RELATION BETWEEN HIGHER OBSTRUCTIONS AND POSTNIKOV INVARIANTS

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Introduction

The problem of extending a continuous map is one of the most important problems in algebraic topology. Many topologists have contributed for the solution of this problem. One of the most powerful methods in the extension problem is the obstruction theory defined first by S. Eilenberg [1] and developed by many others. N. E. Steenrod worked on the primary obstruction and showed that there is a strong connection between obstruction theory and cohomology operations [7].

The main objective of this paper is to establish a certain relation between higher obstructions and the higher order cohomology operations induced by the Postnikov invariants in the sense of F. P. Peterson [4]. But this is done only for the stable range.

1. Obstruction set

By a pair (K, L) we shall mean a connected CW-complex K together with a subcomplex $L \subset K$. Let K^q be a q-skeleton of K and $\bar{K}^q = K^q \cup L$. Let $f : \bar{K}^q \to X$ $(q \geqq 1)$ be a map from $\bar{K}^q$ to a topological space X, which is n-simple for all $n \geqq 1$. Then we define the obstruction cocycle $c^{q+1}(f) \in Z^{q+1}(K, L; \pi_q(X))$ in a usual fashion, where $Z^{q+1}(K, L; \pi_q(X))$ is the $(q+1)$-dimensional group of cocycles of K modulo L with coefficients in $\pi_q(X)$ and $\pi_q(X)$ is the q-dimensional homotopy group of the space X.

Let f_0 and f_1 be maps of $\bar{K}^q$ to X which agree on $\bar{K}^{q-1}$, then we can define a difference cochain $d^q(f_0, f_1) \in C^q(K, L; \pi_q(X))$ in a usual fashion, where $C^q(K, L; \pi_q(X))$ is the q-dimensional cochain group of K modulo L with coefficients in $\pi_q(X)$. Then the following properties are well known. (Cf. Eilenberg

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[1], Olum [3], Steenrod [7])

(1.1) If $f_0 \sim f_1$ (i.e. f_0 and f_1 are homotopic), then $c^{q+1}(f_0) = c^{q+1}(f_1)$.

(1.2) $c^{q+1}(f) = 0$ if and only if f extends over $\bar{K}^{q+1}$.

(1.3) $\delta d^q(f_0, f_1) = c^{q+1}(f_0) - c^{q+1}(f_1)$, where δ is the coboundary operation.

(1.4) If f_0 and $d \in C^q(K, L; \pi^q(X))$ are given, there exists a map f_1 such that $d^q(f_0, f_1) = d$.

(1.5) $d^q(f_0, f_1) + d^q(f_1, f_2) = d^q(f_0, f_2)$.

(1.6) $d^q(f_0, f_1) = 0$ if and only if $f_0 \sim f_1$ relative to $\bar{K}^{q-1}$.

(1.7) If $g : (K, L) \to (K', L')$ is cellular and $f : \bar{K}'^q \to X$, then $g^\# c^{q+1}(f) = c^{q++1}(f \circ g)$,

where $g^\#$ is an induced cochain map of g.

Let $f : \bar{K}^q \to X$, we define $O_q^r(f)$ for $r > q+1$ as a set of all obstruction cocycles $c^r(f')$, where f' is an extension of f over $\bar{K}^{r-1}$. Then $O_q^{q+2}(f)$ is either an empty set or a single cohomology class by (1.3) and (1.4). Therefore, we denote by $\bar{c}^{q+2}(f)$ the cohomology class $O_q^{q+2}(f)$ if f is extendable over $\bar{K}^{q+1}$. Next we define $Z_q^r(f)$ for $r > q+1$ as a set of all $\bar{c}^r(f')$ of an extension $f' : \bar{K}^{r-2} \to X$ of f which is extendable over K^{r-1}. Then we have the following equality.

(1.8) $O_q^r(f) =$ the set of all cocycles whose cohomology classes belong to $Z_q^r(f)$.

Therefore, by (1.2) we have

(1.9) $Z_q^r(f) \ni$ 0-cohomology class if and only if f is extendable over $\bar{K}^r$.

2. Postnikov system

Let X' be a simply connected CW-complex, then there exists a sequence of fiber spaces $p_{n+1} : X_{n+1} \to X_n$ $(n \geqq 2)$ such that

(2.1) X' and X are of the same homotopy type, where X is the inverse limit of X_n,

(2.2) the fiber of $p_{n+1} : X_{n+1} \to X_n$ is a $(\pi_{n+1}(X'), n+1)$-type complex, (i.e. homotopy groups of the fiber vanish except for the $(n+1)$-st homotopy group which is isomorphic to $\pi_{n+1}(X')$)

(2.3) $\pi_q(\pi_n) = 0$ for $q > n$,

$p_{n+1*} : \pi_q(X_{n+1}) \approx \pi_q(X_n)$ for $q \leqq n$, and

(2.4) p_{n+1} is an homeomorphism on the n-skeleton of X_{n+1}, and its inverse over the n-skeleton of X_n is extendable over the $(n+1)$-skeleton of X_n as a cross-section.

