A FUNCTION ALGEBRA ON RIEMANN SURFACES

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1. Introduction. In this note, we treat the problem to determine the conformal structure of the closed surface by the structure of the differentiable function algebra as the normed algebra with a certain norm.

A similar investigation is found in Myers [1]. He concerns himself with determining the Riemannian structure of the compact manifold using a certain normed algebra of differentiable functions.

We have shown in [2] the fact that the Royden's ring as a topological ring determines the quasiconformal structure of the Riemann surface. Thus it is natural to inquire whether *the Royden's ring as a normed ring characterizes the Riemann surface* or not. This problem is positively answered for closed surfaces by reduction to the following: *A topological mapping between two surfaces with the annular maximal dilatation*[1] 1 *is a conformal*[2] *mapping.*

2. Royden's ring. We denote by R an open or closed Riemann surface and by $M(R)$ its Royden's ring, i.e., the normed ring of all bounded continuous functions on R which are absolutely continuous in the sense of Tonelli[3] with finite Dirichlet integrals. The norm of f in $M(R)$ is given by

$$\|f\| = \|f\|_\infty + \sqrt{D[f]}, \tag{1}$$

where $\|f\|_\infty$ denotes the uniform norm $\sup(|f(P)|\ ;\ P \in R)$. Then $M(R)$ is a complete normed ring with respect to the norm (1).

We denote by $C^n \cap M(R)$ the incomplete normed subring of $M(R)$ consisting of all C^n-functions in $M(R)$. The following holds (cf. [2]).

LEMMA 1. *$C^n \cap M(R)$ is dense in $M(R)$* $(n = 1, 2, \ldots)$.

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1) The definition will be given in §3.

2) Here and hereafter the term *conformal* includes both of the *direct* and the *indirect* one.

3) A function $f(x, y)$ on $[a, b;\ c, d]$ is called *absolutely continuous in the sense of Tonelli* if $f(x, y)$ is absolutely continuous in $x \in [a, b]$ for almost every fixed values $y \in [c, d]$ and the corresponding fact holds if x and y are interchanged and further f_x and f_y are locally integrable. The notion is carried over Riemann surfaces using local parameters.

Let A be an annulus which is contained in a simply connected domain D in R and whose boundary consists of two simple closed curves C_0 and C_1. We assume that the simply connected domain $(C_0) \subset D$ bounded by C_0 includes C_1. Define a continuous function $f_A(P)$ on R as follows: $f_A(P)=0$ if $P \in R-(C_0)$, $f_A(P)=1$ if $P \in \overline{(C_1)}$, the closure of the simply connected domain (C_1) in D bounded by C_1, and $f_A(P)$ is harmonic in $(C_0)-\overline{(C_1)}$. Clearly f_A is contained in $M(R)$. We shall call f_A the *fundamental function* with the *base* A. Denote by F_P^R the totality of fundamental functions in $M(R)$ whose bases contain the fixed point P in R. The linear space with real coefficients generated by F_P^R will be denoted by $\tilde{F}_P^R$. We notice that the functions in $\tilde{F}_P^R$ is harmonic at P.

Let $z=x+iy$ be a local parameter at P. We define

$$\mathfrak{M}_{P,z}^R=\{(f_{xx}(P), f_{xy}(P), f_x(P), f_y(P)) \ ; \ f \in \tilde{F}_P^R\}.$$

Then $\mathfrak{M}_{P,z}^R$ is a linear subspace of 4-dimensional real linear space $\mathbf{R}^4$. For this space we can show the following:

LEMMA 2. $\mathfrak{M}_{P,z}^R=\mathbf{R}^4$.

Proof. Let z be valid in a simply connected domain D in R. Then P is represented $a+ib$ in terms of z. Let (ε, η) be a pair of real numbers such that an annulus $B_{(\varepsilon,\eta,r_1,r_2)}=\{Q \ ; \ r_1<|a+ib+\varepsilon+i\eta-z(Q)|<r_2\}$ is contained in D with its closure and that $P \in B_{(\varepsilon,\eta,r_1,r_2)}$. The totality of such pairs (ε, η) contains a punctured disc E in the (ε, η)-plane: $0<|\varepsilon+i\eta|<\min(|z(Q)-z(P)| \ ; Q \in \partial D)$.[4] Let $f(Q)$ be the fundamental function with the base $B_{(\varepsilon,\tau,r_1,r_2)}$. Then

$$f(Q)=\mu(\log r_2-\log|a+ib+\varepsilon+i\eta-z(Q)|),$$
$$\mu=1/(\log r_2-\log r_1),$$

