AN INVESTIGATION ON THE LOGICAL STRUCTURE OF MATHEMATICS (VIII)*

CONSISTENCY OF THE NATURAL-NUMBER THEORY $T_1(N)$

SIGEKATU KURODA

Preliminaries and Consequences

1. Consistency proof and intuitive knowledge

The consistency of the natural number theory was proved, as is well known, by G. Gentzen in 1935 for the first time in such generality that the mathematical induction can be consistently used for any arbitrary predicate of natural numbers, which is well-formed in his system so that every quantifier ranges over all natural numbers. His formulation[1] of the natural number theory will be for simplicity referred to as GN. In GN the natural numbers 0, 1, 2, . . . are represented by special symbols of the system GN. Some predicate of natural numbers, such as $*<*$, $*=*$, etc., and some operations, such as $*+*$, $*\times*$, etc., are also represented by special symbols of GN. These special symbols of GN must be such that their intuitive interpretations are allowed in such a way that the intuitive truth and falsehood of those statements which are construed by variables for natural numbers, the special symbols, and the connectives of propositional logic, without using quantifiers, can be determined by our intuitive knowledge. Conversely, any predicates and operations which have this property can be used as special symbols of GN. From among the statements of the above mentioned form, the "mathematischen Grundsequenzen", such as $\rightarrow a=a$, $\rightarrow a<b \wedge b<c \rightarrow a<c$, etc. are extracted as such statements that are intuitively true, i.e. true for any arbitrary substitution of natural numbers for all the variables occurring in the statements. This is the basis for the fact that the formal system GN is related to our intuition of natural numbers. For this reason

---

Received September 17, 1958.

* This Part (VIII) depends logically only on Part (I), §1–§11 and Part (II), §12–§15, Hamburger Abh. vol. **22**. The superscript such as §11, (I) is the reference to §11, Part (I).

[1] In the sequel we refer to his second formulation given in Semesterber. Münster (1938).

Gentzen's consistency proof of GN can not remain as a mere consistency proof but shows that our formal knowledge obtained by the deduction in GN harmonizes with our intuitive knowledge, or more precisely, that the negation of any "mathematischen Grundsequenz", say A, can never be proved in GN. Indeed, this fact is proved in the same way as in Gentzen's consistency proof by using the "Endsequenz" $A \rightarrow$ instead of the void "Endsequenz" $\rightarrow$. Thus, the importance of Gentzen's consistency proof consists in bringing the formal natural number theory into harmony with the intuitive natural number theory.

In general, the consistency proof of a certain formal system T shows at the same time that T is in harmony, in the above-mentioned sense, with our intuitive knowledge I of a certain kind. The wider the knowledge I the more interesting the consistency of the system T. In particular, it is of primary importance when I contains some part of our knowledge based on the intuition of natural numbers, of the linear continuum, and of the transfinite ordinal numbers (conceived intuitively), which is the basis of the intuitive mathematics.

In proving the consistency of T which harmonizes with I, we can and must make use of the intuitive knowledge I to the best advantage.

Now, this part (VIII) is divided into two Sections A and B. In Section A the theory $T_0(N)$ is defined (§3) and its consistency is proved (§§4-8). In Section B the natural number theory $T_1(N)$ is defined (§8) and its consistency is proved (§§9-18.2). $T_0(N)$ is a theory containing the totality N of all natural numbers as a set of the theory $T_0(N)$ but not the inference by mathematical induction, while $T_1(N)$ is the natural number theory obtained from $T_0(N)$ by adjoining the inference of mathematical induction with respect to any predicate of natural numbers which is construed with variables of no type distinction.[2] Both $T_0(N)$ and $T_1(N)$ are subsystems of UL, and $T_0(N)$ is a subsystem of $T_1(N)$.

The species of sets of $T_0(N)$ consists of the universal constant V, the null constant 0, the set N of all natural numbers, and the elementary sets (§3)

[2] Simple type theory is reasonable and useful in some cases. We can dispense with it in UL, however. For, let E be any variable, independent or dependent, and put $E_1=E$, $E_n=\mathfrak{P}(E_{n-1})$, where $\mathfrak{P}(E_{n-1})$ denotes the power set of E_{n-1}, and $E_{i_1,\dots,i_m}=\langle E_{i_1},\dots,E_{i_m}\rangle$ ($\langle E_i\rangle=E_i$). Then we get a simple type theory in UL of finite order with E as basic type, if we only restrict every variable x, free or bound, to some $T=E_{i_1,\dots,i_m}$ ($1\leq i_1,\dots,1\leq i_m$) in such a way that $x\in T$, $\overset{T}{\forall}x(\forall x.\, x\in T\rightarrow)$, and $\overset{T}{\exists}x(\exists x.\, x\in T\wedge)$. The UL with a kind of universal independent variables is thus sufficient to construct simple type theories as subsystems of UL.

generated by V, 0, and N. The intuitive knowledge on these sets is the basis of the consistency proof of $T_0(N)$.

To begin with, the intuitive conception of the totality N of all natural numbers is well in hand. Namely, that N is the totality of all natural numbers is intuitively nothing but a variant form of the expression of the intuitive apprehension of natural numbers: (i) 0 is a natural number; (ii) if n is a natural number, then the successor n' of n is a natural number; and (iii) there is no other natural number than those which are obtained by the generating procedures (i) and (ii). The natural numbers are represented in $T_0(N)$ by 0, $\{0\}$, $\{\{0\}\}$, $\{\{\{0\}\}\}$, . . . , so that $m \in N$ is intuitively true if and only if m is a numeral in $T_0(N)$ or an object occurring in the indefinitely proceeding sequence 0, $\{0\}$, $\{\{0\}\}$, ...

Second, by the meaning of V and 0 we see that $m \in V$ and $m \notin 0$ are intuitively true and $m \notin V$ and $m \in 0$ are intuitively false for any constant m of $T_0(N)$. Thus the criterion for the intuitive truth and falsehood of the primitive formula $l \in m$ is determined where l is any constant of $T_0(N)$ and m is V, 0, or N.

Third, again by the meaning of V, 0, and N, we see that V, 0, and N are intuitively different each other.

Fourth, by the meaning of the elementary sets and by the intuitive knowledge determined above, we can determine the intuitive truth and falsehood of the primitive formula $l \in m$ and of the equality $l = k$, where l and k are any sets of $T_0(N)$ and m is an elementary constant generated by V, 0, and N.

These are all the intuitive knowledge concerning the constants of $T_0(N)$ which is used in the consistency proof of $T_0(N)$ (§§ 4-7). Moreover, also in the consistency proof of $T_1(N)$ (§§ 9-18.2), we call for no other intuitive knowledge concerning the constants of $T_1(N)$ except these mentioned above. In case of $T_0(N)$ the "intuitive string" (§ 6) of a given $T_0(N)$-proof of contradiction is determined (§ 7) in virtue of this intuitive knowledge, and in case of $T_1(N)$ the same procedure determines the "intuitive part" (§§ 10, 12, 16), instead of intuitive string, of a given $T_1(N)$-proof of contradiction. In this way, the consistency proof of $T_0(N)$ is a preparation for the consistency proof of $T_1(N)$.

Now, the intuitive knowledge we have described above concerning the sets of $T_0(N)$ and $T_1(N)$ is based upon our intuition on natural numbers, on the universe of discourse of $T_0(N)$, and on the intuitive meaning of the elementary sets.

In general, let T be a subsystem of UL and assume that we have determined by virtue of the intuitive meaning of the constants of T a criterion for the truth and falsehood of any primitive formula $l \in m$ and of any equality $l=m$ where l and m are any constants of T. We call this criterion the intuitive knowledge for T.