The sequence of fiber spaces $\{(X_{n+1}, X_n, p_{n+1})\}$ is by definition the Postnikov system in the sense of Moore. (Cf. Moore [2].)

By (2.2) and the classification theorem for fiber spaces with fiber a (π, n)-type space, we obtain a map $k^{n+2} : X_n \to K(\pi_{n+1}(X'), n+2)$ such that the fiber space (X_{n+1}, X_n, p_{n+1}) is equivalent to the induced fiber space of the standard fiber space over $K(\pi_{n+1}(X'), n+2)$ by k^{n+2}, where the standard fiber space over the Eilenberg-MacLane complex $K(\pi_{n+1}(X'), n+2)$ is the fiber space whose total space is contractible. To define the Postnikov invariants, we need a following well-known lemma.

LEMMA 2.5. $\pi(K, K(\pi', n)) \approx H^n(K, \pi')$,

where $\pi(K, K(\pi', n))$ *is a group whose elements are homotopy classes of mappings* $K \to K(\pi', n)$ *with multiplication induced by the group structure of* $K(\pi', n)$, *and* $H^n(K, \pi')$ *is the n-th cohomology group of* K *with coefficients in* π'.

By the above lemma we have a unique cohomology class $k^{n+2}(X)$ in $H^{n+2}(X_n, \pi_{n+1}(X'))$ corresponding to $k^{n+2} : X_n \to K(\pi_{n+1}(X'), n+2)$. $\{k^{n+2}(X)\}$ is by definition the Postnikov invariants of X, and it is known that $k^{n+2}(X)$ is equal to the primary obstruction of the fiber space (X_{n+1}, X_n, p_{n+1}). (Cf. Moore [2].)

Let $f : K^q \to X'$ be any map and $h : X' \to X$ be an homotopy equivalence given in (2.1), then it is easy to see that the obstruction for extending f is equivalent to that for $h \circ f$. Therefore, we can assume that $X' = X$ for the rest of the paper.

Let $f : K^q \to X$ be extendable over K^{q+1}. Then $p'_q \circ f : K^q \to X_q$ is extendable over K since X_q has vanishing homotopy groups $\pi_i(X)$ for $i > q$, where $p'_q : X \to X_q$ is a natural projection. Let $F : K \to X_q$ be any extension of $p'_q \circ f$. Then

LEMMA 2.6. $\bar{c}^{q+2}(f) = F^* k^{q+2}(X)$,

where $F^* : H^{q+2}(X_q, \pi_{q+1}(X)) \to H^{q+2}(K, \pi_{q+1}(X))$ *is the induced homomorphism of* F.

Proof. This is an easy consequence of the fact that $k^{q+2}(X)$ is also the primary obstruction for the fiber space $p'_q : X \to X_q$, and it is equal to the obstruction for extending the cross-section as a continuous map for this fiber space.

Suppose $f : K^q \to X$ be extendable over K^{r+1}, $r \geqq q$, then $p'_r \circ f : K^q \to X_r$ is extendable over K by the same argument above. By Lemma 2.6, we immediately have

LEMMA 2.7. *$Z_q^{r+2}(f)$ is the set of all $F^* k^{r+2}(X)$, where $F : K \to X_r$ is an extension of $p'_r \circ f : K^q \to X_r$.*

3. Higher order cohomology operations

We shall give a brief resume of higher order cohomology operations in the sense of Peterson (for details and proofs see Peterson [4]).

Let $\{(X_{q+1}, X_q, p_{q+1})\}$ be a Postnikov system, and suppose $X_n = K(\pi_n, n)$. Consider the following diagram.

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
 & & \downarrow p_{q+1} & & \\
K(\pi_q, q) & \xrightarrow{i_q} & X_q & \xrightarrow{k^{q+2}} & K(\pi_{q+1}, q+2) \\
 & & \downarrow p_q & & \\
K(\pi_{q-1}, q-1) & \xrightarrow{i_{q-1}} & X_{q-1} & \xrightarrow{k^{q+1}} & K(\pi_q, q+1) \\
\vdots & & \downarrow p_{q-1} & & \vdots \\
 & & \vdots & & \\
 & & \downarrow p_{n+2} & & \\
K(\pi_{n+1}, n+1) & \xrightarrow{i_{n+1}} & X_{n+1} & \xrightarrow{k^{n+3}} & K(\pi_{n+2}, n+3) \\
 & & \downarrow p_{n+1} & & \\
 & & X_n & \xrightarrow{k^{n+2}} & K(\pi_{n+1}, n+2)
\end{array}
$$

Diagram 1.

where $p_q : X_q \to X_{q-1}$ is the fiber map of the Postnikov system, $i_q : K(\pi_q, q) \to X_q$ is the inclusion map of the fiber into the total space of the fibering $X_q \to X_{q-1}$, $k^{q+2} : X_q \to K(\pi_{q+1}, q+2)$ is the map corresponding to the postnikov invariant of X, and $\pi_q = \pi_q(X)$.

