A GENERALIZATION OF SUSPENSION THEOREMS

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Our purpose in this note is to establish a classification theorem for fiberings with a loop-space as fibre. This is deduced by applying a generalized suspension theorem which will be proved in § 1. As a by-product we obtain a proposition concerning fiberings with a loop-space as the total. Throughout this note we shall denote by $\mathfrak{W}$ the category of spaces having the based homotopy type of a CW-complex.

§ 1. Generalized suspension theorems

For a given map $f: X \to Y$, let E_f denote the subspace of $X \times EY$ consisting of the pairs (x, β) such that $f(x) = \beta(1)$ where EY is the space of paths in Y emanating from the base-point y_0, and let C_f denote the space obtained by attaching the reduced cone over X to Y by means of f. Denoting the loop and suspension functors respectively by Ω and S, we have defined in [6, p. 136] $\eta': SE_f \to C_f$ and $\eta: E_f \to \Omega C_f$ by setting, for $(x, \beta) \in E_f$,

$$\eta'((x, \beta), s) = \begin{cases} \beta(2s) & \text{if } 0 \leqq s \leqq \frac{1}{2}, \\ (x, 2-2s) & \text{if } \frac{1}{2} \leqq s \leqq 1, \end{cases}$$

$$\{\eta(x, \beta)\}(s) = \begin{cases} (x, 2s) & \text{if } 0 \leqq s \leqq \frac{1}{2}, \\ \beta(2-2s) & \text{if } \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

These induce *suspensions* $\sigma^* = (\eta')^*: \pi(C_f, Z) \to \pi(SE_f, Z)$ and $\sigma_* = \eta_*: \pi(Z, E_f) \to \pi(Z, \Omega C_f)$ for any space Z, where $\pi(A, B)$ denotes the set of homotopy classes of maps $A \to B$. The following has been established there:

THEOREM 1. *If Y is r-connected and E_f s-connected, then $\sigma^*: H^q(C_f) \to H^q(SE_f)$ is an isomorphism for $q \leqq r+s+1$ and a monomorphism for $q = r+s+2$.*

Received April 26, 1961.

THEOREM 1'. *Let X and C_f be r- and s-connected respectively. Then* σ_*: $\pi_q(E_f)\to\pi_q(\Omega C_f)$ *is an isomorphism for* $1\leqq q\leqq r+s-1$ *and an epimorphism for* $q=r+s$.

These theorems may be generalized as follows:

THEOREM 2. *Let X and Y both belong to $\mathfrak{W}$, and let Y and E_f be r- and s-connected respectively* ($r\geqq 1$, $s\geqq 0$). *Suppose that Z is n-simple for all $n>0$. Then*

(a) σ^*: $\pi(C_f, Z)\to\pi(SE_f, Z)$ *is onto if* $\pi_q(Z)=0$ *for* $q\geqq r+s+2$.

(b) σ^*: $\pi(C_f, Z)\to\pi(SE_f, Z)$ *is* $1-1$ *if* $\pi_q(Z)=0$ *for* $q\geqq r+s+3$.

THEOREM 2'. *Let Z be in $\mathfrak{W}$ and let X aud C_f be r- and s-connected respectively* ($r\geqq 1$, $s\geqq 2$). *Then*

(a) σ_*; $\pi(Z, E_f)\to\pi(Z, \Omega C_f)$ *is* $1-1$ *if the integral cohomology groups* $H^q(Z)$ *are trivial for* $q\geqq r+s$.

(b) σ_*: $\pi(Z, E_f)\to\pi(Z, \Omega C_f)$ *is onto if* $H^q(Z)=0$ *for* $q\geqq r+s+1$.