We now construct all the decompositions of the negation of the formula (I) (§3) and of the defining formulas of all constant sets of T, as is done in §3 for $T_0(N)$. Let D be a "tree" of such a decomposition. Assume that there is a method, as is described for $T_0(N)$ in §§5-7, which assigns an appropriate T-constant to each eigen variable occurring in D in such a way that, if the T-constant is substituted for every free variable in D to which the constant is assigned, we get a tree D^* from D in which there is at least a false D^*-string. By a false D^*-string is meant a D^*-string such that all the formulas of the form $l \in m$, $l \notin m$, $l=m$, or $l \neq m$ carried by the string are false in virtue of the intuitive knowledge for T, mentioned above. If this assumption is fulfilled, then the consistency of T can be proved in the same way as in the consistency proof for $T_0(N)$ in §§5-7. When the consistency of a theory T is proved in this way we say that the theory T is *elementarily consistent.*[3]

Although the intuitive knowledge used in the consistency proof of T is extracted from the meaning of the constants of T, only the intuition on symbols is needed in the consistency proof of T, once the intuitive truth and falsehood are defined. Further, an intuitively true formula of the form $l \in m$, $l \notin m$, $l=m$ or $l \neq m$ is not necessarily T-provable, while, conversely, we have the following theorem.

THEOREM 1. *Let a theory T be elementarily consistent. Let further A be a formula construed by the connectives of propositional logic, primitive formulas $l \in m, \ldots$, and equalities $l=m, \ldots$, where $l, m, \ldots$ are any constant sets of the theory T. If A is intuitively true, then $\neg A$ is T-unprovable.*

This theorem is proved just in the same way as in the consistency proof of T, not applied to an assumed proof of contradiction $\sigma \vdash$ but to an assumed

3) For some theory T it may happen that some restriction of the decomposition of some premises is assumed by the definition of T. For instance, if T contains N as its set, the negative proof constituent associated with the defining formula of N should be prohibited as in $T_0(N)$ in order that T be elementarily consistent.

proof of an assertion $\sigma \vdash A$ instead.

Theorem 1 holds also for any formula B in place of A, which depends on some independent variables, provided that B can be changed into a formula A satisfying the condition in theorem 1 by an appropriate substitution of some constants for the independent variables occurring free in B.

If we apply theorem 1 to $T_0(N)$, we see, as in Gentzen's case, that the harmony between our intuitive and formal knowledge[4] on natural numbers is maintained in $T_0(N)$. This is also true for $T_1(N)$. In order to see this in case of $T_1(N)$, however, we shall have to investigate into the consistency proof of $T_1(N)$.

2. Comparison with Gentzen's proof and some consequences

The method of reduction of an assumed proof of contradiction by replacing mathematical induction by cuts is essentially the same as in Gentzen's proof. The main difference of the consistency proof given in Section B from Gentzen's is caused by the formulation of $T_1(N)$. In Gentzen's formulation the mathematical induction is formulated as a special inference figure while in $T_1(N)$ an application of a mathematical induction in a $T_1(N)$-proof is the use of a negative proof constituent [NN] (§3) associated with the defining formula of N. In Gentzen's case, the "Endformel" of a proof of contradiction is the void sequence while the top sequence of a UL-proof of contradiction consists of a finite number of the negation of defining formulas and of the formula (I) (§3). For the sake of completeness the consistency proof of $T_1(N)$ is given in full detail. Namely, in §11 the method of the replacement by cuts of an [NN**] is stated. Here, contrary to Gentzen's case, some dependent variable (namely, m in (3) and (4), §11) occurs instead of a numeral n in the "Untersequenz" of a "*VJ*-Schussfigur" which is in the "Endstück" of Gentzen's reduction. As a consequence, we have only to make use of the assignment of constants, described in §5, to the eigen variables of [NN**] (see §8) occurring in the intuitive part of a proof of contradiction. The intuitive part (§10) of a proof corresponds to Gentzen's "Endstück", but it does never reach to the bottom formulas of the proof considered. Just as in the same way as in Gentzen's proof, after looking for a "particular cut"

4) By formal knowledge we mean always theorems deduced in UL or in a consistent subsystem of UL, while intuitive knowledge is not necessarily obtained by deductions.

(§13) to which further reduction is to be applied, his method of the reduction of the proof in question is applied to the particular cut (§15). The association of transfinite ordinal numbers to proof is slightly different from Gentzen's. The necessity of this change is mainly caused in order to make lemmas 1 and 2 in §17 valid (see foot-notes 17 and 18). By transfinite induction up to the first ε number the finiteness of the formal procedures is, as in Gentzen's proof, proved (§18), which contradicts the indefiniteness of the intuitive procedures (§10). This proves the consistency of $T_1(N)$ (§9).[5]

In this way, in the consistency proof of $T_1(N)$ the decomposition of the negation of defining formula of a "set for induction" is never used but the definiens of a set for induction is changed into a cut formula. Therefore, theorem 1 holds also for $T_1(N)$ without taking the sets for induction into consideration.

In order to see the significance of this consistency proof we shall here enter into the consideration of the general method of "natural-number-theoretic extension" of a certain consistent subsystem of UL.

Let T_0 be an elementarily consistent subsystem of UL and Σ the species of sets of the theory T_0 and assume that Σ contains N as its member. Let further I be the intuitive knowledge in virtue of which the consistency of T_0 is proved and assume that the part of the knowledge I which is concerned with N coincides with the intuitive conception of the totality of natural numbers. Assume further that in T_0 the use of the affirmative but not the negative proof constituent (see §3) associated with the defining formula of N is allowed. We shall call such a *consistent* subsystem T_0 of UL an *elementary natural-number-theory*.

Let now T_1 be the theory which is obtained from T_0 by adjoining the mathematical induction, i.e. the use of the negative proof constituent associated with the defining formula of N for any set for induction.[6] Then the consistency proof T_1 can be proved by using the consistency proof of T_0 just in the same way as the consistency of $T_1(N)$ is proved by using that of $T_0(N)$. We call the extension T_1 of T_0, described above, the *natural-number-theoretic extension*

---

5) The consistency proof of $T_1(N)$ contains some simplification of that of GN. In particular, the unnecessary complication in Gentzen's proof caused by the "Verdünnung" is completely eliminated.

6) In defining the sets for induction for T_1 a condition similar to that stated in (ii), §8, for $T_1(N)$ is necessary.

of T_0, and express the above fact as

THEOREM 2. *The natural-number-theoretic extension of an (consistent) elementary natural-number-theory is consistent.*

By the same reasoning by which the harmony between our intuitive and formal knowledge on natural numbers in $T_1(N)$ is concluded from that in $T_0(N)$, the same harmony in T_1 is concluded from that in T_0. Thus, as theorem 1, we have

THEOREM 3. *Let T be the natural-number-theoretic extension of an elementary natural-number-theory, and A a formula as in theorem* 1. *If A is intuitively true, then $\neg A$ is T-unprovable.*

Now, let Σ be a finite or infinite closed (§ 11, Part (I)) species of dependent variables, containing at least a constant, and Σ_0 a closed subspecies of Σ, also containing at least a constant. Denote by T_Σ and T_{Σ_0} the subsystems of UL of which Σ and Σ_0 are the species of dependent variables respectively.[7]

Assume that T_{Σ_0} is elementarily consistent and denote by I_0 the intuitive knowledge by which the consistency of T_{Σ_0} is proved. Assume further that by I_0 and by the intuitive meaning[8] of the defining formulas of the dependent variables of Σ we obtain the knowledge I, sufficient to be able to conclude that any arbitrary constant belonging to Σ is intuitively equal to a constant of Σ_0. If these assumptions are fulfilled we call T_Σ an *elementary extension* of T_{Σ_0}. Since in this case we can find by using I a false string of any decomposition of the negation of a defining formula of a constant belonging to Σ, after giving an appropriate association of constants to eigen variables, we can prove elementarily the consistency of T_Σ. Thus:

THEOREM 4. *An elementary extension of an elementarily consistent theory is consistent. In particular, an elementary extension of an elementary natural-number-theory is an elementary natural-number-theory.*

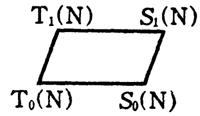
By theorems 2 and 4 we can gradually extend the natural number theory

---

[7] Some metalogical restriction of the decomposition of some premises of T_{Σ_0} may be assumed as the negative proof constituent associated with the defining formula of N is prohibited in $T_0(N)$. If there are such restrictions in T_{Σ_0}, the same restrictions shall be assumed in T_Σ (note that $\Sigma_0 \subseteq \Sigma$).

