noncommutative, Q is a directly irreducible N_1-extension of B. Since Corollary 1.7 shows that Q is a division ring, it follows from Theorem 3.1 that either $B = Q$, or else Q is a field. This completes the proof of the next theorem.

THEOREM 3.4. *Let* A *be a* Φ*-ring, and* B *a division* Φ*-subring such that* A/B *is an* N_1*-extension. Then, either* A *is commutative, or else* $A = B \oplus P$*, where* P *is a (commutative)* N_1*-ring. Furthermore, if* A *is directly irreducible, and* $B \neq A$*, then* A *is a field.*

The theorem and the discussion preceding have the corollary.

COROLLARY 3.5. *If A is a Φ-ring which is an N_1-extension of a Φ-subfield B, then A is commutative.*

§4. **ξ_2-extensions.** The extension A/B is a ξ-extension in case to each $a \in A$ there exist $x = x_a \in A$ and a natural number $n = n_a$ such that $a^n - a^{n+1}x \in B$. If x can be chosen such that x^n commutes with a^n, for every $a \in A$, then a ξ-extension is a ξ'-extension. A ξ-extension is ξ_2, if B contains all idempotents of A, and ξ_2' if it is both ξ_2 and ξ'.

If A is a Φ-ring, where Φ is a commutative ring with identity, and if B is a Φ-subring such that to each $a \in A$ there correspond $p_a(x) \in \Phi[x]$ and a natural number $n = n_a$ such that $a^n - a^{n+1}p_a(a) \in B$, then A/B is a ξ'-extension; it is ξ_2' if $p_e(e) = 0$ for each idempotent $e \in A$. Thus, the results of this section are applicable to these extensions; in particular, they are applicable to N-extensions.

A ring A is a ξ_2'-ring if it is a ξ_2'-extension of $\mathbf{0}$. It is trivial to verify that any ξ_2'-ring is a nil ring, and conversely. If A/B is ξ_2', and if L is any left ideal disjoint from B, then L is nil. To see this, if $a \in L$, and if $a^n - a^{n+1}x \in B$, then $0 = a^n - a^{n+1}x \in B \cap L = \mathbf{0}$. Since $a^n x^n = x^n a^n$, this implies that $e = a^n x^n$ is idempotent. Since $e \in L \cap B = \mathbf{0}$, then $a^n = ea^n = e = 0$, so that L is nil. This fact is used several times below.

THEOREM 4.1. *If A is a ξ_2'-extension of a simple ring B, and if $J(A) \neq A$, then $J(A)$ is nil, and $A - J(A)$ is primitive.*

Proof. Suppose for the moment that $J(A) \supseteqq B$. Then $A - J(A)$ would be a ξ_2'-ring, whence it is a nil ring. This would imply that $A = J(A)$, which is excluded by hypothesis. Hence $J(A) \not\supseteqq B$, so that $J(A) \cap B = \mathbf{0}$, whence $J(A)$ is nil. Now B cannot be contained in every primitive ideal of A, since the intersection of these is $J(A)$. Hence there exists a primitive ideal P which is disjoint from B. Then P is nil, whence $P = J(A)$, and $A - J(A)$ is primitive.

Now suppose that A is a ring with no nil ideals $\neq \mathbf{0}$ which is a ξ_2'-extension of a division subring B. By the theorem, A is primitive, but, as a matter of fact, A is division. The proof of this is similar to the proof of the theorem, except that one considers the modular maximal left ideals (m.m.l.-ideals) of A instead of the primitive ideals. Since A contains no nil left ideals, one concludes that $\mathbf{0}$ is a m.m.l.-ideal, that is, that A is a division ring. This fact is stated in the next theorem.

THEOREM 4.2. *If A is a ring with no nil ideals $\neq 0$, and if A is a ξ_2'-extension of a division subring, then A is a division ring.*

The corollary below is a consequence of the theorem, and of Theorem 3.1.