The proofs will be based on the following simple:

LEMMA. *Let X and Y be in $\mathfrak{W}$ and let f: $X\to Y$ be a given map. Then*

(i) *f is equivalent to an inclusion $X'\subset Y'$ where X' is a subcomplex of a CW-complex Y'.*

(ii) *E_f and C_f also belong to $\mathfrak{W}$.*

Proof of Lemma. By using the mapping cylinder argument we may assume that f is an inclusion. Then we have a commutative diagram

$$\begin{array}{ccc} |X| & \xrightarrow{|f|} & |Y| \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ X & \xrightarrow{f} & Y \end{array}$$

where $|X|$ and $|Y|$ are respectively the geometric realizations (see [4]) of the singular complexes of X, Y and $|f|$ is induced by f. j_1 and j_2 are canonical maps which induce isomorphisms of homotopy groups [4, Theorem 4], and hence homotopy equivalences by [8, Theorem 1]. Taking X', Y' to be $|X|$, $|Y|$, we obtain the first assertion. We shall now prove the second half. In view of [6, Lemmas 6, 9] and (i), we may assume that X is a subcomplex of a CW-complex Y and that f is an inclusion. Obviously, $C_f=Y\cup CX$ is then a CW-complex. On the other hand, E_f is the space of paths in Y emanating from the base-point and ending in X. Milnor's result [5, Theorem 3] implies that E_f belongs to $\mathfrak{W}$. This completes the proof of the lemma.

Proof of Theorem 2. By the preceding lemma, η' is equivalent to an inclusion i: $A \subset B$ where A is a subcomplex of a CW-complex B. Hence, to prove our theorem, it suffices to show that i induces i^*: $\pi(B, Z) \to \pi(A, Z)$ having the property stated in Theorem 2. Theorem 1 and the exactness of the cohomology sequence of the pair (B, A) now imply that $H^q(B, A; G) = 0$ for $q \leqq r+s+2$ and for all coefficient groups G.

Firstly, let $\pi_q(Z) = 0$ for $q \geqq r+s+2$; then $H^{q+1}(B, A; \pi_q(Z)) = 0$ for all $q \geqq 1$. Thus a standard obstruction argument shows that i^* is onto. Secondly, suppose $\pi_q(Z) = 0$ for $q \geqq r+s+3$; in this case $H^q(B, A; \pi_q(Z))$ vanishes for every $q \geqq 1$. Therefore, by the same reasoning, we see that i^* is $1-1$.

Proof of Theorem 2'. Consider the mapping cylinder M of η: $E_f \to \Omega C_f$. We have a homotopy-commutative diagram:

$$\begin{tikzcd} E_f \arrow[r, "\eta"] \arrow[dr] & \Omega C_f \arrow[d] \\ & M \end{tikzcd}$$

where the unlabelled arrows are inclusions and the vertical one is a homotopy equivalence. According to Theorem 1', we have $\pi_q(M, E_f) = 0$ for $1 \leqq q \leqq r+s$. Let $|Z|$ be the geometric realization of the singular complex of Z and j: $|Z| \to Z$ be the canonical map. In case (a), $H^{q-1}(|Z|; \pi_q(M, E_f)) = 0$ for all $q \geqq 2$, and, in case (b), $H^q(|Z|; \pi_q(M, E_f)) = 0$ for all $q \geqq 2$. Observe also that our assumption implies $\pi_1(E_f) = 0$ and hence the pair (M, E_f) is n-simple for every $n \geqq 2$. Noting that j is an equivalence and using the theory of obstructions to compressions as outlined in [2, Theorem 4.4.2], we obtain the desired conclusions.

COROLLARY 3. (J. Stasheff [7, Theorem 2]) *Suppose Y is a r-connected space belonging to $\mathfrak{W}$, and that Z satisfies the condition $\pi_q(Z) = 0$ for $q \geqq 2r+1$ $(r \geqq 1)$. Then $\pi(Y, Z)$ is in $1-1$ correspondence with $\pi(\Omega Y, \Omega Z)$.*

Proof. Taking X to be a point in Theorem 2, we have $E_f = \Omega Y$, $C_f = Y$, and $s=r-1$. The assertion then follows at once. Note that $\sigma^*: \pi(Y, Z) \to \pi(S\Omega Y, Z) \approx \pi(\Omega Y, \Omega Z)$ is seen to be the mapping induced by assigning to $v: Y \to Z$ $\Omega v: \Omega Y \to \Omega Z$.