[8] For intuitive meaning see introduction in Part (I).

$T_1(N)$ consistently. Namely, let $S_0(N)$ is an elementary extension of $T_0(N)$ and $S_1(N)$ the natural-number-theoretic extension of $S_0(N)$. Then by theorems 2 and 4 we see that $S_1(N)$ is a consistent natural-number-theory which is an extension of $T_1(N)$. We shall give an example.



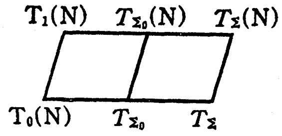
Let Σ_0 be the species of constants, consisting of 0, V, N, ι (identical mapping), all the direct products of any finite number of constants, and all the elementary constants, generated by these constants. Let Σ be the minimal closure[9] (with respect to substitution) of 0, V, N, ι, $\{a_1, \ldots, a_n\}$, $\langle a_1, \ldots, a_n\rangle$, $\sigma ` a$, $\sigma `` a$, D_σ, W_σ, σ^{-1}, $\sigma\circ\tau$, $\sigma\restriction a$, and $a_1\times\ldots\times a_n$.

We see easily that T_{Σ_0} is elementarily consistent (notice the restriction in footnote 3)) and that T_Σ is an elementary extension of T_{Σ_0}. To prove the latter we have only to verify recursively for any given constant of Σ whether it is equal to a constant of Σ_0. For instance, $\iota``N\times V$ is a constant of Σ, and we have the defining formula

(1) $$u\in\iota``N\times V\equiv\exists x.\ x\in N\times V\wedge\langle xu\rangle\in\iota.$$

We see from this that $\iota``N\times V=N\times V$ and that for any given constant u of Σ_0 we can find a false string of any decomposition of (1), and so on.

If we denote by $T_{\Sigma_0}(N)$ and $T_\Sigma(N)$ the natural-number-theoretic extensions of T_{Σ_0} and T_Σ respectively, we obtain the consistent extensions as shown in the figure 2. Since Σ contains all the dependent variables used in the deductions in Part (III), except Un, Un_2, and $Map_2^{a,b}$, which are used in Part (III) only as concepts, we have in particular that *all the* UL-*theorems proved in Part* (III) *are also theorems of the consistent natural-number-theory* $T_\Sigma(N)$. How far we can extend the natural number theory in this way by theorems 2 and 4 will be seen in the precise sense after performing actual deductions, collecting the dependent variables used in them, and examining whether the conditions of theorems 2 and 4 are fulfilled for these dependent variables. In this way, if we wish to construct mathematics consistently, *the proof of consistency must be accompanied step-by-step according*



[9] The words used here such as minimal closure, all the direct products, any finite number etc. can and shall be understood as terms of intuitive mathematics.

as the deductions are proceeding. This is rather a similar situation in mathematical analysis in which the proof of convergence must be accompanied in each step of limes process.

However, it seems not so much speculative to state without precise formulation of the domain of validity that the consistency of the algebraic systems follows to some extent from theorems 2 and 4, so that it is assured to use consistently in these theories the mathematical induction for any sets for induction (see foot-note 6)).[10)]

Since there is no principal difference in the consistency proof of any elementary natural-number-theories as well as in that of any natural-number-theoretic extensions of such theories, we shall denote, without fixing the particular species of sets, any elementary natural-number-theory by $T_0(\mathrm{N})$ and the natural-number-theoretic extension of $T_0(\mathrm{N})$ by $T_1(\mathrm{N})$.

The necessity of the use of transfinite induction up to the first ε-number in the consistency proof of $\mathrm{T}_1(\mathrm{N})$ is not a consequence of Gödel's theorem, since $\mathrm{T}_1(\mathrm{N})$ does not yet contain the theory of recursive functions. Since, however, no special principle is required to extend $\mathrm{T}_1(\mathrm{N})$ consistently to the theory containing the theory of primitive recursive functions, the transfinite induction in the consistency proof of $\mathrm{T}_1(\mathrm{N})$ turns out not to be avoided by Gödel's theorem.

As is shown above, by applying Gentzen's method on any $T_1(\mathrm{N})$, it is proved that any set,[6)] "impredicative" or even self-contradictory, can be consistently used in $\mathrm{T}_1(\mathrm{N})$ as "set for induction". This is another important fact hidden in Gentzen's method.

A. Consistency of the theory $\mathrm{T}_0(\mathrm{N})$

3. Definition of $\mathrm{T}_0(\mathrm{N})$

A set m of a theory T is a dependent variable which is admitted in any proof of T to substitute for the bound variable of any T-proof formula of the form $\neg\forall x F^x$ when we associate[§11, (I)] with it the proof constituent $\overline{\neg F^m}$.

[10)] In this stage of consistency proof the significance of the consistency is still very weak, since the intuitive meaning of the abstract algebraic systems does not used in the consistency proof, so that the harmony between intuitive and formal knowledge can not yet be extended to the algebraic systems considered. Here is not the place to enter into the formulation of this problem precisely. However, these considerations, together with $T_\Sigma(\mathrm{N})$ discussed above, show some examples to combine abstract and concrete mathematics consistently.

The sets of the theory $T_0(N)$ are the universal set V, the null set 0, the set N of all natural numbers, and the elementary sets generated by V, 0, and N. The set N is defined by the formula

(N) $$\forall u\colon u \in N \equiv \forall x\colon 0 \in x \wedge \forall y.\ y \in x \to y' \in x.\ \to u \in x,$$

where y' stands for $\{y\}$. We call 0, 0′, 0″, . . . *numerals*, and denote them also by 0, 1, 2, ...

The species $\triangle\{V, 0, N\}$ of all the elementary sets generated by V, 0 and N are as follows:

(i) If $m_1, \ldots, m_k$ are V, 0, N or any independent variables, then $\{m_1, \ldots, m_k\}$ belongs to $\triangle\{V, 0, N\}$.

(ii) If $m_1, \ldots, m_k$ belong to $\triangle\{V, 0, N\}$, so is also $\{m_1, \ldots, m_k\}$.

We use the characterizing formulas for V and 0:

(V) $$\forall x.\ x \in V$$

(0) $$\forall x.\ x \notin 0$$

as premises in the theory $T_0(N)$. We use the defining formula (N) as premise in the proofs of $T_0(N)$ only under the restriction that the proof constituent associated with (N) should be affirmative, namely of the form

[NA] $$\overline{m \notin N \qquad \forall x\colon 0 \in x \wedge \forall y.\ y \in x \to y' \in x.\ \to m \in x.}$$

The negative proof constituent associated with (N):

[NN] $$\overline{m \in N \qquad \neg\forall x\colon 0 \in x \wedge \forall y.\ y \in x \to y' \in x.\ \to m \in x}$$

is not allowed to use in $T_0(N)$. The other formulas used as premises in $T_0(N)$ are only the formula

(I) $$\forall xyz.\ x = y \wedge x \in z \to y \in z$$

and the defining formula of $\{m_1, \ldots, m_k\}$:

(El) $$\forall u.\ u \in \{m_1, \ldots, m_k\} \equiv .\ u = m_1 \vee \ldots \vee u = m_k,$$

where k is a metalogical number and $m_1, \ldots, m_k$ are any sets of $T_0(N)$. These are all the formulas allowed to use as premises of proofs in $T_0(N)$. The theory $T_0(N)$ is, thus, defined.