COROLLARY 4.3. *If A is a ring with no nil ideals $\neq 0$, and if A is a N_2-extension of a division subring $B \neq A$, then A is a field.*

The last two results can be restated as follows: If A is a ring containing no nonzero idempotents $\neq 1$, and containing no nonzero nil ideals, and if A is a ξ'-extension (resp. N-extension) of a division subring $B \neq A$, then A is a division ring (resp. field.)

The corollary generalizes results on radical extensions of [4] and [5].

If A is a radical extension of an integral domain, then to each $a \in A$ there corresponds a natural number n such that a^n has certain regularity properties. The situation is generalized below.

THEOREM 4.4. *Let A be a ring wiih the property that to each $a \in A$ there corresponds a natural number $n = n_a$ such that a^n is not a proper right divisor of zero in A. Then the set N of nilpotent elements is an ideal, and $A-N$ is an integral domain.*

Proof. Let D denote the set of all right divisors of zero in A. The condition of the theorem implies that $N = D$, so that the theorem follows from Lemma 1.9.

Remark. If A is a ring with a nil ideal N such that $A-N$ is integral, then, of course, $D = N$ in A, and A has the property of the theorem.

Now let A be a radical extension of an integral domain B, that is, such that to each $a \in A$ there corresponds a natural number $n = n_a$ such that $a^n \in B$. Assume that A contains no nil left ideals $\neq 0$, let $x \in A$ be nonnilpotent, and let $y \in L_x = \{a \in A \mid ax = 0\}$. Then, since $y^m x^n = 0$, $m = m_y$. $n = n_x$, since B is integral, and since $x^n \neq 0$, then $y^m = 0$. L_x is therefore nil, so $L_x = 0$. This shows that each $a \in A$ has the property stated in the theorem, and completes the proof of the corollary.

COROLLARY 4.5. *If A is a ring with no nil left ideals $\neq 0$, and if A is a radical extension of an integral domain, then A is an integral domain.*

A commutative integral domain A can be radical over a subring B only under very special circumstances. For then, if A^* and B^* denote the respective quotient fields of A and B, then A^* is radical over B^*. It follows from the work of Kaplansky [Canad. J. Math. vol. 3 (1951) 290-292] that either $A^* = B^*$, or else, A^* has characteristic $p > 0$, and either A^*/B^* is purely inseparable, or else A^* is algebraic over $GF(p)$. It would be interesting to know the corresponding situation for noncommutative integral domains (cf. [5] for some results with added hypotheses on A and B).

5. Generation of primitive rings. If A is a ring, let $T(A)$ denote the subring generated by all nilpotent elements of index two, and let $E(A)$ be the subring generated by all idempotents. If A is primitive, and A/B is s.r., then by Corollary 2.2, B is dense in the finite topology on A, if A is not division. In view of the fact that B contains $T(A)$ when A/B is s.r., it would be interesting to know if any subring of A which contains $0 \neq T(A)$ is dense in A. Positive results abound in special cases, making a counterexample hard to find.

THEOREM 5.1. *If A is a primitive ring with a minimal left ideal, and if A is not a division ring, then $T(A)$ and $E(A)$ are dense in the finite topology on A.* (*Then $T(A)$ and $E(A)$ are primitive rings*).

Let S denote the socle of A. It suffices to show that $T(S) = E(S) = S$, since then density follows from the inclusions $T(A) \supseteqq S$, $E(A) \supseteqq S$. Thus, the theorem is a consequence of the lemma below. (In case A does *not* satisfy the minimum condition, then the theorem follows immediately from Rosenberg's generalization [Proc. Amer. Math. Soc. vol. 7 (1956) p. 897, Corollary 5] of a theorem of Kasch [10]).