for Q in $B_{(\varepsilon,\eta,r_1,r_2)}$. Hence we get

$$(f_{xx}(P), f_{xy}(P), f_x(P), f_y(P))$$
$$=\frac{-\mu}{|\varepsilon+i\eta|^4}(-\varepsilon^2+\eta^2,\ 2\varepsilon\eta,\ -\varepsilon(\varepsilon^2+\eta^2),\ -\eta(\varepsilon^2+\eta^2)),$$

which shows that $\mathfrak{M}_{P,z}^R$ contains the linear subspace $\mathfrak{M}'$ which is generated by

$$\{((\eta^2-\varepsilon^2),\ 2\varepsilon\eta,\ -\varepsilon(\varepsilon^2+\eta^2),\ -\eta(\varepsilon^2+\eta^2)) \ ; \ (\varepsilon, \eta) \in E\}.$$

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[4] For the set D, we denote by ∂D the boundary of D.

It is easy to see, choosing ε and η suitably, that $\mathfrak{M}'$ contains unit vectors $(1, 0, 0, 0), \ldots, (0, 0, 0, 1)$ and hence $\mathfrak{M}^R_{P,z}$ is of 4 dimension. This completes the proof.

Let K be a compact domain in R whose boundary consists of a finite number of closed Jordan curves. First, for a function f in $C^1 \cap M(R)$, the function $\pi_K f$ is defined as follows: $\pi_K f = f$ in $R - K$ and $\pi_K f$ is the harmonic function in K with boundary values f on ∂K. Then by Dirichlet principle and the maximum principle of harmonic functions, we get the following

$$\|\pi_K f\| \leqq \|f\|. \tag{2}$$

By Green's formula, we also have the equality.

$$D[f] = D[\pi_K f] + D[f - \pi_K f]. \tag{3}$$

Thus π_K is a linear operator of $C^1 \cap M(R)$ into $M(R)$ and, using Lemma 1 and the inequality (2), we see that π_K can be extended to the whole $M(R)$ preserving the relations (2) and (3). We shall call π_K the *harmonizer on* $M(R)$ *with respect to* K. Summing up these, we get

LEMMA 3. *The harmonizer* π_K *is a linear operator with* $\pi_K \cdot \pi_K = \pi_K$ *of* $M(R)$ *into* $M(R)$ *possessing the following properties:*

(a) $\pi_K f = f$ *in* $R - K$ *and* $\pi_K f$ *is harmonic in* K,

(b) (2) *and* (3) *hold for all* f *in* $M(R)$,

(c) $\pi_K f = 0$ *if and only if* $f = 0$ *in* $R - K$.

3. Maximal dilatation. Let T be a topological mapping of a Riemann surface R_1 onto another surface R_2. The *annular maximal dilatation* $K^*(T)$ of T is defined by the following

$$K^*(T) = \inf(\lambda\ ;\ \lambda^{-1} \operatorname{mod} A \leqq \operatorname{mod} TA \leqq \lambda \operatorname{mod} A). \tag{4}$$

Here A runs over all annuli with boundary consisting of two Jordan closed curves in R_1 and $\operatorname{mod} A$ denotes the modulus of A. It is clear that $1 \leqq K^*(T) \leqq \infty$. It is known that $K^*(T) \leqq K(T) \leqq e^{\pi K^*(T)}$ holds, where $K(T)$ denotes the maximal dilatation in the sense of Pfluger-Ahlfors, i.e., the one using quadrilaterals instead of annuli in (4). It is well known that $K(T) = 1$ if and only if T is a conformal mapping. We shall prove the corresponding fact for $K^*(T)$.

THEOREM 1. *A topological mapping* T *of* R_1 *onto* R_2 *is conformal if and only if* $K^*(T) = 1$.

Proof. First we show that $f \in \tilde{F}_{TP}^{R_2}$ implies $f \circ T \in \tilde{F}_P^{R_1}$. For this aim, we have only to prove that $f \circ T$ is harmonic on A_1 if f is in $F_{TP}^{R_2}$, where A_1 is the inverse image of the base A_2 of f by T. Let $z = x + iy$ and $w = u + iv$ be uniformizers valid in neighbourhoods of A_1 and A_2, respectively. Let φ_1(resp. φ_2^{-1}) be a conformal mapping of a circular ring A_1^* (resp. A_2) onto A_1 (resp. a circular ring A_2^*).