The proof constituents [NN] used later in $T_1(N)$ corresponds to the inference of mathematical induction. Namely, a set p defined by

$$\forall u.\ u \in p^{x_1, \ldots, x_n} \equiv F^{u; x_1, \ldots, x_n}$$

is substituted for the bound variable x of the right formula of [NN] in order to prove $m \in p$ by the mathematical induction on the element variable u of p (see the deduction in Part (VII)). This inference is later adjoined to the theory $T_0(N)$ to formulate the theory $T_1(N)$, the consistency of which is proved in Section B. Therefore, $T_0(N)$ may be said to be the natural number theory without mathematical induction.

Besides [NA] we have the following proof constituents associated with the premises in $T_0(N)$. Namely, with (V) and (0) are associated

[V] $\qquad \overline{m \notin V},$

[0] $\qquad \overline{m \in 0},$

respectively. For elementary sets we use the composite proof constituents[§17, (II); §3, (IV)] associated with (El):

[El, A] $\qquad \overline{m \notin \{m_1, \ldots, m_k\} \quad m = m_1 \quad \cdots\cdots \quad m = m_k},$

[El, N] $\qquad \overline{m \in \{m_1, \ldots, m_k\} \quad m \neq m_1 \ldots m \neq m_k}.$

With (I), we associate the composite constituent:

[I] $\qquad \overline{m = n \quad m \in l \quad n \notin l},$

where $m, n, m_1, \ldots, m_k$ and l are any independent variables or sets of $T_0(N)$.

With the right formula [NA] is associated the composite proof constituent

[NA*] $\qquad \overline{\begin{array}{c} 0 \notin s \\ \neg \forall y.\ y \in s \rightarrow y' \in s \\ m \in s \end{array}}$

where s is the eigen variable of [NA*] and m an independent variable or a set of $T_0(N)$. With the middle formula of [NA*] is associated

[NA**] $\qquad \overline{m \in s \quad m' \notin s},$

where m is again an independent variable or a set of $T_0(N)$. These are all the decompositions of the premises of the theory $T_0(N)$.

Lastly, we fix the proof constituents associated with the equality. As $m = n$ is the abbreviation of the formula

$$\forall x.\ x \in m \equiv x \in n,$$

we associate to $m=n$ and $m \neq n$ respectively the composite proof constituents

[=A] $$\overline{\begin{matrix} s\in m & s\in n \\ s\notin n & s\notin m, \end{matrix}}$$

s being the eigen variable, and

[=N] $$\overline{\begin{matrix} l\in m & l\notin n \\ l\in n & l\notin m, \end{matrix}}$$

l being an independent variable or any set of $T_0(N)$.

4. Preliminaries for the consistency proof of $T_0(N)$

Let P be a proof of contradiction in $T_0(N)$ so that the top sequence of P consists of any arbitrary premises of $T_0(N)$ without the conclusion formula.

Without loss of generality we may assume for P the following two properties:

(g) There is no cut in P.[§15, (II)]

(h) (Primitive cancelling property.[§14, (II)]) Every P-string contains a pair of formulas of the form $(m\in n,\ m\notin n)$.

For simplicity we shall denote, after each transformation, the transformed proof by the same letter P, unless the contrary is explicitly mentioned.

We perform first the following preliminary transformation.

a*1 (Elimination of trivial free variables[§5, (IV)] from P) We replace every trivial (independent) variable in P by 0 (we may use other constant of $T_0(N)$ instead of 0).

The proof of consistency is based on the intuitive truth which we know on the ground of the recursive definition of elementary sets, of the characteristic property of 0 and V, and of the intuitive perception of the notion of natural numbers. Namely, for any set m of $T_0(N)$ and for any elementary set n of $T_0(N)$ we are sure by the definition of the elementary set whether $m\in n$ or $m\notin n$ holds. We have the same knowledge whether $m=n$ or $m\neq n$ for any sets m and n of $T_0(N)$. We say that $m\notin 0$ and $m\in V$ hold for any m and that $m\in 0$ and $m\notin V$ does not for any m. Lastly, we say that $m\in N$ holds, and $m\notin N$ does not, if and only if m is a numeral.

These intuitive knowledge is partly represented by the theory $T_0(N)$ in the sense that some part of intuitive truth is provable in $T_0(N)$, for instance, $0 \in N$, $0' \in N$, . . . , $V \in V$ etc. On the contrary, for instance, $V \notin N$ holds, but is not provable in $T_0(N)$. The consistency proof of $T_0(N)$ amounts to show that the formal knowledge of a $T_0(N)$-provable formula of the form $m \in n$ or $m \notin n$ for $T_0(N)$-constants m and n coincides with the intuitive knowledge that one of the $T_0(N)$ formulas $m \in n$ and $m \notin n$ is intuitively true and the other false.

5. Assignment of constant to eigen variable

Now, using our intuitive knowledge, we shall describe a way to assign a constant to every eigen variable of *P*. The proof constituent in *P* which has an eigen variable is, as listed in § 3, either [NA*] associated with the right formula of [NA], or [=A] associated with $m=n$. We proceed in *P* from above to below; so let *E* be one of the highest *P*-constituent with an eigen variable. Then, no free variable occurs in *E* except the eigen variable of *E*.

b*1 If *E* is [NA*], we assign N to the eigen variable of [NA*].

b*2 If *E* is [=A], and if the constants m and n are intuitively equal we assign any constant, for instance, 0 to the eigen variable [=A].

b*3 If *E* is [=A] and if m and n are intuitively different constants, we determine a constant l to be assigned to the eigen variable s of [=A] in such a way that if s is replaced in [=A] by l the two formulas either on the left or on the right hand side of [=A] become both false. This is possible owing to the fact that there are, according to our intuitive knowledge, indefinitely many number of numerals as well as of elementary sets different from numerals. In detail:

(i) If m is V or 0, then n is different from V or 0 respectively. We select l in such a way that $l \in n$ or $l \notin n$ does not hold, respectively;

(ii) If m is N and n is an elementary set, then l is determined as such a numeral that $l \notin n$.

(iii) If n is V, 0 or N, we determine l as in (i) and (ii) symmetrically with respect to m and n;

(iv) If m and n are (different) elementary sets, we select as l either an element of m which does not belong to n or an element of n which does not belong to m.

Thus the constant l is determined in each case.

Next, let E be a P-constituent with an eigen variable and assume that a constant has already been assigned to any eigen variable occurring over E. Then we replace every eigen variable w over E in P, together with all free variables w occurring under the eigen variable w, by the constant that has been assigned to w. The figure we get from P by this replacement is denoted by P^*. Let E^* be the figure similarly obtained from E. Then there occurs no free variable in E^* except the eigen variable, say v, of E. We assign a constant to v in a similar way to that described above.

In this way, proceeding downwards from above, we assign constants to all eigen variables in P, so that no free variable occurs in P^* at the final stage. The figure P^* is no more a proof. However, any string of P^* contains a pair of formulas of the form $(m^* \in n^*,\ m^* \notin n^*)$, owing to the property (h) of P.