Similarly, by taking Y to be a point in Theorem 2′, we have

COROLLARY 3′. *Let X be r-connected and let Z be a CW-complex with* $\dim Z \leqq 2r$. *Then $\pi(Z, X)$ is in $1-1$ correspondence with $\pi(SZ, SX)$.*

§ 2. Applications

First, the notations introduced in [6] will be used throughout; in particular, we consider $\bar{\sigma}^*: \pi(C_f, Z) \to \pi(E_f, \Omega Z)$ given by $\bar{\sigma}^*\{v\} = \{\eta_{f,u}\}$, where $v: C_f \to Z$, $u = v \circ P' f: Y \to C_f \to Z$, and $\eta_{f,u}: E_f \to \Omega Z$ is defined by

$$\{\eta_{f,u}(x, \beta)\}(s) = \begin{cases} v(x, 2s) & \text{if } 0 \leqq s \leqq \frac{1}{2}, \\ u\beta(2-2s) & \text{if } \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

Note that $\bar{\sigma}^*$ corresponds to $-\sigma^*$ (see [6, Lemma 14]).

Next, let $F \xrightarrow{i} X \xrightarrow{f} Y$ and $F' \xrightarrow{i'} X' \xrightarrow{f'} Y'$ be two fiberings with fibres F, F' respectively, and let the following diagram be commutative:

$$\begin{array}{ccccc} F & \xrightarrow{i} & X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \bar{h}} & & \downarrow{\scriptstyle g} \\ F' & \xrightarrow[i']{} & X' & \xrightarrow[f']{} & Y' \end{array}$$

If g, $\bar{h}$, h are all (weak) homotopy equivalences, we say that $(h, \bar{h}, g): f \to f'$ is a *(weak) equivalence* between two fiberings.

Finally, we define $\{Y, Z\}$ to be the set of equivalence classes of elements in $\pi(Y, Z)$, in which we call u_1, u_2: $Y \to Z$ *equivalent* if and only if there exist homotopy equivalences $h: Z \to Z$, $k: Y \to Y$ satisfying $u_2 \simeq h u_1 k$.

Under these definitions, the main result obtained by applying Theorem 2 is stated as follows:

THEOREM 4. *Let Y and Z belong to $\mathfrak{W}$, and let Y be $(r-1)$-connected $(r \geqq 2)$. Suppose further that $\pi_q(Z) \neq 0$ only for $s+1 \leqq q \leqq r+s-1$ $(s \geqq 1)$ and that $s+1 \geqq r$. Then the equivalence classes of fiberings in $\mathfrak{W}$, with Y as base and with fibre ΩZ, are in $1-1$ correspondence with $\{Y, Z\}$.*

We list an immediate consequence of this theorem which seems to be well known:

COROLLARY. *Let Y be a 1-connected space in $\mathfrak{W}$, and let Z be a space of type $(\pi, n+1)$ which also belongs to $\mathfrak{W}$. Then the classes of fiberings in $\mathfrak{W}$ having Y as base and ΩZ as fibre are in $1-1$ correspondence with the equivalence classes in $H^{n+1}(Y, \pi)$ under Θ, where Θ is the group of automorphisms of $H^{n+1}(Y, \pi)$ determined by homotopy equivalences of Y and automorphisms of π.*

The $1-1$ correspondence in Theorem 4 is established by assigning to u: $Y \to Z$ the class of the fibering Pu: $E_u \to Y$ induced by u from $EZ \to Z$. The fact that this is really $1-1$ will be readily seen by combining several lemmas below.

The following result is due to T. Ganea [3, Lemma 2.1]:

LEMMA 1. *Let $F \xrightarrow{i} X \xrightarrow{f} Y$ be a fibering with X and Y both in $\mathfrak{W}$. Suppose Y is $(r-1)$-connected and that $\pi_q(F) \neq 0$ only for $s \leqq q \leqq r+s-2$ $(r \geqq 2, s \geqq 1)$. Suppose further that there exists a weak equivalence θ: $F \to \Omega Z$, where Z is a 1-connected space. Then we can find a map u: $Y \to Z$ such that there exists a weak equivalence $(\xi, \bar{\xi}, 1)$: $f \to Pu$ with $\xi \simeq \theta$.*

