ON THE HECKE-LANDAU L-SERIES

To ZYOITI SUETUNA on his 60th Birthday

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§ 1. Introduction

Let k be an algebraic number field of degree $n = r_1 + 2r_2$ with r_1 real conjugates $k^{(l)}$ $(1 \leqq l \leqq r_1)$ and r_2 pairs of complex conjugates $k^{(m)}$, $k^{(m+r_2)}$ $(r_1 + 1 \leqq m \leqq r_1 + r_2)$. Let $\mathfrak{o}$ be the integral domain consisting of all integers in k. We introduce a generalized module $\tilde{\mathfrak{f}}$ composed of an ordinal integral ideal $\mathfrak{f}$ in k and an infinite part $\mathfrak{f}_\infty$ which is a product of some infinite prime spots $\mathfrak{p}_\infty^{(l)}$, say,

$$\tilde{\mathfrak{f}} = \mathfrak{f} \cdot \mathfrak{f}_\infty, \qquad \mathfrak{f}_\infty = \mathfrak{p}_\infty^{(1)} \mathfrak{p}_\infty^{(2)} \cdots \mathfrak{p}_\infty^{(q)} \qquad (0 \leqq q \leqq r_1). \tag{1}$$

For $\alpha \in k$, the (multiplicative) congruence

$$\alpha \equiv 1 \qquad (\text{mod } \tilde{\mathfrak{f}}) \tag{2}$$

means that $\alpha \equiv 1$ (mod $\mathfrak{f}$) and α is $\mathfrak{f}_\infty$-positive namely $\alpha^{(1)} > 0$, $\alpha^{(2)} > 0, \ldots,$ $\alpha^{(q)} > 0$. Let A be the multiplicative group constituted by ideals in k prime to $\mathfrak{f}$ and S be the group of principal ideals generated by α satisfying (2). From an abelian character of the group A/S, we can define a character χ mod $\tilde{\mathfrak{f}}$ in a similar way as in the rational case. Let $\tilde{\mathfrak{g}}$ be a divisor of $\tilde{\mathfrak{f}}$. We say that χ is also defined by $\tilde{\mathfrak{g}}$, whenever the assumption $\alpha \equiv 1$ (mod $\tilde{\mathfrak{g}}$), $(\alpha, \mathfrak{f}) = \mathfrak{o}$, entails the conclusion $\chi(\alpha) = 1$. There exists the minimal (with respect to the number of prime factors) generalized module which defines χ. This is called the conductor of χ. If the conductor of χ mod $\tilde{\mathfrak{f}}$ is $\tilde{\mathfrak{f}}$ itself, then χ is called a primitive character mod $\tilde{\mathfrak{f}}$.

From now on let χ be a primitive character mod $\tilde{\mathfrak{f}}$. Let $\mathfrak{d}$ be the ramification ideal (different) of k. Let $\mathfrak{K}$ be an absolute ideal class of k. We denote by $\hat{\mathfrak{K}}$ the ideal class $\mathfrak{K}^{-1}\mathfrak{K}^*$ where $\mathfrak{K}^*$ is an absolute ideal class containing $\mathfrak{d}\mathfrak{f}$. Let $s = \sigma + it$ be a complex variable. Let $L(s, \mathfrak{K}, \chi)$ and $L(s, \chi)$ be respectively the functions defined by

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$$\sum_{\mathfrak{a}\in\mathfrak{K},\,\mathfrak{a}\neq 0} \chi(\mathfrak{a})/N(\mathfrak{a})^s, \qquad \sum_{\mathfrak{a},\,\mathfrak{a}\neq 0} \chi(\mathfrak{a})/N(\mathfrak{a})^s$$

for $\sigma > 1$, the summation running over all non-zero integral ideals in $\mathfrak{K}$ and in k respectively. Similarly we define that

$$\zeta_k(s, \mathfrak{K}) = \sum_{\mathfrak{a}\in\mathfrak{K},\,\mathfrak{a}\neq 0} 1/N(\mathfrak{a})^s, \qquad \zeta_k(s) = \sum_{\mathfrak{a},\,\mathfrak{a}\neq 0} 1/N(\mathfrak{a})^s$$

for $\sigma > 1$. We put

$$A(\chi) = \pi^{-n} d N(\mathfrak{f}),$$

where $d = N(\mathfrak{d})$ is the discriminant of k. For convenience, we put

$$a_p = \begin{cases} 1 & 1 \leqq p \leqq q \\ 0 & q+1 \leqq p \leqq n, \end{cases}$$