6. Conditions for intuitive string

We shall prove in the next §7 that, starting from the top sequence, we can descend along an appropriate P-string called an ***intuitive string*** which satisfies the following conditions c*1, (i-iv) and c*2 (i-iv):

c*1 An affirmative P-formula of the form $m \in n$ can belong to the string only if $m^* \in n^*$ does not hold, namely either if

(i) n^* is 0,

(ii) n^* is N and m^* is not a numeral, or

(iii) n^* is an elementary set and m^* is not an element of n^*.

c*2 A negative formula of the form $m \notin n$ can belong to the string only if $m^* \notin n^*$ does not hold, namely either if

(i) n^* is V,

(ii) n^* is N and m^* is a numeral, or

(iii) n^* is an elementary set and m^* is an element of n^*.

c*1 An affirmative formula of the form $m = n$ can belong to the string only if

(iv) m^* and n^* are intuitively different.

c*2 A negative formula of the form $m = n$ can belong to the string only if

(iv) m^* and n^* are intuitively equal.

7. Determination of intuitive string

We have only to determine the direction of the intuitive string. When we are at the top of P, the conditions c*1-2 are trivially fulfilled. Assume, therefore, that we are directly upon a P-constituent E and that the conditions c*1-2 are satisfied over E along the string through E. We shall determine the direction to descend so as to go through false formulas. In the subsequent description we refer to the proof constituents written in § 3.

d*1 If E is [V] or [0], there is no alternative and the conditions (c*2, i) and (c*1, i) are satisfied, respectively, for the formula carried by E^*.

d*2 If E is [I] and if m^* and n^* are different, then we proceed to $m=n$, (c*1, iv),

d*3 If E is [I] and if m^* and n^* are intuitively equal, then we proceed to

(i) $n \notin l$, when l^* is V (c*2, i),

(ii) $m \in l$, when l^* is 0 (c*1, i),

(iii) $n \notin l$, when l^* is N, and $m^*(=n^*)$ is a numeral (c*2, ii),

(iv) $m \in l$, when l^* is N, and $m^*(=n^*)$ is not a numeral (c*1, ii),

(v) $n \notin l$, when l^* is an elementary set, and $m^*(=n^*)$ is an element of l^* (c*2, iii),

(vi) $m \in l$, when l^* is an elementary set, and $m^*(=n^*)$ is not an element of l^* (c*1, iii).

d*4 If E is [El, N], we proceed to

(i) $m \in \{m_1, \ldots, m_k\}$, when m^* is different from $m_1^*, \ldots, m_k^*$ (c*1, iii),

(ii) $m \neq m_i$, when m_i^* and m^* are intuitively equal (c*2, iv).

d*5 If E is [El, A] we proceed to

(i) $m \notin \{m_1, \ldots, m_k\}$, when m^* is intuitively equal to one of $m_1^*, \ldots, m_k^*$ (c*2, iii),

(ii) $m=m_1, \ldots, m=m_k$, when m^* is different from $m_1^*, \ldots, m_k^*$ (c*1, iv),

d*6 If E is [NA], then we proceed to

(i) $m \notin$N, when m^* is a numeral (c*2, ii),

(ii) the right, when m^* is not a numeral.

d*7 If E is [NA*], then there is no alternative. We have $s^*=$N by b*1 and m^* is not a numeral by d*6, (ii). Therefore the condition is satisfied (c*1, ii and c*2, ii).

d*8 If E is [NA**], then $s^*=$N, and m^* is not a numeral, as in d*7. So we proceed to $m \in s$ (c*1, ii).

d*9 If E is [=A], then $m=n$ is over E. Therefore, m^* and n^* are different by assumption. Hence s^* is determined by b*3. In b*3 the constant s^* has been so determined that either the left two formulas $s^*\in m^*$ and $s^*\notin n^*$ of E^* or the right $s^*\in n^*$ and $s^*\notin m^*$ are both false. We proceed to the direction where there are false formulas.

d*10 If E is [=N], then there is $m\neq n$ over E, whence $m^*=n^*$ by assumption. Therefore we proceed to the left or to the right, according as $l^*\in m^*$ or $l^*\notin m^*$ is false.

In each case we see that the conditions c*1 and c*2 are fulfilled. Therefore the intuitive string would be indefinitely long since we encounter in the intuitive string no pair of formulas of the form $m\in n$ and $m\notin n$, contradicting the assumed property (h) of P. Therefore there is no $T_0(N)$-proof of contradiction i.e. $T_0(N)$ is consistent.

B. Consistency of the theory $T_1(N)$

8. Definition of $T_1(N)$

The species of sets of $T_1(N)$ is the extension of that of $T_0(N)$ which we get by adjoining to the latter the *sets for induction* defined below. Any set for induction is allowed to use in $T_1(N)$ as set in a proof of $T_1(N)$ only in the way defined below.

First, we define the sets for induction as follows:

(i) Any set of the theory $T_0(N)$ is a set for induction.

(ii) Let $M=M^{x_1,\dots,x_n}$ be a dependent variable defined by the defining formula

$$(M) \qquad \forall u.\ u\in M\equiv F^{u;\,x_1,\dots,x_n}.$$

The variable M is a set for induction, exactly if in the definiens F of M occur, besides independent variables, only those dependent variables which are sets of the theory $T_0(N)$.

Next, we explain the way of use of a set for induction in a proof of $T_1(N)$. The proof constituent [NN] (cf. §3) associated with the defining formula of N is allowed to use in a proof of $T_1(N)$, and only any arbitrary set M for induction is allowed to substitute for the bound variable x of the right formula of [NN] in the proof constituent

$$[\mathrm{NN}^*] \qquad \overline{\neg\ 0\in M\wedge\forall y.\ y\in M\rightarrow y'\in M.\ \rightarrow m\in M}$$

associated with it. With the formula of [NN*] is associated the composite proof constituent

[NN**] $\dfrac{0\in M \quad \dfrac{s\notin M}{s'\in M} \quad m\notin M}{}$

where s is the eigen variable of [NN**].

Now, the proof constituents

[MA] $\overline{a\notin M \qquad F^{a}}$,

[MN] $\overline{a\in M \qquad \neg F^{a}}$,

associated with the defining formula of M are only allowed to use under [NN**] in such a way that the left formula $a\notin M$ or $a\in M$ of [MA] or [MN] makes the cancelling pair with a formula of [NN**] (so that a must be 0, s, s', or m)[11].

The sets for induction are not allowed to use in a proof of $T_1(N)$ in other way than described above, so that in other places of a proof of $T_1(N)$ any dependent variables except the sets of $T_0(N)$ are used only as concepts[(X)] or as sets for understanding.

All the decompositions of the premises of $T_1(N)$ have been given in this §8 and in §3, except the further decompositions of F^a and $\neg F^a$ in [MA] and [MN], respectively.

Thus the theory $T_1(N)$ is defined.

9. Outline of the proof of consistency of $T_1(N)$

Let P be a proof of contradiction in $T_1(N)$. We assume for P only the property (h) in §4, so that cuts may occur in P.[12] However, since we may assume that any dependent variable occurring in P is a set of $T_1(N)$ (concepts can be eliminated), the dependent variables occurring in any cut formula of P are exclusively sets of $T_0(N)$.

The outline of the proof of consistency of $T_1(N)$ is as follows. We first eliminate all trivial variables from P as in a*1, §4. We perform two kinds of procedures on P alternatingly, the one being intuitive procedures, quite similar to Section A, and the other being formal procedures, similar to those adopted by Gentzen in the proof of consistency of the natural number theory in his

[11] This condition does not restrict the usual way of inference by mathematical induction.