Putting $T^* = \varphi_2 \circ T \circ \varphi_1$ and considering T^* as a topological mapping of A_1^* onto A_2^*, we see that $K^*(T^*) = 1$. Thus we may assume $A_1^* : r_1 < |z^*| < r_2$, $A_2^* : r_1 < |w^*| < r_2$. Let A_1^* be divided into A_{11}^* and A_{12}^* by a concentric circle l_1^* and let A_{21}^*, A_{22}^* and l_2^* be their images under T^*. As we have

$$\text{mod } A_2^* = \text{mod } A_1^* = \text{mod } A_{11}^* + \text{mod } A_{12}^*,$$

and

$$\text{mod } A_{1k}^* = \text{mod } A_{2k}^* \qquad (k = 1, 2),$$

we get

$$\text{mod } A_2^* = \text{mod } A_{21}^* + \text{mod } A_{22}^*,$$

which shows l_2^* is the concentric circle with the same radius as l_1^*. Hence we see that

$$|T^* z^*| = |z^*|. \tag{5}$$

Since, obviously, $f(\varphi_2^{-1}(w^*))$ is a harmonic measure of $|w^*| = r_1$ with respect to A_2^*, we have

$$f(\varphi_2^{-1}(w^*)) = \log (k/|w^*|), \tag{6}$$

where μ and k are suitable constants. By using (5) and (6),

$$f \circ T(z) = f \circ \varphi_2^{-1} \circ T^* \circ \varphi_1^{-1}(z) = \log (k/|T^* \circ \varphi_1^{-1}(z)|)$$
$$= \mu \log (k/|\varphi_1^{-1}(z)|),$$

which shows $f \circ T$ is harmonic in A_1.

Next we show that u and v are in class C^1, where $u(z)$ and $v(z)$ are the local equations of $T : w = Tz = u(z) + iv(z)$. Let a point $z = x + iy$ be fixed. Putting, for example,

$$\Delta u = \Delta u(\Delta x) = u(x + \Delta x, y) - u(x, y),$$

we get, for f in $\tilde{F}_{Tz}^{R_2}$,

(7) $$\frac{1}{\Delta x}(f\circ T(x+\Delta x, y)-f\circ T(x, y))$$
$$=f_u(u+\theta\Delta u,\ v+\theta\Delta v)\frac{\Delta u}{\Delta x}+f_v(u+\theta\Delta u,\ v+\theta\Delta v)\frac{\Delta v}{\Delta x},\quad 0\leqq\theta\leqq 1.$$

Now we can see that

$$-\infty<\varliminf_{\Delta x\to 0}\frac{\Delta v}{\Delta x}\leqq\varlimsup_{\Delta x\to 0}\frac{\Delta v}{\Delta x}<\infty.$$

Contrary to the assertion, assume that there exists a sequence $\{\Delta x_n\}\to 0$ such that $\lim_{n\to\infty}\frac{\Delta v(\Delta x_n)}{\Delta x_n}=\infty$. By Lemma 2, there exists f in $\tilde{F}_{Tz}^{R_2}$ satisfying $(f_u(Tz), f_v(Tz))=(1, 1)$ or $(-1, 1)$. As f and $f\circ T$ are harmonic at Tz and z, respectively, we arrived at the following contradiction: $\lim_{n\to\infty}\frac{\Delta u(\Delta x_n)}{\Delta x_n}=-\infty$ and at the same time $=\infty$. Thus $\varlimsup_{\Delta x\to 0}\frac{\Delta v}{\Delta x}<\infty$. Similarly, we get $\varliminf\frac{\Delta v}{\Delta x}>-\infty$. Again choosing f in $\tilde{F}_{Tz}^{R_2}$ such that $(f_u(Tz), f_v(Tz))=(1, 0)$, we get from (7) and from the above argument that

$$\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\frac{\partial f\circ T}{\partial x}(z).$$

Hence $u_x(z)$ and similarly $v_x(z)$ must exist. From (7) it follows that

(7)′ $$\frac{\partial}{\partial x}f\circ T(z)=f_u(u, v)u_x(z)+f_v(u, v)v_x(z).$$

By the similar argument as used in showing the existence of $\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$, the continuity of u_x and v_x can be easily proved. We get the existence and the continuity of u_y and v_y, similarly.

Applying the similar argument to (7)′, we have the existence of u_{xx}, u_{xy}, v_{xx} and v_{xy} and their continuity and also for u_{yy} and v_{yy}.