where q has the same meaning as in (1). Further we define that

$$\Gamma(s, \chi) = \int_0^\infty \cdots \int \exp\Big(-\sum_{p=1}^n z_p\Big) \prod_{p=1}^n z_p^{(s+a_p)/2} \frac{dz_1 dz_2 \cdots dz_{r+1}}{z_1 z_2 \cdots z_{r+1}}$$

for $\sigma > 0$, where $r_1 + r_2 = r + 1$ and

$$z_p = z_{p+r_2} \qquad (r_1 + 1 \leqq p \leqq r_1 + r_2). \tag{3}$$

We shall know in § 3 that

$$\Gamma(s, \chi) = 2^{-r_2 s} \Gamma\Big(\frac{s+1}{2}\Big)^q \Gamma\Big(\frac{s}{2}\Big)^{r_1 - q} \Gamma(s)^{r_2}. \tag{4}$$

Now we put

$$\phi(s, \chi) = \frac{(2\pi)^{r_2}}{\sqrt{d}} A(\chi)^{s/2} \Gamma(s, \chi) L(s, \chi).$$

This function is regular for all s with one exception $s = 1$ (simple pole) in the case of the Dedekind zeta-function $\zeta_k(s)$ ($\mathfrak{f} = \mathfrak{o}$, χ principal), moreover it satisfies the functional equation

$$\phi(s, \chi) = I(\chi) \phi(1 - s, \bar{\chi}) \tag{5}$$

where $I(\chi)$ will be defined in § 2.

For an integral ideal $\mathfrak{a}$ we define that

$$\Gamma(s, \chi, \mathfrak{a}) = \int \cdots \int \exp\Big(-\sum_{p=1}^n z_p\Big) \prod_{p=1}^n z_p^{(s+a_p)/2} \frac{dz_1 dz_2 \cdots dz_{r+1}}{z_1 z_2 \cdots z_{r+1}}$$

$$z_p > 0, \ \prod_{p=1}^n z_p \geqq N(\mathfrak{a})^2 / A(\chi) \tag{6}$$

for $\sigma > 0$ with (3). As we shall prove later, (6) and the series

$$\psi(s, \chi) = \frac{(2\pi)^{r_2}}{\sqrt{d}} A(\chi)^{s/2} \sum_{\mathfrak{a}, \mathfrak{a} \neq 0} \chi(\mathfrak{a}) \Gamma(s, \chi, \mathfrak{a}) / N(\mathfrak{a})^s \tag{7}$$

(the summation runs over all non-zero integral ideals in k) are absolutely convergent for all s and represent integral functions. Further we obtain

$$\varphi(s, \chi) = -\frac{2^{r_1+r_2}\pi^{r_2}Rh}{w\sqrt{d}} \frac{E(\chi)}{s(1-s)} + \psi(s, \chi) + I(\chi)\psi(1-s, \bar{\chi}), \tag{8}$$

where R is the regulator of k, w is the number of roots of unity contained in k, h is the class number of k, and

$$E(\chi) = \begin{cases} 1 & \text{if } \mathfrak{f} = \mathfrak{o},\ \chi \text{ principal} \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$I(\chi) I(\bar{\chi}) = 1 \tag{9}$$

(which will be proved in §2), (5) can be derived from (8), so that (8) is finer than (5). In the case of the Riemann zeta-function, (8) implies

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = -\frac{1}{s(1-s)}$$
$$+ \pi^{-s/2}\sum_{n=1}^{\infty} n^{-s}\int_{\pi n^2}^{\infty} e^{-z} z^{(s/2)-1} dz + \pi^{-(1-s)/2}\sum_{n=1}^{\infty} n^{-1+s}\int_{\pi n^2}^{\infty} e^{-z} z^{((1-s)/2)-1} dz.$$

In this paper we shall prove (7) and (8).

§2. On the Gauss sum

For every $\xi \neq 0$ in k, $\eta = \eta(\xi)$ is defined such that

$$\eta \equiv 1 \pmod{\mathfrak{f}}, \qquad \eta \equiv \xi \pmod{\mathfrak{f}_\infty}.$$