Proof. By assumption, $\pi_q(Z) \approx \pi_{q-1}(\Omega Z) \approx \pi_{q-1}(F) = 0$ for $q \geqq r+s$. Further, since F and E_f are equivalent to each other (see [6, Theorem 1]), E_f is $(s-1)$-connected. Thus Theorem 2, (a) implies that $\bar{\sigma}^{*}$: $\pi(C_f, Z) \to \pi(E_f, \Omega Z)$ is onto, i.e. there exists a map v: $C_f \to Z$ such that $\eta_{f,u} \simeq \theta\Psi$ where u is the composition $Y \xrightarrow{P'f} C_f \xrightarrow{v} Z$ and Ψ: $E_f \to F$ the canonical equivalence. We have a homotopy-commutative diagram:

$$\begin{tikzcd} E_f \arrow[r, "Pf"] \arrow[d, dashed, shift right, "\theta\Psi"'] \arrow[d, dashed, shift left, "\eta_{f,u}"] & X \arrow[r, dashed, "f"] \arrow[d, dashed, "\bar{\xi}"] & Y \arrow[r, dashed, "P'f"] \arrow[d, dash, dotted] & C_f \arrow[d, dashed, "v"] \\ \Omega Z \arrow[r, "Iu"'] & E_u \arrow[r, "Pu"'] & Y \arrow[r, dashed, "u"'] & Z \end{tikzcd}$$

in which the middle square is commutative, and so $\bar{\xi}$ induces ξ: $F \to \Omega Z$. Applying the five lemma to the diagram of homotopy groups derived from the above one, we conclude that $\bar{\xi}$ is a weak equivalence. Since a simple computation shows that

$$\xi(x)(s) = v(x, s) \qquad \text{for } x \in F,\ 0 \leqq s \leqq 1,$$

$$\{\eta_{f,u} \circ \Phi(x)\}(s) = \begin{cases} v(x, 2s) & \text{for } 0 \leqq s \leqq \frac{1}{2}, \\ z_0 = \text{base-point of } Z & \text{for } \frac{1}{2} \leqq s \leqq 1, \end{cases}$$

where Φ: $F \to E_f$ is a canonical inverse of Ψ given by $\Phi(x) = (x, e)$, e being the constant path at y_0, we have $\theta \simeq \xi$, as asserted.

Next, in order to examine the extent to which u is determined, we need

LEMMA 2. *Let h: $Z \to Z'$ be a map; then $\sigma^* h_* = h_* \sigma^*$, $\bar{\sigma}^* h_* = (\Omega h)_* \bar{\sigma}^*$.*

LEMMA 3. *In Lemma 1 we assume moreover that Z belongs to $\mathfrak{W}$ and that $s+1 \geqq r$ (this is always the case when $r=2$). Suppose there is given the commutative diagram*

$$\begin{array}{ccccccc} F & \xrightarrow{i} & X & \xrightarrow{f} & Y & & \\ \downarrow{\scriptstyle \xi_j} & & \downarrow{\scriptstyle \bar{\xi}_j} & & & & \\ \Omega Z & \xrightarrow[Iu_j]{} & E_{u_j} & \xrightarrow[Pu_j]{} & Y & \xrightarrow[u_j]{} & Z \end{array} \qquad j = 1, 2,$$

in which vertical maps are (weak) equivalences. Then there exists an equivalence h: $Z \to Z$ such that $u_2 \simeq hu_1$ and $\xi_2 \overset{w}{\simeq} \Omega h \circ \xi_1$.

Proof. Let $\tilde{\xi}_1$ be a homotopy inverse of ξ_1; since Z is s-connected and $r+s \leqq 2s+1$, we can apply Corollary 3 to obtain a map h: $Z \to Z$ such that $\Omega h \simeq \xi_2 \circ \tilde{\xi}_1$: $\Omega Z \to \Omega Z$. Consider v_j: $C_f \to Z$ given by $(j=1, 2)$

$$v_j(y) = u_j(y) \qquad \text{for } y \in Y,$$
$$v_j(x, s) = \gamma_j(x)(s) \qquad \text{for } x \in X,\ 0 \leqq s \leqq 1,$$