LEMMA 5.2. (a) *If A is a simple ring containing a nontrivial idempotent, then $T(A) = A$. If, in addition,* (b) *A is an algebra over a field $\Phi \neq GF(2)$, or* (c) *if A contains a minimal left ideal, then $E(A) = A$.*

Proof. (a) Let $\mathscr{T}$ denote the additive subgroup generated by all nilpotent elements of index two, and let, for any subset S of A, $[S, S]$ denote the additive subgroup generated by all $[a, b] = ab - ba$, $a, b \in S$. If $u, v \in \mathscr{T}$ are nilpotent of index two, then so is

$$w = (1+u)v(1-u),$$

Then,

$$[u, v] = w + uvu - v \in \mathscr{T}.$$

It easily follows from this that $\mathscr{T}$ is a Lie ring with respect to $[a, b]$. Then Amitsur's [1, Lemma 2] shows that $\mathscr{T} \supseteqq [A, A]$, so that $T(A) \supseteqq [A, A]$. If e is any nontrivial idempotent in A, then eAf, $fAe \subseteqq \mathscr{T} \subseteqq T(A)$, where (formally) $f = 1 - e$. But $T(A)$ also contains the product

$$eAe = e(AfA)e = (eAf)(fAe),$$

similarly, $fAf \subseteqq T(A)$. Then $T(A) = A = eAe + eAf + fAe + fAf$, as needed.

(b) In this case Amitsur's [1, Theorem 1] states that A contains no non-invariant noncentral subalgebras $\neq A$, unless A is 4-dimensional over a field F of characteristic two. Since $E(A)$, $T(A)$ are invariant noncentral subalgebras, equality $E(A) = T(A) = A$ follows when $\dim A/F \neq 4$. In this exceptional case, A is a simple matrix algebra. A general property of arbitrary matrix algebras $A = R_n$, $n > 1$, implied by [7, Prop. 1] is that $E(A) = T(A) = A$. This latter result also suffices for the case (c), since A is then locally a complete matrix ring R_n, $n > 1$, by Litoff's theorem [8, p. 90].

THEOREM 5.3. *Let A be an algebraic algebra over the field Φ.* (a) *If A is primitive, but not division, then $E(A)$ and $T(A)$ are dense in the finite topology on A. (Then $E(A)$ and $T(A)$ are primitive algebras.)* (b) *If A is semisimple, so is $E(A)$.*

Proof. (a) The proof is analogous to that of Theorem 2.1. Adopting the terminology there, with $B = E(A)$ (resp. $B = T(A)$), if $\bar{e}$ is any element in a complete set of matrix units for $\bar{U}$, by [9, p. 239 ff.], there exists an element f in a complete set of matrix units in U such that $\bar{f} = \bar{e}$. If $\bar{e}^2 = \bar{e}$ (resp. $\bar{e}^2 = 0$), then, since $f \in E(A)$ (resp. $f \in T(A)$), it follows that $\bar{e} \in \bar{Q}$. Since any automorphism of $\bar{U}$ maps a complete set of matrix units onto another complete set, this latter assertion shows that $\bar{Q}$ contains all conjugates of $\bar{e}$. Since $\bar{U} = D_n$, $n > 1$, by [7, Prop. 1], $\bar{U}$ is generated by the conjugates of $\bar{e}$, so that $\bar{U} = \bar{Q}$. The rest of the proof is unchanged.

(b) It is not hard to show that a subring (subalgebra) B of a semisimple ring (algebra) A is itself semisimple, if each homomorphism of A which maps A onto a primitive ring (algebra) also maps B onto a primitive ring (algebra).

(The proof of this is related to that of Corollary 2.2). Thus, if A is a semi-simple algebraic algebra, and if P is any primitive ideal of A, then [9, p. 239 ff.] shows that the canonical map $A \to A-P$ maps $B=E(A)$ onto $E(A-P)$. If $A-P$ is not division, then $E(A-P)$ is primitive by (a), while if $A-P$ is division, since it is an algebraic division algebra, every nonzero subalgebra is a division algebra. Thus $E(A-P)$ is primitive in this case too, and the semisimplicity of B follows from the remark above.

Relating to Lemma 5.2 is the question whether $T(A)=A$ in a simple ring (algebra) A implies the equality $E(A)=A$.

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