Let $\mathfrak{a}$ be any ideal (fractional or integral) in k and $\xi \in \mathfrak{a}$. We define

$$\psi(\mathfrak{a}, \xi) = \begin{cases} \chi\left(\frac{\xi}{\mathfrak{a}}\eta(\xi)\right) & \xi \neq 0 \\ 0 & \xi = 0,\ \mathfrak{f} \neq \mathfrak{o} \\ \bar{\chi}(\mathfrak{a}) & \xi = 0,\ \mathfrak{f} = \mathfrak{o} \end{cases} \tag{10}$$

and put

$$\psi(\xi) = \psi(\mathfrak{o}, \xi). \tag{11}$$

When χ is replaced by $\bar{\chi}$ in (10) and (11), we write $\bar{\psi}$ instead of ψ. If η_l $(1 \leqq l \leqq q)$ is an integer in k such that

$$\eta_l \equiv 1 \quad (\mathrm{mod}\ \mathfrak{f})$$
$$\eta_l^{(l)} < 0, \quad \eta_l^{(m)} > 0 \quad (m \neq l,\ 1 \leqq m \leqq q),$$

then

$$\chi(\eta_l) = -1.$$

Were it $\chi(\eta_l) = 1$, χ would be defined by $\tilde{\mathfrak{f}}_l$ where $\tilde{\mathfrak{f}}_l = \mathfrak{f} \cdot \mathfrak{f}_{l\infty}$, $\mathfrak{f}_{l\infty} = \mathfrak{f}_\infty / \mathfrak{p}_\infty^{(l)}$. Indeed, if $\alpha \equiv 1$ (mod $\tilde{\mathfrak{f}}_l$) then α or $\alpha\eta_l$ is congruent to 1 mod $\tilde{\mathfrak{f}}$, whence it follows that $\chi(\alpha)$ or $\chi(\alpha\eta_l)$ is equal to 1 and this implies $\chi(\alpha) = 1$. If we write for $\xi \in k$

$$P(\xi) = \begin{cases} \xi^{(1)}\xi^{(2)} \cdots \xi^{(q)} & q > 0 \\ 1 & q = 0, \end{cases}$$

then we can prove that

$$\chi(\eta(\xi)) = \operatorname{sgn} P(\xi) \tag{12}$$

by the aid of auxiliary integers η_l $(1 \leqq l \leqq q)$ (see [2], p. 75).

We take λ, μ such that

$$\lambda \quad \mathfrak{f}_\infty\text{-positive}, \quad \lambda = \mathfrak{d}\mathfrak{f} \cdot \mathfrak{g}, \quad (\mathfrak{g}, \mathfrak{f}) = \mathfrak{o},$$
$$\mu \quad \mathfrak{f}_\infty\text{-positive}, \quad \mu = \mathfrak{g} \cdot \mathfrak{y}, \quad (\mathfrak{y}, \mathfrak{f}) = \mathfrak{o},$$

where $\mathfrak{g}$ and $\mathfrak{y}$ are integral ideals in k, and set

$$F(\chi) = \chi(\mathfrak{y}) \sum_\beta \chi(\beta) \exp\left\{2\pi i S\left(\frac{\beta\mu}{\lambda}\right)\right\}, \tag{13}$$

where β runs over a complete system of residues mod $\mathfrak{f}$ which are all $\mathfrak{f}_\infty$-positive. By the definition of $\mathfrak{d}$ it is obvious that $\sum_\beta$ is independent of the choice of a system. If $\nu \in (\mathfrak{d}\mathfrak{f})^{-1}$, then (see [2], p. 76) we get, from (13),

$$\sum_\beta \chi(\beta) \exp\{2\pi i S(\beta\nu)\} = \begin{cases} \bar{\chi}(\eta(\nu)\nu\mathfrak{d}\mathfrak{f}) F(\chi) & \nu \neq 0 \\ \bar{\chi}(\mathfrak{d}\mathfrak{f}) F(\chi) & \nu = 0. \end{cases} \tag{14}$$

We denote by $F(\nu, \chi)$ the left-hand side of (14). There exists a number ν_0 in k such that

$$\nu_0 \quad \mathfrak{f}_\infty\text{-positive}, \quad \nu_0 = (\mathfrak{d}\mathfrak{f})^{-1}\mathfrak{n}_0, \quad (\mathfrak{n}_0, \mathfrak{f}) = \mathfrak{o}, \tag{15}$$

where $\mathfrak{n}_0$ is an integral ideal in k. Since

$$\bar{\chi}(\eta(\nu_0)\nu_0\mathfrak{d}\mathfrak{f}) \neq 0,$$

$F(\chi)$ is independent of choices of λ and μ.