Assume that each $k^{q+2}(X)$ is a suspension for the rest of this section. Applying the functor $X \to \pi(K, X)$ to the diagram 1, we have the diagram 2, by using Lemma 2.5.

$$
\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
 & & \big\downarrow p_{q+1} & & \\
H^q(K, \pi_q) & \xrightarrow{i_q} & \pi(K, X_q) & \xrightarrow{k^{q+2}} & H^{q+2}(K, \pi_{q+1}) \\
 & & \big\downarrow p_q & & \\
H^{q-1}(K, \pi_{q-1}) & \xrightarrow{i_{q-1}} & \pi(K, X_{q-1}) & \xrightarrow{k^{q+1}} & H^{q+1}(K, \pi_q) \\
\vdots & & \big\downarrow p_{q-1} & & \vdots \\
 & & \vdots & & \\
 & & \big\downarrow p_{n+2} & & \\
H^{n+1}(K, \pi_{n+1}) & \xrightarrow{i_{n+1}} & \pi(K, X_{n+1}) & \xrightarrow{k^{n+3}} & H^{n+3}(K, \pi_{n+2}) \\
 & & \big\downarrow p_{n+1} & & \\
 & & H^n(K, \pi_n) & \xrightarrow{k^{n+2}} & H^{n+2}(K, \pi_{n+1})
\end{array}
$$

Diagram 2.

In the above diagram we have the following properties.

(3.1) Each object in the diagram 2 is a group and each map is a homomorphism.

(3.2) The sequence

$$H^q(K, \pi_q) \xrightarrow{i_q} \pi(K, X_q) \xrightarrow{p_q} \pi(K, X_{q-1}) \xrightarrow{k^{q+1}} H^{q+1}(K, \pi_q)$$

is exact.

(3.3) $H^n(K, \pi_n) \xrightarrow{k^{n+2}} H^{n+2}(K, \pi_{n+1})$ is the ordinary cohomology operation corresponding to $k^{n+2}(X) \in H^{n+2}(\pi_n, n, \pi_{n+1})$, where $H^{n+2}(\pi_n, n, \pi_{n+1})$ is the $(n+2)$-nd cohomology group of the Eilenberg-MacLane complex $K(\pi_n, n)$ with coefficients in π_{n+1}.

Let $x \in H^n(K, \pi_n)$ be an element such that $k^{n+2}(x) = 0$. By (3.2) we have an element $y \in \pi(K, X_{n+1})$ such that $p_{n+1}(y) = x$. Then $k^{n+3}(y)$ is unique modulo the image of $H^{n+1}(K, \pi_{n+1})$ under the mapping $k^{n+3} \circ i_{n+1}$. We denote by $[k^{n+3}]$ the operation $x \to k^{n+3}(y)$ modulo Image $(k^{n+3} \circ i_{n+1})$.

In the same fashion we define $[k^{q+2}]$ from the Kernel $([k^{q+1}])$ into $H^{q+2}(K, \pi_{q+1})$ modulo a certain subgroup $L^{q+2}(k^{q+2})$. Then we have the following lemma.

LEMMA 3.4. *If* $x \in$ Kernel $([k^{q+1}])$, *then* $[k^{q+2}]x$ *is the set of all* $F^* k^{q+2}(X)$, *where* $F: K \to X_q$ *is a map such that* $p_{n+1} \circ \cdots \circ p_q \circ F = \bar{x}$, *and* $\bar{x}: K \to K(\pi_n, n)$ *is a map corresponding to* $x \in H^n(K, \pi_n)$.

The proof of the lemma will follow immediately from the definition of $[k^{q+2}]$.

LEMMA 3.5. $x \in$ Kernel $([k^{q+1}])$ *if and only if there exists a map* $G : K \to X_q$ *such that* $p_{n+1} \circ \cdots \circ p_q \circ G = \bar{x}$ *where* $\bar{x} : K \to K(\pi_n, n)$ *is a map corresponding to* $x \in H^n(K, \pi_n)$.

Proof. By Lemma 3.4 we set $p_q \circ G = F$, then we immediately have $F^* k^{q+1}(X) = 0$. The converse is also straight forward.