[12] Since cuts may occur in P, the top sequence of P may be void.

formulation. By the intuitive procedures we conclude, as before, that the procedures can be applicable indefinitely, while by the formal procedures that the procedures must come to an end after a finite number of applications of the procedures. This contradiction of the finiteness and the indefiniteness of the procedures proves the non-existence of the proof of contradiction in $T_1(N)$. For the proof of the finiteness of the formal procedures, an intuitive transfinite induction up to the first ε number is used in the same way as in Gentzen's proof.

10. Intuitive procedures

From the top of the proof P we descend downwards along (perhaps more than one) strings of P as follows. When we arrive at a position of P, directly under which a proof constituent given in §3, occurs, we proceed downwards exactly in the same way described in §7. Thereby, we assign a constant to every eigen variable of the P-constituent we are passing, just in the same way described in §5.

When we encounter a cut we go down to both cut formulas, provided that the cut is not primitive, i.e. that the cut is not of the form $\overline{m \in n \quad m \notin n}$. If the cut is primitive, i.e. of the above form, then m and n are by assumption sets of $T_0(N)$, and moreover m^* and n^* (the notation $*$ is as in §5) are constants. Therefore one of $m^* \in n^*$ and $m^* \notin n^*$ is intuitively true and the other false. We proceed to the direction of the false formula.

When we encounter [NN], then we descend to the left formula $m \in \mathrm{N}$ of [NN], if m^* is not a numeral; otherwise to the right.

When we encounter [NN*] we go down one step simply, and we stop when we arrive at a position which is either directly upon an [NN**] or upon a P-constituent associated with an (imprimitive) cut formula.

We call the part of the proof P we have thus travelled *intuitive part* of P, and a P-constituent directly under a bottom formula of the intuitive part of P a *boundary P-constituent* (of the intuitive part of P). The intuitive part of P is ramified at each place where we have crossed an imprimitive cut. But each string of the intuitive part of P can not reach to the bottom formula of P by the same reason as in Section A. Therefore, each string of the intuitive part ends at the formula directly under which there is a boundary constituent.

Our first intuitive procedure is thus finished. If all the boundary constituents are associated with imprimitive cut formulas we perform next the formal procedure described in § 15, and if there is at least one boundary constituent which is an [NN**] we perform the formal procedure described in the next § 11.

11. Replacement of induction by cuts

Let A be a bottom formula of the intuitive part of P and assume that [NN**] occurs directly under A as boundary constituent. Let the part of P situated under A be of the form:

$$(1) \qquad \begin{array}{ccc} \overline{0 \in M} & \overline{s \notin M} & \overline{m \notin M} \\ \vdots & s' \in M & \vdots \\ (1) & (s)\vdots & (3) \\ & (2) & \end{array}$$

where m^* is a numeral by our construction of the intuitive part in § 10. We replace this part of P by the figure

$$(2) \qquad \begin{array}{cc} \overline{0 \in M} & \overline{m \notin M} \\ \vdots & \vdots \\ (1) & (3) \end{array}$$

if $m^*=0$, or by the figure

$$(3) \qquad \begin{array}{llllll} \overline{0 \in M} & \overline{0 \notin M} & & & & \\ \vdots & \overline{0' \in M} & \overline{0' \notin M} & & & \\ (1) & (0)\vdots & \overline{0'' \in M} & \overline{0'' \notin M} & & \\ & (2) & (0')\vdots & \overline{\quad\ldots\quad} & \ldots & \\ & & (2) & & n \notin M & \\ & & & & m \in M & m \notin M\ , \\ & & & & (n)\vdots & \vdots \\ & & & & (2) & (3) \end{array}$$

if $m^* \neq 0$, where $m^*=n'$ and (0), $(0')$, . . . , (n) indicate the substitution of 0, $0'$, . . . , n, respectively, for the variable s in the part of P indicated by (2) in the figure (1).

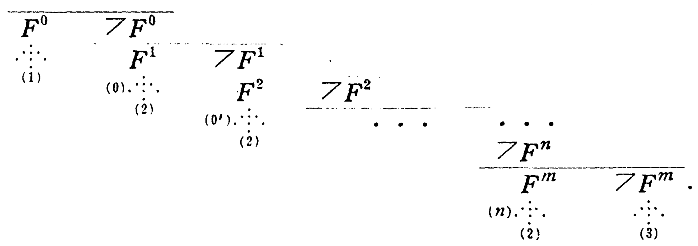
After this transformation P is no more a proof, since P may perhaps not have the cancelling property: for instance, some strings through $m \notin M$ may not contain cancelling pairs, unless m is replaced by m^*. However, we retain the associations of P-constituents with P-formulas in a natural manner as they were in the proof P before the transformation, the association of an [NN**] with the formula of [NN*] being naturally cut off, when the [NN**] are replaced by cuts.

Let P^* be, as before, the figure obtained from P in the same way as in Section A by replacing the independent variables, to which constants have already been assigned in the intuitive procedures, by these constants. The figure P^* also is not a proof, since the independent variable restriction [§11, (I)] is destroyed by the substitution of variables. However, P^* has the primitive cancelling property (h), which states that each P^*-string contains a pair of formulas of the form $m^* \in n^*$ and $m^* \notin n^*$.

Remark. In the sequal, as above, we consider transformed figures P and the corresponding figures P^* and we shall use only the association property of P and the property (h) of P^*. We need not other proof properties of P and P^*. So we call a figure P with the association property and the corresponding figure P^* with the property (h) also proofs and use the same terminology defined concerning a proof, for instance, the association of a proof constituent with a proof formula, cancelling pair of a proof string, superfluous formulas etc.

Now, we replace the formulas $a \in M$ and $a \notin M$ $(a=0, \ldots, n, m)$ in the figure (3) [the figure (2) is a special case of (3)] by their definiens F^a and $\neg F^a$ respectively. We obtain thus from (3) the figure

(4)



Under the formulas F^a and $\neg F^a$ of (4) there are P-constituents [MA] and [MN] (see §6), of which the right formula is F^a and $\neg F^a$, respectively, so that these [MA] and [MN] are superfluous, since we have over them the same formulas. We, therefore, erase all these [MA] and [MN] from P, together with the whole part which lie under the left formula of [MA] and [MN], and re-associate each P-constituent E which was associated with a right formula of these [MA] and [MN] to the same cut formula which is now found in (4) over E. After this transformation the association of P-constituents with P-formulas is defined and P^* has the property (h).

12. Repetition of procedures

By the formal transformation described in § 11 an [NN**] in P changes into a finite number of cuts in P^*. Let A be the bottom formula of the intuitive part of P directly under which the [NN**] occurred. We can then extend the original intuitive part of P downwards beyond A, since now under A occur cuts. This extension is performed in the same way as is described in § 10. Any string of the extended intuitive part of P can not reach to a bottom formula of P by the same reason, as before, that any string of the extended intuitive part of P^* can contain no primitive cancelling pair.

After the above transformation, if there remains still an [NN**] as boundary constituent, we perform[13] again the formal procedure of § 11, and then again an extension of the intuitive part, stated just above. Proceeding successively in this way, we finally obtain a "proof", say P_0, in which no [NN**] is found as boundary constituents of the intuitive part of P_0, so that all the boundary P_0-constituents are associated with imprimitive cut formulas.[14]

We shall perform another formal procedure, described in § 15, on P_0. Before doing this we need some preparation.

13. Existence of a particular cut

As is mentioned above, P_0 has the property that directly under any bottom formula of the intuitive part of P_0 there is a boundary P_0-constituent which is associated with an imprimitive cut formula. In virtue of this property of P_0 we shall prove that there is a "particular cut" in the intuitive part of P_0.