Let ρ_j $(1\leqq j\leqq N(\mathfrak{f}))$ be a complete system of residues mod $\mathfrak{f}$ which are all $\mathfrak{f}_\infty$-positive. We put

$$\nu_j=\nu_0\rho_j, \qquad \mathfrak{n}_j=\nu_j\mathfrak{d}\mathfrak{f}.$$

Since the number of $\mathfrak{n}_j$ satisfying $(\mathfrak{n}_j,\mathfrak{f})=\mathfrak{o}$ is $\varphi(\mathfrak{f})$, we get

$$\sum_{j=1}^{N(\mathfrak{f})}|F(\nu_j,\chi)|^2=\varphi(\mathfrak{f})\,|F(\chi)|^2 \tag{16}$$

by (14). On the other hand,

$$\sum_{j=1}^{N(\mathfrak{f})}|F(\nu_j,\chi)|^2=\sum_{\beta_1}\sum_{\beta_2}\chi(\beta_1)\bar{\chi}(\beta_2)\sum_{j=1}^{N(\mathfrak{f})}\exp\{2\pi iS((\beta_1-\beta_2)\nu_j)\}. \tag{17}$$

Now we prove that if $\alpha\in(\mathfrak{d}\mathfrak{f})^{-1}$ then

$$\sum_{j=1}^{N(\mathfrak{f})}\exp\{2\pi iS(\alpha\rho_i)\}=\begin{cases}N(\mathfrak{f}) & \mathfrak{f}\mid\alpha\mathfrak{d}\mathfrak{f}\\ 0 & \mathfrak{f}\nmid\alpha\mathfrak{d}\mathfrak{f}.\end{cases} \tag{18}$$

The first part is obvious. To prove the second part, we denote by T the left-hand side of (18) and put $\alpha\mathfrak{d}\mathfrak{f}=\mathfrak{g}$. If $\mathfrak{f}\nmid\mathfrak{g}$, then α does not belong to $\mathfrak{d}^{-1}$. By the definition of $\mathfrak{d}^{-1}$ there is an integer γ such that $\exp\{2\pi iS(\gamma\alpha)\}\neq 1$. Since

$$\exp\{2\pi iS(\alpha\gamma)\}T=\sum_{j=1}^{N(\mathfrak{f})}\exp\{2\pi iS(\alpha(\gamma+\rho_j))\}=T,$$

we obtain $T=0$ provided that $\mathfrak{f}\nmid\mathfrak{g}$. It follows from (18) that

$$\sum_{j=1}^{N(\mathfrak{f})}\exp\{2\pi iS((\beta_1-\beta_2)\nu_0\rho_j)\}=\begin{cases}N(\mathfrak{f}) & \beta_1\equiv\beta_2 \pmod{\mathfrak{f}}\\ 0 & \beta_1\not\equiv\beta_2 \pmod{\mathfrak{f}},\end{cases}$$

whence follows

$$\sum_{j=1}^{N(\mathfrak{f})}|F(\nu_j,\chi)|^2=\sum_{\beta}\chi(\beta)\bar{\chi}(\beta)N(\mathfrak{f})=\varphi(\mathfrak{f})N(\mathfrak{f})$$

by (17). This combined with (16), we obtain

$$|F(\chi)|=\sqrt{N(\mathfrak{f})} \tag{19}$$

(see [3], p. 213). Now we define

$$I(\chi)=(-i)^q F(\chi)/\sqrt{N(\mathfrak{f})}. \tag{20}$$

Since $\chi(\eta(\nu_0))=1$ by (12) and (15),

$$\sum_{\beta}\chi(\beta)\exp\{2\pi iS(\beta\nu_0)\}=\bar{\chi}(\eta(\nu_0)\mathfrak{n}_0)F(\chi)=\bar{\chi}(\mathfrak{n}_0)F(\chi). \tag{21}$$

Similarly, since $\chi(\eta(-\nu_0))=(-1)^q$,

$$\sum_{\beta}\bar{\chi}(\beta)\exp\{2\pi iS(-\beta\nu_0)\}=\chi(\eta(-\nu_0)\mathfrak{n}_0)F(\bar{\chi})=(-1)^q\chi(\mathfrak{n}_0)F(\bar{\chi}). \tag{22}$$

Because of $\chi(\mathfrak{n}_0)\neq 0$ it follows from (21) and (22) that $F(\chi)$ and $(-1)^qF(\bar{\chi})$ are conjugate, so that

$$\overline{I(\chi)}=I(\bar{\chi}).$$

Since $|I(\chi)|=1$ by (19) and (20), this implies (9).