4. Main theorems

Let X be $(n-1)$-connected $(n \geqq 2)$. Suppose that its Postnikov invariants $k^{q+2}(X)$ are suspensions for $q+2 \leqq r$. Then each $k^{q+2}(X)$ defines a higher order cohomology operation $[k^{q+2}(X)]$ for $q+2 \leqq r$. Let $b^n(X) \in H^n(X, \pi_n(X))$ be the element corresponding to the inverse of the Hurewicz isomorphism, i.e. the fundamental cohomology class of X.

THEOREM 4.1. *Let* X, $[k^{q+2}(X)]$ *be as above. Let* $f : K^n \to X$ *be extendable over* K^{q+1} *for* $n \leqq q \leqq r-2$. *Then* $Z_n^{q+2}(f)$ *is a coset of* $H^{q+2}(K, \pi_{q+1}(X))$ *by* $L^{q+2}(k^{q+2}(X))$, *and* $f^*(b^n(X))$ *is in the image of* $i^* : H^n(K) \to H^n(K^n)$, *where* i^* *is an isomorphism into induced by the injection* $i : K^n \to K$. *Furthermore,* $i^{*-1} \circ f^*(b^n(X))$ *is in the* Kernel $([k^{q+1}(X)])$, *and the following equality holds.*

$$Z_n^{q+2}(f) = [k^{q+2}(X)]\{i^{*-1} \circ f^*(b^n(X))\}.$$

Proof. Since f is extendable over K^{q+1} $(q \geqq n)$, $f^*(b^n(X))$ is in the image of i^*, and by Lemma 3.5 we have $i^{*-1} \circ f^*(b^n(X)) \in$ Kernel $([k^{q+1}(X)])$. Therefore, by Lemma 3.4 we have $[k^{q+2}(X)]\{i^{*-1} \circ f^*(b^n(X))\}$ is the set of all $F^* k^{q+2}(X)$, where $F : K \to X_q$ is a map such that $p_{n+1} \circ \cdots \circ p_q \circ F$ is an extension of $p'_n \circ f$. It is obvious that $Z_n^{q+2}(f)$ coincides with $[k^{n+2}(X)]\{i^{*-1} \circ f^*(b^n(X))\}$ by Lemma 2.7.

Remark. If X is $(n-1)$-connected $(n \geqq 2)$, then $k^{q+2}(X)$ are suspensions for $q+2 \leqq 2n-1$.

COROLLARY 4.2. Let X be $(n-1)$-connected $(n \geqq 3)$, and $f : K^n \to X$ be extendable over K^{n+1}. Then

$$\bar{c}^{n+2}(f) = Sq^2\{i^{*-1} \circ f^*(b^n(X))\}.$$

Proof. Let $\phi : \pi_n(X) \to \pi_{n+1}(X)$ be the pairing induced by composing the essential map of S^{n+1} to S^n, then it induces the squaring operation Sq^2 : $H^n(K, \pi_n(X)) \to H^{n+2}(K, \pi_{n+1}(X))$. It is well-known that $k^{n+2}(X) = Sq^2(b^n(X))$. (Cf. J. H. C. Whitehead [8].) Therefore, the corollary is proved.

For the next theorem we need the following diagarm.

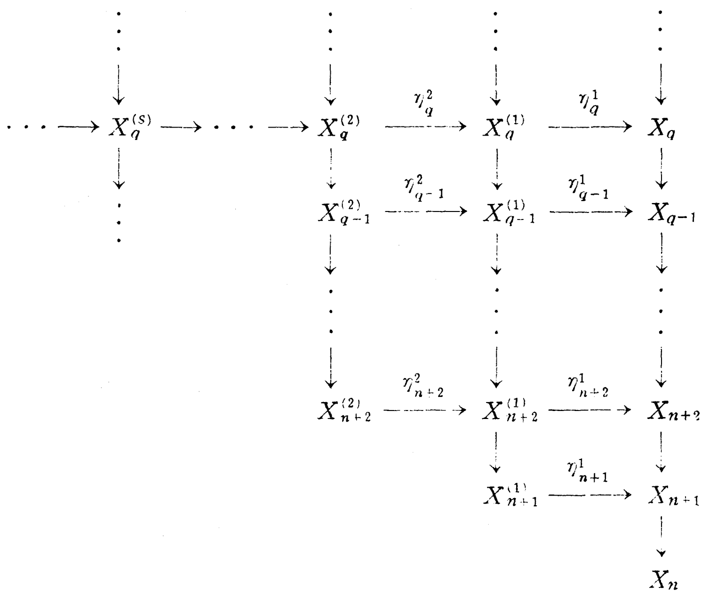


Diagram 3.

The above diagram is constructed as follows.

