A PROPERTY OF SOME POINCARÉ THETA-SERIES

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1. Consider circles $c_\nu (\nu = \pm 1, \pm 2, \ldots)$ with centers ξ_ν on the real axis of the z-plane such that they are disjoint from each other and cluster to infinity $z = \infty$ from the both sides of the real axis. Here, without loss of generality, we may assume that $\xi_{-\nu-1} < \xi_{-\nu} < 0 < \xi_\nu < \xi_{\nu+1}$ for every positive integer ν. Let B be the fundamental domain, bounded by $c_\nu (\nu = \pm 1, \pm 2, \ldots)$, of the properly discontinuous group Γ generated by the hyperbolic linear transformations with real coefficients

$$z' = S_\nu(z) = \frac{\alpha_\nu z + \beta_\nu}{\gamma_\nu z + \delta_\nu}, \qquad (\nu = \pm 1, \pm 2, \ldots), \tag{1}$$

each of which for every ν transforms the outside of $c_{-\nu}$ into the inside of c_ν.

Consider the Poincaré theta-series of (-2)-dimension

$$\Theta(z) = \sum_{\Gamma} H[S(z)] \frac{dS(z)}{dz}, \tag{2}$$

where the kernelfunction $H(z)$ is a real rational function whose poles are in the set $\bar{B} = B \cup (\bigcup_{\nu=-\infty}^{\infty} c_\nu)$. It is well known that the series (2) converges absolutely and uniformly in the complement D of the set of singular points of Γ, with respect to the z-plane, and defines a function meromorphic in D. For each transformation of Γ, we have the well known differential invariant

$$\Theta(S(z))\, dS(z) = \Theta(z)\, dz. \tag{3}$$

This invariant (3) is called an automorphic differential. The function

$$I(z) = \int_{z_0}^{z} \Theta(z)\, dz$$

is obtained by integrating the automorphic differential along an arbitrary path in D.

Now, if we choose as a kernelfunction

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$$H(z)=\frac{1}{z-a}-\frac{1}{z-b} \qquad (a<b, \text{ real}, a, b \in \bar{B}),$$

then we obtain the following analytic representation of $I(z)$:

$$(4) \qquad I(z)=\sum_{\Gamma}\log\left[\frac{S(z)-a}{S(z)-b}:\frac{S(z_0)-a}{S(z_0)-b}\right]=\sum_{\Gamma}\log\left[\frac{z-S(a)}{z-S(b)}:\frac{z_0-S(a)}{z_0-S(b)}\right].$$

In what follows, we assume that z_0 is the origin $z=0$ for convenience.

The following two cases (i) and (ii) occur according to the positions of a and b.

(i) The case where a and b are congruent with respect to some generator of Γ, that is, $b=S_\nu(a)$ for some ν. In this case, the poles of different terms of (4) are canceled each other in pairs and we obtain a finite integral in D. Moreover, we can easily see that $I(z)$ depends on the pole $J_\nu=-\frac{\delta_\nu}{\gamma_\nu}$ of $S_\nu(z)$ but does not depend on a. If we denote such an $I(z)$ by

$$\varphi_\nu(z)=\int\Theta(z, J_\nu)\,dz,$$

then we have a sequence of functions $\{\varphi_\nu(z)\}$ $(\nu=1, 2, 3, \ldots)$. If $\xi_\nu=-\xi_{-\nu}$ and if the radius of c_ν equals that of $c_{-\nu}$, then the function $\varphi_\nu(z)$ is a real elementary normal integral of the first kind in the sense of L. Myrberg [2]. We call $\varphi_\nu(z)$ a real normal integral of the first kind.

By an easy computation (Burnside [1], P. J. Myrberg [4]) we obtain the relations

$$\int_{c_\nu} d\varphi_\nu=2\pi i, \qquad \int_{c_\mu} d\varphi_\nu=0 \qquad (\mu\neq\nu).$$

If γ_ν is a Jordan curve which joins two equivalent points on circles $c_{-\nu}$ and c_ν in the upper half of B, then the period

$$\tau_{\nu\mu}=\int_{\gamma_\mu} d\varphi_\nu$$

of φ_ν along γ_μ is real.