For any ideal $\mathfrak{a}$ in k (fractional or integral), we put

$$c(\mathfrak{a})=\{dN(\mathfrak{a})^2N(\mathfrak{f})\}^{-1/n}. \tag{23}$$

Let t_p $(1\leqq p\leqq n)$ be real variables satisfying $t_p=t_{p+r_2}$ $(r_1+1\leqq p\leqq r_1+r_2)$. If we define

$$\Theta(t\,;\,\mathfrak{a},\chi)=\sum_{\xi\in\mathfrak{a}}\psi(\mathfrak{a},\xi)P(\xi)\exp\{-\pi c(\mathfrak{a})\sum_{p=1}^{n}t_p|\xi^{(p)}|^2\},$$

then we have the following generalized Hecke's Θ-formula

$$\Theta(t\,;\,\mathfrak{a},\chi)=I(\chi)c(\mathfrak{a})^{-q}\prod_{p=1}^{n}t_p^{-1/2-a_p}\Theta\Big(\frac{1}{t}\,;\,\frac{1}{\mathfrak{a}\mathfrak{f}\mathfrak{d}},\bar{\chi}\Big), \tag{24}$$

which is due to Suetuna (see [5], p. 78). Landau's formula is somewhat complicated, because he does not use fractional ideals.

§ 3. Integral representation

Let c be a positive and $\xi\neq 0$ be in k. Since

$$\Gamma\Big(\frac{s+1}{2}\Big)(\pi c)^{-(s+1)/2}|\xi^{(p)}|^{-s-1}=\int_0^\infty\exp(-\pi c|\xi^{(p)}|^2t_p)\,t_p^{((s+1)/2)-1}dt_p \qquad (1\leqq p\leqq q)$$

$$\Gamma\Big(\frac{s}{2}\Big)(\pi c)^{-s/2}|\xi^{(p)}|^{-s}=\int_0^\infty\exp(-\pi c|\xi^{(p)}|^2t_p)\,t_p^{(s/2)-1}dt_p \qquad (q+1\leqq p\leqq r_1)$$

$$\Gamma(s)(2\pi c)^{-s}|\xi^{(p)}\xi^{(p+r_2)}|^{-s}=\int_0^\infty\exp(-2\pi c|\xi^{(p)}|^2t_p)\,t_p^{s-1}dt_p \qquad (r_1+1\leqq p\leqq r_1+r_2)$$

for $\sigma>0$, we have

$$\begin{aligned}&(\pi c)^{-(ns+q)/2}2^{-r_2s}\Gamma\Big(\frac{s+1}{2}\Big)^q\Gamma\Big(\frac{s}{2}\Big)^{r_1-q}\Gamma(s)^{r_2}\frac{\chi(\eta(\xi))}{|N(\xi)|^s}\\&=P(\xi)\int_0^\infty\!\!\cdots\!\int\exp(-\pi c\sum_{p=1}^{n}|\xi^{(p)}|^2t_p)\prod_{p=1}^{n}t_p^{(s+a_p)/2}\frac{dt_1dt_2\cdots dt_{r+1}}{t_1t_2\cdots t_{r+1}}.\end{aligned} \tag{25}$$

If we put, in (25), $c=\pi^{-1}$, $\xi=1$ and $t_p=z_p$, then we obtain (4), so that the existence of the integral (6) is also established. Similarly, we have

$$(\pi c)^{-(ns+q)/2}2^{-r_2 s}\Gamma\left(\frac{s+1}{2}\right)^q\Gamma\left(\frac{s}{2}\right)^{r_1-q}\Gamma(s)^{r_2}\frac{1}{|N(\xi)|^s}$$
$$=|P(\xi)|\int_0^\infty\cdots\int\exp\left(-\pi c\sum_{p=1}^n|\xi^{(p)}|^2 t_p\right)\prod_{p=1}^n t_p^{(s+a_p)/2}\frac{dt_1\,dt_2\cdots dt_{r+1}}{t_1 t_2\cdots t_{r+1}} \qquad (26)$$

for $\sigma>0$.

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ $(r=r_1+r_2-1)$ be a system of fundamental units. For brevity, we use $Q=n2^{r_1-1}R$ which is the absolute value of the following determinant

$$\begin{vmatrix} 1, & 2\log|\varepsilon_1^{(1)}|, & \ldots, & 2\log|\varepsilon_r^{(1)}| \\ 1, & 2\log|\varepsilon_1^{(2)}|, & \ldots, & 2\log|\varepsilon_r^{(2)}| \\ \cdots & \cdots & \cdots & \cdots \\ 1, & 2\log|\varepsilon_1^{(r+1)}|, & \ldots, & 2\log|\varepsilon_r^{(r+1)}| \end{vmatrix}.$$