Finally we obtain

(8) $$\triangle(f\circ T(z))=f_{uu}(Tz)(u_x^2+u_y^2-v_x^2-v_y^2)+2f_{uv}(T)(u_xv_x+u_yv_y)$$
$$+f_u(Tz)\triangle u+f_v(Tz)\triangle v,$$

where $\triangle$ is Laplacian. By Lemma 2, we get

(9) $$\triangle u=\triangle v=0$$
$$u_x^2+u_y^2=v_x^2+v_y^2,\ u_xv_x+u_yv_y=0,$$

which implies the Cauchy-Riemann relation for (u, v) or $(u, -v)$ which shows that Tz is a direct or an indirect conformal mapping. This completes the proof of Theorem 1.

4. Algebras of differentiable functions. Here we state our main result in this note.

THEOREM 2. *Two closed Riemann surfaces R_1 and R_2 are conformally equivalent if and only if their Royden's rings $M(R_1)$ and $M(R_2)$ are isometrically isomorphic.*

In other words, *the normed ring theoretic structure of Royden's ring determines the conformal structure of the closed surface.*

Proof. The necessity of our condition is evident. So we have only to show that an isometric isomorphism σ of $M(R_1)$ onto $M(R_2)$ is induced by a direct or indirect conformal mapping T of R_2 onto R_1.

Let R_j^* be the character space of $M(R_j)$, i.e., the totality of homomorphisms of $M(R_j)$ onto the complex number field preserving the positiveness. Then there exists a natural correspondence T of R_2^* onto R_1^* induced by σ : $T\chi(f) = \chi(f^\sigma)$ for $\chi \in R_2^*$, $f \in M(R_1)$. But, for compact spaces R_k, it is easy to see that $R_k^* = R_k$. Here we consider $P \in R_k$ as a character defined by $P(f) = f(P)$ for $f \in M(R_k)$. Moreover the topology of R_k as a Riemann surface is coincident with the weak* topology $\sigma(R_k^*, M(R_k))$ of $R_k^* = R_k$. Thus, by definition it is clear that T is a topological mapping.

Let A_2 be an annulus with boundary consisting of two Jordan curves. Let $TA_2 = A_1$. We shall prove that $\operatorname{mod} A_1 = \operatorname{mod} A_2$, or $K^*(T) = 1$.

For the aim, we notice that

$$\|f^\sigma\|_\infty = \|f\|_\infty \qquad \text{for } f \text{ in } M(R_1). \tag{10}$$

In fact, $\|f\|_\infty = \sup(|\lambda|\ ;\ \lambda \in S(f))$, where $S(f)$ is the *spectra* of f in $M(R_1)$, that is, the totality of complex numbers such that $f - \lambda$ is not inversible. Clearly, $S(f) = S(f^\sigma)$, so (10) follows. Thus by the isometricity of σ with respect to the norm (1), we get

$$D[f^\sigma] = D[f]. \tag{11}$$

Let f_2 be the fundamental function with the base A_2 and put $\tilde{f}_1 = f_2^{\sigma^{-1}}$. Obviously, $\pi_{A_1}\tilde{f}_1 = f_1$ is a fundamental function with the base A_1. Putting $\tilde{f}_2 = f_1^\sigma$, we have $\pi_{A_2}\tilde{f}_2 = f_2$. By (3) and (11), it holds

$$D[f_1] = D[f_1^\sigma] = D[\tilde{f}_2] \geqq D[\pi_{A_2}\tilde{f}_2] = D[f_2] = D[\tilde{f}_1^\sigma]$$
$$= D[\tilde{f}_1] \geqq D[\pi_{A_1}\tilde{f}_1] = D[f_1].$$

Thus we get $D[f_1] = D[f_2]$. As $\text{mod}\, A_j = 2\pi/D[f_j]$, we get $\text{mod}\, A_1 = \text{mod}\, A_2$ or $K^*(T) = 1$.

By Theorem 1, the topological mapping T is conformal. This completes the proof of Theorem 2.

COROLLARY. *Two closed Riemann surfaces R_1 and R_2 are conformally equivalent if and only if $C^n(R_1)$ and $C^n(R_2)$ are isometrically isomorphic, where $C^n(R_j)$ denotes the incomplete normed ring of all functions in the class C^n with the norm* (1). *Here n is an arbitrary positive integer.*

Proof. Let σ be an isometric isomorphism of $C^n(R_1)$ onto $C^n(R_2)$. Then by Lemma 1, σ can be extended to the isometric isomorphism of $M(R_1)$ onto $M(R_2)$. Thus R_1 and R_2 are conformally equivalent.

The converse is obvious. This completes the proof.

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