(ii) The case where a and b are not congruent for any generator of Γ. The poles of different terms of (4) cannot be canceled each other. We denote such an intgeral $I(z)$ by $\chi_{ab}(z)$ and call it a real normal integral of the third kind (P. J. Myrberg [3], [4], [5]). It has the following properties:

1° $\chi_{ab}(z)$ is regular in B except at a and b, where it has logarithmic poles with residues -1 and 1 respectively.

2° The periods of $\chi_{ab}(z)$ along c_ν and γ_ν are

$$\int_{c_\nu} d\chi_{ab}(z)=0 \qquad (\nu=\pm 1,\ \pm 2,\ \ldots),$$

and

$$\int_{\gamma_\nu} d\chi_{ab}(z)=\varphi_\nu(b)-\varphi_\nu(a) \qquad (\nu=1,\ 2,\ \ldots).$$

2. Let B_0 be the upper half of the fundamental domain B. Any branches of $\varphi_\nu(z)$ and $\chi_{ab}(z)$ are single-valued and regular in B_0 by the monodromy theorem. We take the branches of $\varphi_\nu(z)$ and $\chi_{ab}(z)$ such that $\varphi_\nu(0)=0$ and $\chi_{ab}(0)=0$ and denote them by $\varphi_\nu(z)$ and $\chi_{ab}(z)$ again. Let us consider the images of B_0 by them. The function $\varphi_\nu(z)$ is real on the intersection of $\overline{B}$ with the part of the real axis between $c_{-\nu}$ and c_ν. The imaginary part of $\varphi_\nu(z)$ increases by π, when z describes the upper half circumference of c_ν or $c_{-\nu}$. According as the origin $z=0$ is contained in the interval $[a, b]$ or not, $\chi_{ab}(z)$ is real on the real axis in B inside or outside $[a, b]$. The imaginary part of $\chi_{ab}(z)$ increases by $-\pi$ or π in the former and by π or $-\pi$ in the latter respectively, when z passes through $z=a$ or $z=b$ in the positive direction.

We see that $w=\varphi_\nu(z)=u_\nu(z)+iv_\nu(z)$ maps B_0 conformally onto the rectangle $a_\nu<u_\nu<a_\nu+\tau_{\nu\nu}$, $0<v_\nu<\pi$ with vertical slits starting from the upper and lower sides and corresponding to the upper halves of all c_μ except for $\mu=\nu$. And $w=\chi_{ab}(z)=u_{ab}(z)+iv_{ab}(z)$ maps B_0 conformally onto the strip domain $-\infty<u_{ab}<\infty$, $0<v_{ab}<\pi$ with vertical slits starting from the upper and the lower sides and corresponding to the upper halves of $c_\mu(\mu=\pm 1,\ \pm 2,\ \ldots)$.

As to these slits, there are two cases: these slits cluster to a point from the both sides or not. In the former case we say that the type of $\varphi_\nu(z)$ or $\chi_{ab}(z)$ is parabolic.

In the following, we shall give some results concerning the type of $\varphi_\nu(z)$ or $\chi_{ab}(z)$. These results are analogues of theorems due to L. Myrberg [2].

3. Mapping the upper half plane $\mathrm{Im}(z)\geqq 0$ onto the unit circle $|z_1|\leqq 1$ conformally, we use the notations S'_ν, Γ', B'_0, $\{c'_\nu\}$ $(\nu=\pm 1,\ \pm 2,\ \ldots)$, a' and b' for the corresponding ones in the z_1-plane for simplicity and denote by $P^{(0)}_\infty$ the point on $|z_1|=1$ corresponding to $z=\infty$. Let us denote by α the intersection of the boundary of B'_0 and the circular arc $\widehat{a'b'}$ on $|z_1|=1$ not containing $P^{(0)}_\infty$.