After changing the variables in the right-hand side of (25) by

$$t_p=u|\varepsilon_1^{(p)}|^{2x_1}\cdots|\varepsilon_r^{(p)}|^{2x_r} \qquad (1\leqq p\leqq r+1), \qquad (27)$$

we put $c=c(\mathfrak{a})$ (see (23)) and multiply both sides of (25) by $\psi(\mathfrak{a},\xi)$ and construct the summation $\sum_{(\xi)\equiv\mathfrak{a},\,\xi\neq0}$, then we obtain for $\sigma>1$

$$\{\pi c(\mathfrak{a})\}^{-q/2}A(\chi)^{s/2}\Gamma(s,\chi)L(s,\mathfrak{K},\chi)$$
$$=Q\int_0^\infty u^{((ns+q)/2)-1}du\int_{-\infty}^\infty\cdots\int\sum_{(\xi)\equiv\mathfrak{a},\,\xi\neq0}\psi(\mathfrak{a},\xi)P(\xi)$$
$$\times\exp\left\{-\pi c(\mathfrak{a})u\sum_{p=1}^n|\xi^{(p)}\varepsilon_1^{(p)x_1}\cdots\varepsilon_r^{(p)x_r}|^2\right\}|P(\varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_r^{x_r})|$$
$$\times\,dx_1\,dx_2\cdots dx_r. \qquad (28)$$

provided that $\mathfrak{a}\in\mathfrak{K}^{-1}$, since

$$\left|\frac{\partial(t_1, t_2, \ldots, t_{r+1})}{\partial(u, x_1, \cdots, x_r)}\right|=\frac{t_1 t_2\cdots t_{r+1}}{u}Q.$$

Similarly, from (26) we obtain

$$\{\pi c(\mathfrak{a})\}^{-q/2}A(\chi)^{s/2}\Gamma(s,\chi)\zeta_k(s,\mathfrak{K})$$
$$=Q\int_0^\infty u^{((ns+q)/2)-1}du\int_{-\infty}^\infty\cdots\int\sum_{(\xi)\equiv\mathfrak{a},\,\xi\neq0}|P(\xi)|$$
$$\times\exp\left\{-\pi c(\mathfrak{a})u\sum_{p=1}^n|\xi^{(p)}\varepsilon_1^{(p)x_1}\cdots\varepsilon_r^{(p)x_r}|^2\right\}|P(\varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_r^{x_r})|$$
$$\times\,dx_1\,dx_2\cdots dx_r \qquad (29)$$

for $\sigma > 1$.

Using the Θ-formula (24) and proceeding on with the computation in the same way as Landau, we get from (28) the following formula for $\sigma > 1$.

$$
\begin{aligned}
&A(\chi)^{s/2}\Gamma(s, \chi)L(s, \mathfrak{K}, \chi)\\
&\quad = -\frac{2Q}{nw}E_0\Big(\bar{\chi}(\mathfrak{a})\frac{1}{s} + \chi\Big(\frac{1}{\mathfrak{a}\mathfrak{f}\mathfrak{d}}\Big)\frac{1}{1-s}\Big)\\
&\quad\quad + \frac{Q}{w}\{\pi c(\mathfrak{a})\}^{q/2}\int_{-1/2}^{1/2}\!\!\cdots\!\int |P(\varepsilon_1^{y_1}\varepsilon_2^{y_2}\cdots\varepsilon_r^{y_r})|\,dy_1dy_2\cdots dy_r\\
&\quad \times\int_1^\infty u^{((ns+q)/2)-1}\{-\psi(\mathfrak{a}, 0)P(0) + \Theta(u|\varepsilon_2^{2y_1}\cdots\varepsilon_r^{2y_r}|;\ \mathfrak{a}, \chi)\}\,du\\
&\quad\quad + \frac{Q}{w}\Big\{\pi c\Big(\frac{1}{\mathfrak{a}\mathfrak{f}\mathfrak{d}}\Big)\Big\}^{q/2} I(\chi)\int_{-1/2}^{1/2}\!\!\cdots\!\int |P(\varepsilon_1^{y_1}\varepsilon_2^{y_2}\cdots\varepsilon_r^{y_r})|\,dy_1dy_2\cdots dy_r\\
&\quad \times\int_1^\infty u^{((n(1-s)+q)/2)-1}\Big\{-\bar{\psi}\Big(\frac{1}{\mathfrak{a}\mathfrak{f}\mathfrak{d}}, 0\Big)P(0) + \Theta\Big((u|\varepsilon_1^{2y_1}\cdots\varepsilon_r^{2y_r}|;\ \frac{1}{\mathfrak{a}\mathfrak{f}\mathfrak{d}}, \bar{\chi}\Big)\Big\}\,du
\end{aligned}
\tag{30}
$$

where

$$
E_0 = \begin{cases} 1 & q = 0,\ \mathfrak{f} = \mathfrak{o} \text{ namely } \tilde{\mathfrak{f}} = \mathfrak{o} \\ 0 & \text{otherwise.} \end{cases}
$$