where $\bar{\xi}_j(x) = (f(x), \gamma_j(x))$. Then it follows from the proof of Lemma 1 that $\Phi^* \bar{\sigma}^* \{v_j\} = \{\xi_j\}$. This leads to the following:

$$\begin{aligned} \Phi^* \bar{\sigma}^* h_* \{v_1\} &= \Phi^* (\Omega h)_* \bar{\sigma}^* \{v_1\} \qquad \text{by Lemma 2,} \\ &= (\Omega h)_* \Phi^* \bar{\sigma}^* \{v_1\} = (\Omega h)_* \{\xi_1\} \\ &= \{\xi_2\} = \Phi^* \bar{\sigma}^* \{v_2\}. \end{aligned}$$

Because $\bar{\sigma}^*$ is $1-1$ by Theorem 2, (b) and Φ^* is an isomorphism, we have $\{hv_1\} = \{v_2\}$, whence, composing $P'f$ to the right, $\{hu_1\} = \{u_2\}$.

LEMMA 4. *Let u: $Y \to Z$ be a map and let h: $Z \to Z'$, k: $Y' \to Y$ be homotopy equivalences. Then Pu: $E_u \to Y$ and $P(huk)$: $E_{huk} \to Y'$ are equivalent.*

Proof. Let $\tilde{k}$ be a homotopy inverse of k. Consider then the following homotopy-commutative diagram

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "\tilde{k}"'] & Z \arrow[d, "h"] \\ Y' \arrow[r, dashed, "huk"'] & Z' \end{tikzcd}$$

This transformation gives rise to the map χ: $E_u \to E_{huk}$ which is defined by

$$\chi(y, \gamma) = (\tilde{k}(y), \gamma'(y)) \qquad \text{for } y \in Y,\ \gamma \in EZ,\ u(y) = \gamma(1),$$

where $\gamma'(y)$ is the path in Z' given by, using a homotopy H_t: $Y \to Z'$ with $H_0 = hu$, $H_1 = huk\tilde{k}$, $\gamma'(y)(s) = h\gamma(2s)$ for $0 \leqq s \leqq \frac{1}{2}$, $= H_{2s-1}(y)$ for $\frac{1}{2} \leqq s \leqq 1$. It follows from Lemma 6 in [6] that χ is a homotopy equivalence. Thus $(\Omega h, \chi, \tilde{k})$: $Pu \to P(huk)$ is a desired equivalence.

LEMMA 5. *Suppose there is given the commutative diagram*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, dashed, "\xi"] & X \arrow[r, "f"] \arrow[d, dashed, "\bar{\xi}"] & Y \arrow[d, dashed, "\varphi"] & \\ \Omega Z \arrow[r, "Iu"'] & E_u \arrow[r, "Pu"'] & Y' \arrow[r, "u"'] & Z \end{tikzcd}$$

where the top line is a fibering. Then the above diagram admits a factorization:

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "\xi"] & X \arrow[r, "f"] \arrow[d, "\bar{\varphi}"] & Y & \\ \Omega Z \arrow[r, dashed, "I(u\varphi)"] \arrow[d, dashed, no head] & E_{u\varphi} \arrow[r, "P(u\varphi)"] \arrow[d, "h"] & Y \arrow[r, "u\varphi"] \arrow[d, "\varphi"] & Z \\ \Omega Z \arrow[r, "Iu"'] & E_u \arrow[r, "Pu"'] & Y' \arrow[r, "u"'] & Z \end{tikzcd}$$

Proof. If we write down $\bar{\xi}$ as $\bar{\xi}(x) = (\varphi f(x), \gamma(x))$ for $x \in X$, $\gamma(x) \in EZ$ with $u\varphi f(x) = \gamma(x)(1)$, then it suffices to consider $\bar{\varphi}$, h given by $\bar{\varphi}(x) = (f(x), \gamma(x))$, and $h(y, \gamma) = (\varphi(y), \gamma)$, $y \in Y$, $\gamma \in EZ$.