$\eta^1_{n+1} : X^{(1)}_{n+1} \to X_{n+1}$ is the injection of the fiber $X^{(1)}_{n+1}$ of the fiber space $p_{n+1} : X_{n+1} \to X_n$. $X^{(1)}_s \to X^{(1)}_{s-1}$ $(s \geqq n+2)$ is the induced fiber space of $X_s \to X_{s-1}$ by $\eta^1_{s-1} : X^{(1)}_{s-1} \to X_{s-1}$, and $\eta^1_s : X^{(1)}_s \to X_s$ is the induced fiber space map. Then the sequence of fiber spaces $\{X^{(1)}_s \to X^{(1)}_{s-1}\}$ forms the Postnikov system of the space $X^{(1)}$ which is obtained from X by killing the n-th homotopy group $\pi_n(X)$. We obtain inductively $X^{(t)}_s$ by the same construction, and the sequence of fiber spaces $\{X^{(t)}_s \to X^{(t)}_{s-1}\}$ is the Postnikov system of $X^{(t)}$ which is obtained from $X^{(t-1)}$ by killing the $(n+t-1)$-th homotopy group of $X^{(t)}$.

Let $\xi^t_q : X^{(t)}_q \to X_q$ be the composition of the maps η_q's. Then $\xi^{t*}_q k^{q+2}(X)$ is the Postnikov invariant of $X^{(t)}$, and it is a suspension as far as $k^{q+2}(X)$ is. By using the above notations, we obtain the following theorem.

THEOREM 4.3. *Let $f : K^{s-1} \to X$ be extendable over K^{q+1}, $(n \leqq s \leqq q \leqq 2n-3)$.*

Let $f_0, f_1 : K^s \to X$ *be two extensions of* f *which are extendable over* K^{q+1}. *Then* $Z_s^{q+2}(f_0)$ *and* $Z_s^{q+2}(f_1)$ *are cosets of* $H^{q+2}(K, \pi_{q+1}(X))$ *by the subgroup* $L^{q+1}(\xi_q^{s-n*} k^{q+2}(X))$, *the difference cocycle* $d^s(f_0, f_1) \in \mathrm{Kernel}\ [\xi_{q-1}^{s-n*} k^{q+1}(X)]$, *and the following relation holds.*

$$Z_s^{q+2}(f_0) - Z_s^{q+2}(f_1) = [\xi_q^{s-n*} k^{q+2}(X)] d^s(f_0, f_1).$$

5. Proof of the theorem 4.3

We divide the theorem into two parts.

$(\mathrm{I}_{q,s})$ Let $f : K^s \to X$ be extendable over K^{q+1} $(n \leqq s \leqq q \leqq 2n-3)$, then $Z_s^{q+2}(f)$ is a coset of $H^{q+2}(K, \pi_{q+1}(X))$ by the subgroup $L^{q+2}(\xi_q^{s-n*} k^{q+2}(X))$.

$(\mathrm{II}_{q,s})$ Let $f : K^{s-1} \to X$ be extendable over K^{q+1} $(n \leqq s \leqq q \leqq 2n-3)$. Let f_0, $f_1 : K \to X$ be two extensions of f which are extendable over K^{q+1}. Then $d^s(f_0, f_1) \in \mathrm{Kernel}\ ([\xi_{q-1}^{s-n*} k^{q+2}(X)])$ and

$$Z_s^{q+2}(f_0) - Z_s^{q+2}(f_1) = [\xi_q^{s-n*} k^{q+2}(X)] d^s(f_0, f_1).$$

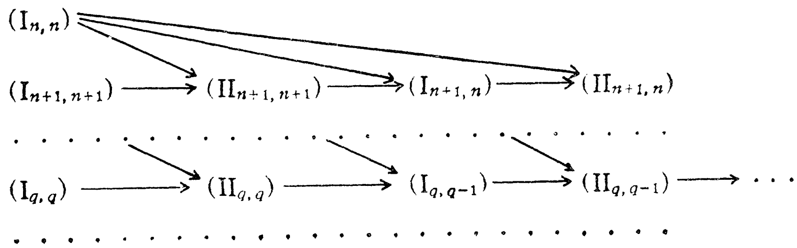
We shall prove $(\mathrm{I}_{q,s})$ and $(\mathrm{II}_{q,s})$ by double induction on (q, s) $(n \leqq s \leqq q \leqq 2n-3)$.

If $s = q$, $Z_q^{q+2}(f_0)$ and $Z_q^{q+2}(f_1)$ are single cohomology classes $\bar{c}^{q+2}(f_0)$ and $\bar{c}^{q+2}(f_1)$, and $\xi_q^{q-n*} k^{q+2}(X) \in H^{q+2}(X_q^{(q-n)}, \pi_{q+1}(X)) \approx H^{q+2}(\pi_q(X), q; \pi_{q+1}(X))$. Therefore, $[\xi_q^{q-n*} k^{q+2}(X)]$ is a usual cohomology operation, and $L^{q+2}(\xi_q^{q-n*} k^{q+2}(X))$ is the zero group. Therefore, $(I_{q,q})$ is proved for any q.