A cut formula in the intuitive part of P_0 is called ***particular*** if a boundary P_0-constituent is associated to it and a cut in the intuitive part is called ***particular***, if both cut formulas of the cut are particular. Thereby it is to be noticed that by our construction both cut formulas of an imprimitive cut belong or do not belong at the same time to the intuitive part of P_0.

In proving the above assertion we consider all the imprimitive cuts occurring in the intuitive part of P_0. Among them there is at least one particular

---

13) In order to fix the [NN**] on which the next procedure is applied, we select the boundary [NN**] with minimum lexicographic P-order §6, (1). Wherever there is an ambiguity in other places concerning the position of application of procedure, one may refer to the lexicographic P-order, which is, however, not mentioned explicitly in each case.

14) See the remark in § 11. See also § 18.

cut formula by the property of P_0 mentioned above. Let C_1 be one of the particular cut formulas at the highest position in P_0, i.e. let C_1 be a particular cut formula over which there is no particular cut formula. If the other cut formula $\neg C_1$ is also particular, then the cut $\overline{C_1 \ \neg C_1}$ is a particular cut. Otherwise, there should be a particular cut formula, say C_2, under $\neg C_1$. Otherwise, any string of the intuitive part of P_0 through $\neg C_1$ could touch no boundary constituent, contrary to the property of P_0. If $\neg C_2$ is not particular we can find a particular cut formula C_3 under $\neg C_2$. Proceeding in this way we arrive finally at a particular cut, namely a cut of which both cut formulas are particular.

14. Definition of the height of proof formula

Let P be a proof in $T_1(N)$. The *height of a P-formula* is defined as the maximum of the reduced degrees of the cuts and [NN**] of which the horizontal lines are situated over the P-formula. Herein, by the *reduced degree* of a cut or [NN**] is meant the number of the logical operators $\wedge$ and $\forall$ contained in a cut formula of the cut or in the definiens of the set for induction occurring in the [NN**], respectively; formulas being considered to be written by the primitive logical operators. Since the formulas attached to the same P-constituent have the same height, the height is an invariant of a P-constituent. The height of a P-top formula or the P-top sequence is 0. When the P-top sequence is vacant, the height of the vacant P-top sequence is also 0.

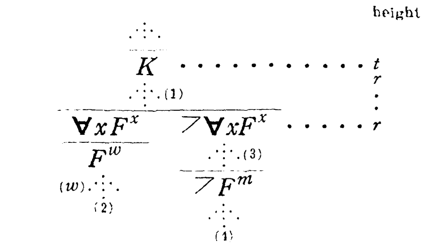
15. Formal transformation of a particular cut

Now, we return to our transformed proof P_0 and perform the following transformation by making use of the particular cut and the height of the P_0-formulas.

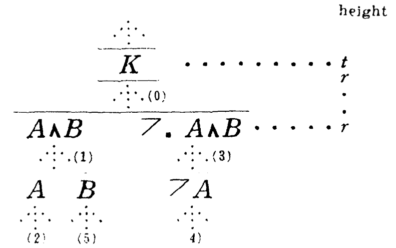
In § 13 it is shown that there is a particular cut in the intuitive part of P_0. Let $\overline{C \ \neg C}$ be one of them, and r the P_0-height of C. Since the particular cut is imprimitive, $r>0$. Hence, there is a P_0-formula (eventually the vacant P_0-top sequence) over C of which the height is smaller than r. Let K be the lowest P_0-formula over C which has a smaller height, say t, than r.

Since C is imprimitive, C is either of the form $\forall x F^x$ or $A \wedge B$; the cut formula C being assumed to be affirmative without loss of generality. Accordingly P_0 has either one of the following forms:

(a)

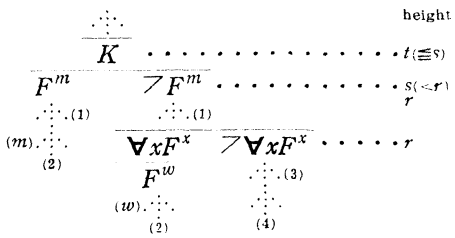


(b)

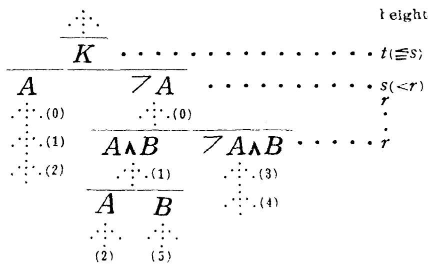


We transform (a) and (b) to (a′) and (b′), respectively as follows. Namely, a cut of reduced degree one smaller than that of the particular cut is inserted at the place where the height of formulas decreases from r to t. Other changes of the figure are done in order to preserve the proof properties.

(a′)



(b′)



(To the right of the above figures the heights of formulas are indicated. The inequalities between these heights shown there will be proved later in § 18.1.)

Denoting by P_1 the transformed figures (a′) and (b′), we retain the associations of P_1-constituents with P_1-formulas in the natural manner as they were in

P_0. Notice that th les which are eigen variables of P_0-constituents under K and over the particular cut of the figures (a) and (b), respectively.[15] If this is really the case, the independent variable restriction will not be preserved in P_1 also for these variables. This is, however, irrelevant for our proof (see the remark in § 11). It is clear that P_1^* has the property (h). It is to be noticed, thereby, that $(F^m)^*$ and A^* contain no eigen variables, since $\overline{\neg F^m}$ in (a) and $\overline{\neg A}$ in (b) are boundary P_0-constituents, which have no eigen variables.

16. Repetition of the procedures

After the transformation in § 15 of P_0 to P_1, we perform the intuitive procedure on P_1 as follows. Using the same assignment of constants to eigen variables as that in P_0, we observe P_1^* and determine the intuitive part of P_1.

If the reduced degree of the particular cut $C \neg C$, i.e. the cut $\overline{\forall x F^x \; \forall x F^x}$ in (a) or $A \wedge B \; \neg . \; A \wedge B$ in (b), is 1, then the inserted cut in (a′) and (b′) is primitive. Therefore, the intuitive part of P_1 must run through only one of the two primitive cut formulas of the inserted cut. On the contrary, if the reduced degree of the particular cut $\overline{C \; \neg C}$ is greater than 1, then the intuitive part of P_1 run through both cut formulas of the inserted cuts. Thereby, the remark given at the end of § 15 is to be noticed.

Next, since a boundary constituent of P_0 is omitted in P_1 between the parts indicated by (1) and (2) as well as by (3) and (4) in the figures (a′) and (b′), the intuitive part of P_1 may be extented. We perform this extension by the way described in § 10 without changing the constants already assigned to independent variables in P_0. In this way we determine the intuitive part of P_1.

Now, if there is an [NN**] in P_1 as boundary constituent of the intuitive part of P_1, we repeat the transformations, described in §§ 11, 12, with respect to P_1. Then, we get from P_1 a figure in which there is no [NN**] as boundary constituent. On this figure we apply the transformation in § 15, after looking for a particular cut.

In this way we perform the intuitive and formal procedures alternatingly.

---

15) In order to avoid the confusion which may arise by the transformation in this § 15 about the substitution of eigen variables by the constants assigned to them, we should remember for each P_1-formula the position where it was in P_0, or else we should transform the proof of each step of our transformations to a proof with distinct eigen variables.

As is mentioned in § 9 our procedures should continue indefinitely in virtue of the property of our intuitive procedures. On the contrary, we shall prove that our procedures should come to an end after a finite number of repetitions of the formal procedures.

17. Association of transfinite ordinal numbers with proofs

Let P be a proof in $T_1(N)$. We shall define a method to associate, from below to above, successively an ordinal number with each column[16] of a P-constituent, with each P-constituent, and with the P-top sequence. We denote, thereby, by $\alpha\dot{+}\beta$ the natural sum of the ordinal numbers α and β, i.e. the sum in the decreasing order of the summands. Mentioning this, we define the association as follows:

(i) We associate the number 1 with all P-bottom formulas.