Finally, we shall prove

THEOREM 5. *Let f: $\Omega W \to \Omega V$ be a map with W, V in $\mathfrak{W}$. Suppose further that $\pi_i(W) \neq 0$ only for $n+1 \leqq i \leqq 2n$ and $\pi_j(V) \neq 0$ only for $q+2 \leqq j \leqq n+q+1$ $(2 \leqq n \leqq q+1)$. If E_f is of the same homotopy type as a loop-space of a space in $\mathfrak{W}$, then f is homotopic to Ωg for some g: $W \to V$.*

Proof. Let $\theta: E_f \to \Omega Z$ be an equivalence. For $q = n-1$, the assertion is trivial by Corollary 3. Hence we may assume that $q \geqq n$. Since E_f is $(n-1)$-connected and $\pi_i(Z) = 0$ for $i \geqq n+q+1$, we can apply Theorem 2, (a) to find a map $v: C_f \to Z$ such that there exists a homotopy-commutative diagram

$$\begin{tikzcd}
E_f \arrow[r, "Pf"] \arrow[d, shift right, "\theta"'] \arrow[d, shift left, "\eta_{f,u}"] & \Omega W \arrow[r, "f"] \arrow[d, "\xi_{f,u}"] & \Omega V \arrow[r, "P'f"] & C_f \arrow[d, "v"] \\
\Omega Z \arrow[r, "Iu"'] & E_u \arrow[r, "Pu"'] & \Omega V \arrow[r, "u"'] & Z
\end{tikzcd}$$

in which $\theta \simeq \eta_{f,u}$, $u = v \circ P'f$. By the five lemma we see that $\xi_{f,u}$ is a homotopy equivalence. Choose a homotopy inverse $\rho: E_u \to \Omega W$ of $\xi_{f,u}$. Observe that Z and E_u are n- and $(n-1)$-connected respectively. Thus Theorem 2 yields again a map $v': C_u \to W$ such that $\eta_{u,w} \simeq \rho$ holds in the homotopy-commutative diagram

$$\begin{tikzcd}
E_u \arrow[r, "Pu"] \arrow[d, shift right, "\rho"'] \arrow[d, shift left, "\eta_{u,w}"] & \Omega V \arrow[r, "u"] \arrow[d, "\xi_{u,w}"] & Z \arrow[r, "P'u"] & C_u \arrow[d, "v'"] \\
\Omega W \arrow[r, "Iw"'] & E_w \arrow[r, "Pw"'] & Z \arrow[r, "w"'] & W
\end{tikzcd}$$

where $w = v' \circ P'u$. As above, $\xi_{u,w}$ is seen to be an equivalence. Let τ: $E_w \to \Omega V$ be a homotopy inverse of $\xi_{u,w}$. Since E_w is q-connected, Theorem 2 shows that there exists a homotopy-commutative diagram

$$\begin{tikzcd}
\Omega W \arrow[r, "Iw"] & E_w \arrow[r, "Pw"] \arrow[d, shift right, "\tau"'] \arrow[d, shift left, "\eta_{w,t}"] & Z \arrow[r, "w"] \arrow[d, "\xi_{w,t}"] & W \arrow[r, "P'w"] & C_w \arrow[d] \\
\Omega W \arrow[r, "\omega\circ\Omega t"'] & \Omega V \arrow[r, "It"'] & E_t \arrow[r, "Pt"'] & W \arrow[r, "t"'] & V
\end{tikzcd}$$

where $\tau \simeq \eta_{w,t}$ and ω is the involution induced by reversing loop-parameter. Combining the above three diagrams with one another we see that $f \simeq \omega \circ \Omega t$. Corollary 3 implies that $\omega \simeq \Omega s$ for some $s: V \to V$, whence $f \simeq \Omega g$ for $g = s \circ t$.

Remark. By a similar argument as in Theorem 5 we may obtain a result due to Berstein and Hilton ([1, Lemma 3.6]): *Let $f: S^{m-1} \to S^q$, $m \geqq q+1 \geqq 3$. If $C_f = S^q \bigcup_f e^m$ is of the same homotopy type as a suspension, then f is homotopic to Sg for some $g: S^{m-2} \to S^{q-1}$.*

Added in proof. After the submission of this note, Hilton's paper "On excision and principal fibrations" (Comment. Math. Helv. **35**, 1961, Fasc. 2) appeared, where he has obtained essentially the same results as our Theorems 2 and 2′.

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