We shall prove $(\mathrm{I}_{q,s})$ and $(\mathrm{II}_{q,s})$ in two steps.

(A) Assuming $(\mathrm{I}_{q',s'})$, $(\mathrm{II}_{q',s'})$ for $q' < q$, $n \leqq s' \leqq q'$ and $(\mathrm{I}_{q,s'})$, $(\mathrm{II}_{q,s'+1})$ for $s' \geqq s$, we deduce $(\mathrm{II}_{q,s})$.

(B) Assuming $(\mathrm{I}_{q',s'})$, $(\mathrm{II}_{q',s'})$ for $q' < q$, $n \leqq s' \leqq q'$ and $(\mathrm{I}_{q,s'})$, $(\mathrm{II}_{q,s'})$ for $s' \geqq s$, we deduce $(\mathrm{I}_{q,s-1})$.



Step (B)

LEMMA 5.1. $L^{q+2}(\xi_q^{s-n-1*}k^{q+2}(X)) = \text{Image } [\xi_q^{s-n*}k^{q+2}(X)]$, *where we consider* Image $[\xi_q^{s-n*}k^{q+2}(X)]$ *as a union of cosets in* $H^{q+2}(K, \pi_{q+1}(X))$ *by* $L^{q+2}(\xi_q^{s-n*}k^{q+2}(X))$.

This follows immediately from the definition.

Step (B) follows from $(\mathrm{I}_{q,s})$, $(\mathrm{II}_{q,s})$, and Lemma 5.1.

Step (A)

By our assumption f_0, f_1 are extendable over K^{q+1}, therefore, $Z_s^{q'+2}(f_0) = Z_s^{q'+2}(f_1) = 0$ for $q' < q$. By $(\mathrm{II}_{q',s})$ $(q' < q)$ we have $d^s(f_0, f_1) \in$ Kernel $[\xi_{q-1}^{s-n*}k^{q+1}(X)]$. Next, we shall prove the equality.

We can assume that $f(K^{n-1})$ is a single point without loss of generality. Let K_0 be obtained from K by shrinking K^{n-1} to a single point v_0. Let $h : K \to K_0$ be the shrinking map.

Since $\pi_i(X_q) = 0$ for $i > q$, and f_0 is extendable over K^{q+1}, $p'_q \circ f_0 : K^n \to X_q$ is extendable over K. Let $F : K \to X_q$ be an extension of $p'_q \circ f_0$. Then there exists a map $g : K_0 \to X_q$ such that $g \circ h = F$.

Take a s-cell Δ_i^s in the interior of each s-cell σ_i^s of K except one common vertex. Let $K_0 \vee X_q^{(s-n)}$ be a union of K_0 and $X_q^{(s-n)}$ with a single point in common. We shall construct a map $k : H^{q+2} \to K_0 \vee X_q^{(s-n)}$ as follows.

Let $k_s : K^s \to K_0 \vee X_q^{(s-n)}$ be a map such that

(1) $$k_s | K^{s-1} = i_1 \circ h | K^{s-1},$$

where $i_1 : K_0 \to K_0 \vee X_q^{(s-n)}$ is the inclusion map,

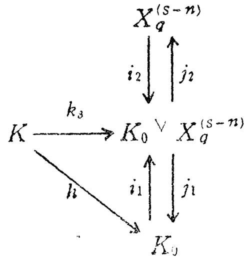
(2) k_s maps $\sigma_i^s - \Delta_i^s$ homeomorphically onto $\sigma_i^s - v_0$ in K_0,

(3) k_s maps the boundary of Δ_i^s to v_0 in K_0, and

(4) k_s maps $(\Delta_i^s$, boundary of $\Delta_i^s)$ into $(X_q^{(s-n)}, v_0)$ by a map corresponding to $d^s(f_1, f_0)(\sigma_i^s) \in \pi_s(X) \approx \pi_s(X_q^{(s-n)})$.

Then $c^{s+1}(k_s) = \delta d^s(f_1, f_0) = 0$. Therefore, k_s is extendable over K^{s+1}.