Put

$$\alpha_\nu = S'_\nu(\alpha), \qquad (\nu = 0, \ \pm 1, \ \pm 2, \ldots)$$

where $S'_{-\nu}$ is the inverse of S'_ν and S'_0 the identical transformation. Obviously, Γ' is a fuchsoid group with fundamental domain B'_0.

Construct a single-valued bounded harmonic function $r_\alpha(z_1)$ in $|z_1|<1$ such that

$$r_\alpha(z_1) = \begin{cases} \pi \text{ on the set } \bigcup\limits_{\nu=-\infty}^{\infty} \alpha_\nu, \\ 0 \text{ on the complementary set } \{|z_1|=1\} - \bigcup\limits_{\nu=-\infty}^{\infty} \alpha_\nu. \end{cases}$$

Then $r_\alpha(z_1)$ is an automorphic function with respect to Γ' and $0<r_\alpha(z_1)<\pi$ in B'_0. We now prove the following

LEMMA. *If for some* α

$$\lim_{z_1 \to P_\infty^{(0)}} r_\alpha(z_1) = 0$$

along the radius, then also $\lim\limits_{z_1 \to P_\infty^{(0)}} r_\alpha(z_1) = 0$ *along the radius for any* α.

It means that $\lim\limits_{z_1 \to P_\infty^{(0)}} r_\alpha(z_1) = 0$ is independent of α.

Proof. We use a similar argument as in L. Myrberg [2]. Let $r_\alpha^{(\mu)}(z_1)$ be the multiple by π of the harmonic measure of α_μ with respect to $|z_1|<1$. Then we obtain

$$r_\alpha(z_1) = \sum_{\mu=-\infty}^{\infty} r_\alpha^{(\mu)}(z_1).$$

Let $R^{(0)}$ be the radius of $|z_1|<1$ terminating in the point $P_\infty^{(0)}$. It is obvious that $R^{(0)}$ lies in B'_0. Then the value $r_\alpha^{(\mu)}(P^{(0)})$ at a point $P^{(0)}$ on $R^{(0)}$ is equal to $r_\alpha^{(0)}(P^{(\mu)})$, where $P^{(\mu)} = S'_\mu(P^{(0)})$. If

(5) $$z_2 = \frac{az_1+b}{cz_1+d}, \qquad (ad-bc=1)$$

is the linear transformation which makes $|z_1|<1$ invariant and transforms $P^{(\mu)}$ into the origin, then $r_\alpha^{(0)}(z_1)$ is transformed into $r_\alpha^{(0)}(z_2)$, which assumes π on the image $\bar{\alpha}$ of α by (5) and zero in the complement of $\bar{\alpha}$ with respect to $|z_2|=1$. Denote by $\bar{l}$ and l the lengths of $\bar{\alpha}$ and α respectively. Then we obtain

$$r_\alpha^{(0)}(0) = \frac{1}{2}\bar{l}, \ (\bar{l} = \int_\alpha \frac{1}{|cz_1+d|^2}|dz_1|).$$

Whence follows

$$\frac{l}{2}\min_{z_1\in\alpha}\frac{1}{|cz_1+d|^2}\leqq r_{\alpha}^{(0)}(0)\leqq\frac{l}{2}\max_{z_1\in\alpha}\frac{1}{|cz_1+d|^2}.$$

Since $r_{\alpha}^{(0)}(0)=r_{\alpha}^{(0)}(P^{(\mu)})=r_{\alpha}^{(\mu)}(P^{(0)})$, we obtain

$$\frac{l}{2|c|^2}\min_{z_1\in\alpha}\frac{1}{\left|z_1+\dfrac{d}{c}\right|^2}\leqq r_{\alpha}^{(\mu)}(P^{(0)})\leqq\frac{l}{2|c|^2}\max_{z_1\in\alpha}\frac{1}{\left|z_1+\dfrac{d}{c}\right|^2}. \tag{6}$$