Similarly we know from (29) that the integral

$$
\begin{aligned}
&\frac{Q}{w}\{\pi c(\mathfrak{a})\}^{q/2}\int_{-1/2}^{1/2}\!\!\cdots\!\int |P(\varepsilon_1^{y_1}\varepsilon_2^{y_2}\cdots\varepsilon_r^{y_r})|\,dy_1dy_2\cdots dy_r\\
&\quad\times\int_1^\infty u^{((n\sigma+q)/2)-1}\sum_{\xi\in\mathfrak{a},\ \xi\neq 0}|P(\xi)|\\
&\quad\times\exp\{-\pi c(\mathfrak{a})u\sum_{p=1}^{n}|\xi^{(p)}|^2\cdot|\varepsilon_1^{(p)y_1}\varepsilon_2^{(p)y_2}\cdots\varepsilon_r^{(p)y_r}|^2\}\,du
\end{aligned}
\tag{31}
$$

exists for $\sigma > 1$. Since (31) is a monotone increasing function of σ, two integrals of the right-hand side of (30) are absolutely convergent for all s and represent integral functions.

§4. Analogue to Siegel's formulation

The first integral of (30) is equal to

$$
\begin{aligned}
&\frac{Q}{w}\{\pi c(\mathfrak{a})\}^{q/2}\int_1^\infty u^{((ns+q)/2)-1}du\int_{-1/2}^{1/2}\!\!\cdots\!\int |P(\varepsilon_1^{y_1}\varepsilon_2^{y_2}\cdots\varepsilon_r^{y_r})|\\
&\quad\times\sum_{\lambda\in\mathfrak{a},\ \lambda\neq 0}\psi(\mathfrak{a}, \lambda)P(\lambda)\exp\{-\pi c(\mathfrak{a})u\sum_{p=1}^{n}|\lambda^{(p)}|^2\cdot|\varepsilon_1^{(p)y_1}\varepsilon_2^{(p)y_2}\cdots\varepsilon_r^{(p)y_r}|^2\}\\
&\quad\times dy_1dy_2\cdots dy_r,
\end{aligned}
\tag{32}
$$

by the convergency of (31). If we put

$$\lambda = \xi\rho\varepsilon_1^{b_1} \cdots \varepsilon_r^{b_r},$$

where ρ is a root of unity and b_j $(1 \leqq j \leqq r)$ is an integer, then we obtain, using (12),

$$\psi(\mathfrak{a}, \lambda)P(\lambda) = |P(\varepsilon_1^{b_1}\varepsilon_2^{b_2} \cdots \varepsilon_r^{b_r})|\psi(\mathfrak{a}, \xi)P(\xi),$$

and (32) turns out to be equal to

$$Q\{\pi c(\mathfrak{a})\}^{q/2}\int_1^\infty u^{((ns+q)/2)-1}du \sum_{b_1, b_2, \ldots, b_r=-\infty}^{\infty} \int_{-1/2}^{1/2}\cdots\int |P(\varepsilon_1^{b_1+y_1} \cdots \varepsilon_r^{b_r+y_r})|$$
$$\times \sum_{(\xi)\subseteqq\mathfrak{a},\, \xi\neq 0} \psi(\mathfrak{a}, \xi)P(\xi) \exp\{-\pi c(\mathfrak{a})u\sum_{p=1}^{n}|\xi^{(p)}\varepsilon_1^{(p)b_1+y_1} \cdots \varepsilon_r^{(p)b_r+y_r}|^2\}$$
$$\times dy_1 dy_2 \cdots dy_r$$
$$= Q\{\pi c(\mathfrak{a})\}^{q/2}\int_1^\infty u^{((ns+q)/2)-1}du\int_{-\infty}^{\infty}\cdots\int \sum_{(\xi)\subseteqq\mathfrak{a},\, \xi\neq 0} \psi(\mathfrak{a}, \xi)P(\xi)$$
$$\times \exp\{-\pi c(\mathfrak{a})u\sum_{p=1}^{n}|\xi^{(p)}\varepsilon_1^{(p)x_1} \cdots \varepsilon_r^{(p)x_r}|^2\} \cdot |P(\varepsilon_1^{x_1}\varepsilon_2^{x_2} \cdots \varepsilon_r^{x_r})|$$
$$\times dx_1 dx_2 \cdots dx_r. \tag{33}$$