Consider the following diagram.



where j_1, j_2 are natural projections, and i_1, i_2 are inclusion maps. Then it is easily seen that

$$j_1 \circ k_s \sim h \quad \text{on } K^s, \text{ and}$$
$$d^s(k_s, i_1 \circ h) = d^s(f_1, f_0).$$

Suppose k_s be extendable ober K^{r+1}, $r < q$. Since

$$\pi_i(K_0 \vee X_q^{(s-n)}) \approx \pi_i(K_0) + \pi_i(X_q^{(s-n)}) \qquad \text{for} \quad i \leqq n+s-2,$$

$Z_s^{r+2}(k_s)$ is decomposed into two terms, i.e. one in coefficients in $\pi_{r+1}(K_0)$ and the other in coefficients in $\pi_{r+1}(X_q^{(s-n)})$ for $r < q \leqq 2n-3$. The first term is zero since $j_1 \circ k_s : K^s \to K_0$ is homotopic to h, and h is defined over K. By $(\mathrm{II}_{r,s})$ $(r<q)$ we have

$$\begin{aligned} Z_s^{r+2}(k_s) - Z_s^{r+2}(i_1 \circ h) &= [\hat{\xi}_r^{s-n*} k^{r+2}(X)] d^s(k_s, i_1 \circ h) \\ &= [\hat{\xi}_r^{s-n*} k^{r+2}(X)] d^s(f_0, f_1). \end{aligned}$$

But $Z_s^{r+2}(i_1 \circ h) = 0$, and $[\hat{\xi}_r^{s-n*} k^{r+2}(X)] d^s(f_0, f_1) = 0$ by our assumption. Therefore $Z_s^{r+2}(k_s) = 0$ for $r < q$, and k_s is extendable over K^{q+1}.

Since $q+1 \leqq n+s-2$, $Z_s^{q+2}(k_s) = Z_s^{q+2}(j_1 \circ k_s) + Z_s^{q+2}(j_2 \circ k_s)$. $Z_s^{q+2}(j_1 \circ k_s) = Z_s^{q+2}(h) = 0$ and $Z_s^{q+2}(j_2 \circ k) = 0$ because $\pi_{q+1}(X_q^{(s-n)}) = 0$. Therefore, $Z_s^{q+2}(k_s) = 0$, and k_s is extendable over K^{q+2}.

Let $k' : K^{q+2} \to K_0 \vee X_q^{(s-n)}$ be an extension of k_s. We shall modify k' to $k : K^{q+2} \to K_0 \vee X_q^{(s-n)}$ so that $j_1 \circ k \sim h$. Let $k'_{r+1} : K^{r+1} \to K_0 \vee X_q^{(s-n)}$, $r \leqq q$, be extension of k_s such that $j_1 \circ k'_{r+1} \sim h$ on K^{r+1}, and $j_2 \circ k'_{r+1} \sim j_2 \circ k'$. Then it is clear that k'_{r+1} is extendable over K^{r+2}. Let k''_{r+2} be an extension of k'_{r+1} over K^{r+2}. There exists a map $k'_{r+2} : K^{r+2} \to K_0 \vee X_q^{(s-n)}$ such that $d^{r+2}(k'_{r+2}, k''_{r+2}) = i_{1*} d^{r+2}(j_1 \circ k'', h) + i_{2*} d^{r+2}(j_2 \circ k''_{r+2}, j_2 \circ k')$, where i_{1*} (resp. i_{2*}) is induced by the coefficients homomorphism $i_{1*} : \pi_{r+2}(K_0) \to \pi_{r+2}(K_0 \vee X_q^{(s-n)})$ (resp. $i_{2*} : \pi_{r+2}(X_q^{(s-n)}) \to \pi_{r+2}(K_0 \vee X_q^{(s-n)})$). Then it is clear by the simple calculation of the difference cochain that $j_1 \circ k'_{r+2} \sim h$ and $j_2 \circ k'_{r+2} \sim j_2 \circ k'$ on K^{r+2}. Therefore, finally we have the desired map $k : K^{q+2} \to K_0 \vee X_q^{(s-n)}$ such that $j_1 \circ k \sim h$.

Then we have the following properties.

$$(5.1) \qquad j_1 \circ k \sim h$$

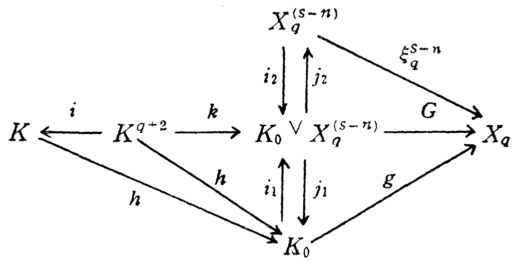
$$(5.2) \qquad (j_2 \circ k)^* b^s(X_q^{(s-n)}) = i^* d^s(f_1, f_0),$$

where $b^s(X_q^{(s-n)}) \in H^s(X_q^{(s-n)}, \pi_s(X))$ is the fundamental cohomology class and

$i^* : H^s(K) \to H^s(K^{q+2})$ is the induced homomorphism by the inclusion $i : K^{q+2} \to K$.

(5.2) follows from the fact that k is an extension of k_S.

Consider the following diagram.



where $G|K_0 = g$ and $G|X_q^{(s-n)} = \xi_q^{s-n}$.

Then we have the following properties.