If the symmetric point $P_1^{(\mu)}=-\dfrac{d}{c}$ of $P^{(\mu)}$ is sufficiently near $P_\infty^{(\mu)}=S'_\mu(P_\infty^{(0)})$ then the distance of $P_1^{(\mu)}$ from a point z_1 on α can be estimated as follows:

$$0<d_{(\alpha)}<\left|z_1+\frac{d}{c}\right|<3,$$

where $d_{(\alpha)}$ is independent of μ. Hence, from (6), we obtain

$$\frac{l}{2|c|^2 3^2}\leqq r_{\alpha}^{(\mu)}(P^{(0)})\leqq\frac{l}{2|c|^2 d_\alpha^2}.$$

For another α', we can also get a similar inequality for $r_{\alpha'}^{(\mu)}(P^{(0)})$. Consequently,

$$c(\alpha,\alpha')\leqq\frac{r_{\alpha'}^{(\mu)}(P^{(0)})}{r_{\alpha}^{(\mu)}(P^{(0)})}\leqq c'(\alpha,\alpha'),$$

where $c(\alpha,\alpha')$ and $c'(\alpha,\alpha')$ are constants independent of μ. Therefore

$$c(\alpha,\alpha')\sum_{\mu=-\infty}^{\infty}r_{\alpha}^{(\mu)}(P^{(0)})\leqq\sum_{\mu=-\infty}^{\infty}r_{\alpha'}^{(\mu)}(P^{(0)})\leqq c'(\alpha,\alpha')\sum_{\mu=-\infty}^{\infty}r_{\alpha}^{(\mu)}(P^{(0)});$$

i.e.

$$c(\alpha,\alpha')\ r_\alpha(P^{(0)})\leqq r_{\alpha'}(P^{(0)})\leqq c'(\alpha,\alpha')\ r_\alpha(P^{(0)}).$$

Hence we see that $r_\alpha(z_1)$ and $r_{\alpha'}(z_1)$ have simultaneously the radial limit zero along $R^{(0)}$. Thus our lemma is proved.

If we map $|z_1|\leqq 1$ conformally onto the upper half plane $\mathrm{Im}(z)\geqq 0$, then $r_\alpha(z_1)$ is transformed into a bounded harmonic function $r_{\tilde{\alpha}}(z)$ which takes π on $\tilde{\alpha}$ and 0 on its complement with respect to the real axis, where $\tilde{\alpha}$ is the image of α. According as the origin $z=0$ is contained in $\tilde{\alpha}$ or not, the imaginary part $v_{ab}(z)$ of $\chi_{ab}(z)=u_{ab}(z)+iv_{ab}(z)$ in B_0 is equal to $\pi-r_{\tilde{\alpha}}(z)$ or $r_{\tilde{\alpha}}(z)$. The radius $R^{(0)}$ in the proof of Lemma is transformed into a part of the imaginary axis of the z-plane. Hence, if $\lim\limits_{z_1\to P_\infty^{(0)}} r_\alpha(z_1)=0$ along $R^{(0)}$, then we obtain $\lim\limits_{z\to\infty}$

$v_{ab}(z) = \pi$ or 0 along the imaginary axis. Since, by Lemma, the existence of the radial limit zero is independent of the parameters a and b, we get

THEOREM 1. *If $\chi_{ab}(z)$ is parabolic with respect to some (a, b) $(-\infty < a < b < \infty)$, then $\chi_{ab}(z)$ is also parabolic with respect to any pair (a, b).*

In the case where a and b are congruent with respect to some generator $S_\nu(z) \in \Gamma$, we can prove the following by the same method as above

THEOREM 2. *Whether the type of $\varphi_\nu(z)$ is parabolic or not is independent of ν; more precisely $\varphi_\nu(z)$, $(\nu = 1, 2, \ldots)$ are all parabolic or all not parabolic.*

As an immediate consequence, we have

THEOREM 3. *In order that the types of $\varphi_\nu(z)$ and $\chi_{ab}(z)$ be parabolic, it is necessary and sufficient that, for some α, $\lim_{P^{(0)} \to P^{(0)}_\infty} r_\alpha(P^{(0)}) = 0$ along the radius $R^{(0)}$*

This contains Theorem 3 of L. Myrberg [2].

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