Since the summation is absolutely and uniformly convergent for

$$2^a \leqq u \leqq 2^{a+1}, \quad a_j \leqq x_j \leqq a_j + 1 \quad (1 \leqq j \leqq r),$$

where a is a non-negative integer and a_j is an integer, (33) is equal to

$$Q\{\pi c(\mathfrak{a})\}^{q/2} \sum_{(\xi)\subseteqq\mathfrak{a},\, \xi\neq 0} \psi(\mathfrak{a}, \xi)P(\xi)\int_1^\infty u^{((ns+q)/2)-1}du$$
$$\times \int_{-\infty}^{\infty}\cdots\int \exp\{-\pi c(\mathfrak{a})u\sum_{p=1}^{n}|\xi^{(p)}\varepsilon_1^{(p)x_1} \cdots \varepsilon_r^{(p)x_r}|^2\} \cdot |P(\varepsilon_1^{x_1}\varepsilon_2^{x_2} \cdots \varepsilon_r^{x_r})|$$
$$\times dx_1 dx_2 \cdots dx_r. \tag{34}$$

By transformation of (27), (34) is changed into

$$\{\pi c(\mathfrak{a})\}^{q/2} \sum_{(\xi)\subseteqq\mathfrak{a},\, \xi\neq 0} \psi(\mathfrak{a}, \xi)P(\xi)\int_{t_p>0,\, t_1 t_2\cdots t_n \geqq 1}\cdots\int \exp\{-\pi c(\mathfrak{a})\sum_{p=1}^{n}|\xi^{(p)}|^2 t_p\}$$
$$\times (\prod_{p=1}^{n} t_p^{(s+a_p)/2})\frac{dt_1 dt_2 \cdots dt_{r+1}}{t_1 t_2 \cdots t_{r+1}}. \tag{35}$$

If $\xi = \mathfrak{a}\mathfrak{b}$, then

$$N(\xi) = N(\mathfrak{a})N(\mathfrak{b})$$

and

$$\psi(\mathfrak{a}, \xi)P(\xi) = \chi(\mathfrak{b}\eta(\xi))P(\xi) = \chi(\mathfrak{b})|P(\xi)|.$$

Now we put

$$\pi c(\mathfrak{a})\,|\xi^{(p)}|^2 t_p = z_p \qquad (1 \leqq p \leqq r+1).$$

Inserting these in (35), we can prove that the first integral of (30) is equal to

$$A(\chi)^{s/2} \sum_{\mathfrak{b}\in\mathfrak{K},\ \mathfrak{b}\neq 0} \frac{\chi(\mathfrak{b})}{N(\mathfrak{b})^s} \Gamma(s,\ \chi,\ \mathfrak{b}).$$

Similarly we can prove that (31) is equal to

$$A(\chi)^{\sigma/2} \sum_{\mathfrak{b}\in\mathfrak{K},\ \mathfrak{b}\neq 0} \frac{1}{N(\mathfrak{b})^\sigma} \Gamma(\sigma,\ \chi,\ \mathfrak{b}),$$

so that this is also a monotone increasing function of σ $(-\infty < \sigma < \infty)$. Hence (7) is proved. We repeat the same argument with respect to the second integral of (30), and finally we obtain

$$\begin{aligned} &A(\chi)^{s/2}\Gamma(s,\ \chi)L(s,\ \mathfrak{K},\ \chi) \\ &= -\frac{2Q}{nw}E_0\Big(\chi(\mathfrak{K})\frac{1}{s} + \overline{\chi}(\hat{\mathfrak{K}})\frac{1}{1-s}\Big) \\ &\quad + A(\chi)^{s/2} \sum_{\mathfrak{b}\in\mathfrak{K},\ \mathfrak{b}\neq 0} \frac{\chi(\mathfrak{b})}{N(\mathfrak{b})^s}\Gamma(s,\ \chi,\ \mathfrak{b}) \\ &\quad + A(\chi)^{(1-s)/2} I(\chi) \sum_{\mathfrak{b}\in\hat{\mathfrak{K}},\ \mathfrak{b}\neq 0} \frac{\overline{\chi}(\mathfrak{b})}{N(\mathfrak{b})^{1-s}}\Gamma(1-s,\ \overline{\chi},\ \mathfrak{b}), \end{aligned}$$

whence follows (8) immediately.

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