(5.3) $$G \circ i_1 \circ h = F \circ i.$$

(5.4) $$G \circ i_2 = \xi_q^{s-n}.$$

(5.5) $$(i_1 \circ h)^* - k^* = -(i_2 \circ j_2 \circ k)^*$$

(5.5) follows from (5.1).

By the same argument used in the beginning of this section $p'_q \circ f_1 : K^s \to X_q$ is extendable over K. Let F' be an extension of $p'_q \circ f_1$. Then

$$\begin{aligned} d^s(G \circ k, F') &= d^s(G \circ k, F) + d^s(F, F') \\ &= G_* d^s(k, i_1 \circ h) + d^s(f_0, f_1) \\ &= d^s(f_1, f_0) + d^s(f_0, f_1) = 0. \end{aligned}$$

Therefore,

(5.6) $$G \circ k \sim F' \quad \text{on } K^s.$$

By Lemma 2.6,

$$\bar{c}^{q+2}(f_0) = F^* k^{q+2}(X), \text{ and}$$
$$\bar{c}^{q+2}(f_1) = F'^* k^{q+2}(X).$$

By (5.6) we have $i^*\bar{c}^{q+2}(f_1)$ and $(G \circ k)^* k^{q+2}(X)$ belong to the same obstruction set $Z_S^{q+2}(f_1|K^{q+2}) = i^* Z_S^{q+2}(f_1)$. Therefore, by (5.3), (5.5), and (5.4)

$$\begin{aligned} i^*(Z_S^{q+2}(f_0) - Z_S^{q+2}(f_1)) &\ni (F \circ i)^* k^{q+2}(X) - (G \circ k)^* k^{q+2}(X) \\ &= ((i_1 \circ h)^* - k^*) G^* k^{q+2}(X) = -(i_2 \circ j_2 \circ k)^* G^* k^{q+2}(X) \\ &= -(j_2 \circ k)^* i_2^* G^* k^{q+2}(X) = -(j_2 \circ k)^* \xi_q^{s-n*} k^{q+2}(X). \end{aligned}$$

On the other hand, $\xi_q^{s-n*}k^{q+2}(X) \in [\xi_q^{s-n*}k^{q+2}(X)]b^s(X_q^{(s-n)})$ by definition. Therefore, by (5.2)

$$\begin{aligned} i^*(Z_s^{q+2}(f_0) - Z_s^{q+2}(f_1)) &= -(j_2 \circ k)^*[\xi_q^{s-n*}k^{q+2}(X)]b^s(X_q^{(s-n)}) \\ &= -[\xi_q^{s-n*}k^{q+2}(X)](j_2 \circ k)^*b^s(X_q^{(s-n)}) \\ &= -[\xi_q^{s-n*}k^{q+2}(X)]i^*d^s(f_1, f_0) \\ &= i^*[\xi_q^{s-n*}k^{q+2}(X)]d^s(f_0, f_1). \end{aligned}$$

Since i^* is an isomorphism into, we have the desired equality, i.e.

$$Z_s^{q+2}(f_0) - Z_s^{q+2}(f_1) = [\xi_q^{s-n*}k^{q+2}(X)]d^s(f_0, f_1).$$

6. Supplementary result

THEOREM 6.1. *Let X be 2-connected CW-complex. Let $f: K^{q-1} \to X$ be a map ($q \geqq 2$) which is extendable over K^{p+1}. Let f_0 and f_1 be two extensions of f over K^q which are extendable over K^{q+1}. Then*

$$\bar{c}^{q+2}(f_0) - \bar{c}^{q+2}(f_1) = Sq^2 d^q(f_0, f_1),$$

where $Sq^2 : H^q(K, \pi_q(X)) \to H^{q+2}(K, \pi_{q+1}(X))$ is the squaring operation.

Proof. With a little modification of the proof of Theorem 4.3, we can prove

$$\bar{c}^{q+2}(f_0) - \bar{c}^{q+2}(f_1) = [\xi_q^{q-3*}k^{q+2}(X)]d^q(f_0, f_1).$$

$\xi_q^{q-3*}k^{q+2}(X)$ belongs to $H^{q+2}(X_q^{(q-3)}, \pi_{q+1}(X))$ and X is 2-connected. Therefore, $X_q^{(q-3)}$ has only one non vanishing homotopy group $\pi_q(X)$ in dimension q, and $\xi_q^{q-3*}k^{q+2}(X)$ is the Eilenberg-MacLane invariant of the space $X_{q+1}^{(q-3)}$, which has two non-vanishing homotopy groups in dimensions q and $q+1$. Therefore, it is well-known that

$$\xi_q^{q-3*}k^{q+2}(X) = Sq^2 b^q(X_{q+1}^{(q-3)}),$$

i.e. $[\xi_q^{q-3*}k^{q+2}(X)] = Sq^2$.

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