(ii) Let E be a P-constituent, and $\alpha_1, \alpha_2, \ldots$ the numbers associated with the columns attached to E. We put

$$\alpha_i=\omega^{\beta_i}\dot{+}\ldots, \qquad (i=1, 2, \ldots),$$
$$\alpha=\alpha_1\dot{+}\alpha_2\dot{+}\ldots,$$
$$\beta=\mathrm{Max}(\beta_1, \beta_2, \ldots).$$

(ii, a) If E is different from [NN**] we associate $\alpha\dot{+}1$ with E.

(ii, b) If E is [NN**] we associate $\omega^{\beta+1}+\alpha$ with E.

(iii) Denoting by ε_0 the first ε-number, define $\varphi(n; \gamma)$ for $0\leq n<\omega$, $0\leq\gamma<\varepsilon_0$ recursively by

$$\varphi(0;\ \gamma)=\gamma, \qquad \varphi(n+1,\ \gamma)=\omega^{\varphi(n;\gamma)}$$

If α is the ordinal number associated with a P-constituent E, and if k is the difference of the heights between the formulas directly upon and under the horizontal line of E, we associate $\varphi(k, \alpha)$ with the formula directly upon E.

The ordinal number which is, thus, associated with the P-top sequence is called the *ordinal number of the proof P*.

The following lemmas are the direct consequences from the definition of the association of an ordinal number with a $T_1(N)$-proof.

LEMMA 1. *Let P and P′ be* $T_1(N)$*-proofs. Assume that if we place the*

[16] The ordinal number associated to a column of a P-constituent is called also the ordinal number associated to a formula which is in the column.

proof P upon the proof P' they coincide completely except the parts of P and P' which are situated under a coinciding formula, say A, of P and P'. Assume further that the ordinal number associated with the A in P' is smaller than that in P. Then the ordinal number of P' is smaller than that of P.

Proof. If A is a top formula of P, accordingly of P', the lemma is evident. Assume that A is not a top formula. Let E and E' be the P- and P'-constituents, respectively, which carry the A. By (ii, a) and (ii, b) and by the assumption of the lemma the number associated with E' in P' is smaller than that associated with E in P.[17] Then by (iii) and by the first assumption of the lemma the number associated with the P'-formula which is directly upon E' is smaller than the number associated with the P-formula which is directly upon E. This proves the lemma by the induction of the number of horizontal lines which are over the formula A in P.

LEMMA 2. *If a $T_1(N)$-proof P is transformed to a $T_1(N)$-proof P' by an erasing and connected method,*[§12, (II)] *then the ordinal number of P' is smaller than that of P.*

Proof. Let A be the P-formula directly under which the P-constituent, say E, occurs which is to be erased by the transformation. Then, P' is the proof obtained from P by erasing E, together with the parts of P which lie under all the columns but at most one attached to E, and by connecting the part of P which remains under the erased E directly under the formula A.

Assume first that the whole part of P which is under A has been erased by the transformation so that A becomes a P'-bottom formula. By the definition of the association of ordinal numbers to proof formulas the number associated with a bottom formula of a proof is 1 and the number associated to a formula which is not a bottom formula is greater than 1.[18] Hence lemma 2 follows from lemma 1.

Assume second that there is the remaining part of P which is to be connected in P' directly under A. The P'-number of A is smaller than the P-number of A. This is evident if E is neither a cut nor an [NN**], or if E is a cut or an [NN**] but the P-height of A is not smaller than the degree of E. We can prove the same in case of E being a cut or an [NN**], and the P-height

17) In order to make this inference valid, the α of $\omega^{\beta+1}+\alpha$ in (ii, b) is necessary.
18) In order to make this inference valid, the 1 of $\alpha+1$ in (ii, a) is necessary.

of A smaller than the degree of E. Hence lemma 2 follows from lemma 1.[19)]

In the same way we have

LEMMA 3. *Let P be a* $T_1(N)$*-proof and A a P-formula. If we transform P to another proof P' by an erasing and connecting method applied under the P-formula A, then the P'-number of A is smaller than the P-number of A.*

18. Proof for finiteness of the procedures

Let P be a $T_1(N)$-proof for contradiction. We shall prove that the procedures described in §§ 10-16 must come to an end after a finite number of repetitions of these procedures. The ordinal number of a proof remains the same when the intuitive procedures described in § 10 are applied. To prove the finiteness of our procedures, it is, therefore, sufficient to prove that the ordinal number of a proof becomes smaller if we apply the formal transformations in § 11 and § 15.

18.1. For § 11.

Let P be a proof in which an [NN**] occurs as boundary constituent of the intuitive part of P. Let the figure (1) in § 11 be the part of P which consists of the [NN**] as a boundary constituent and of the P-constituents under the [NN**]. By the procedures described in § 11 the figure (1) is first transformed to the figure (3) [the figure (2) can be treated as a special case of the figure (3)], and then to the figure (4). Lastly, by the erasing and connecting method the P-constituents [MA] and [MN] associated with the defining formula of M are erased from the figure (4), as is described at the end of § 11. We shall prove that the ordinal number of P becomes smaller by this transformation. Since by lemma 2 it is clear that the ordinal number decreases by the last transformation of erasing [MA] and [MN], we have only to prove that, denoting by P' the figure which is obtained from P by replacing the figure (1) by the figure (4), the ordinal number of P' is smaller than that of P.

Let λ be the P-number of the constituent [NN**] in the figure (1) and λ' the P'-number of the cut $\overline{F^0 \quad \neg F^0}$ in the figure (4). Since the transformation of P to P' is performed exclusively under the P-formula directly upon the

---

19) Owing to lemma 2 we may transform the proofs appearing in each step of our transformation to the proofs with properties (a), (b), (c′), and (d), given in Part (II).

[NN**] of P, it is, by lemma 1, sufficient to prove $\lambda'<\lambda$.[20] It is, however, to be noticed thereby that the P'-heights of all the P'-cuts which are explicitly written in the figure (4) are equal to the P-height of the [NN**] in the figure (1), owing to the definition of the height of constituents.

Now, let α_1, α_2, α_3 be the P-numbers associated with the left, middle, and right columns of the [NN**] in the figure (1). Then, by the definition (ii, b) in § 17, we have

$$\lambda=\omega^{\beta+1}+\alpha_1\dot{+}\alpha_2\dot{+}\alpha_3 \tag{5}$$

where $\beta=\mathrm{Max}(\beta_1, \beta_2, \beta_3)$ and $\alpha_i=\omega^{\beta_i}\dot{+}\ldots$ $(i=1, 2, 3)$.

On the other hand, the P'-number of the P'-formulas $F^0, F^1, F^2, \ldots, F^m$, $\neg F^m$ in the figure (4) are respectively equal to $\alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_2, \alpha_3$. Since the P'-heights of the cuts written in figure (4) are all equal, as is mentioned above, the ordinal number λ' is a natural sum of α_1, α_2, α_3 and 1 with a finite number of repetitions. Therefore we have in virtue of (5)

$$\lambda'\leq\omega^{\beta}\dot{+}\ldots<\omega^{\beta+1}<\lambda,$$

which proves our assertion.

18.2. For § 15.

Let λ and λ' be the ordinal numbers of the formula K before and after the transformation (a)→(a′) or (b)→(b′) in § 15 respectively. We shall prove $\lambda'<\lambda$. Then, by lemma 1, we see that the ordinal number of the transformed proof is smaller than that of the proof before the transformation.

We prove $\lambda'<\lambda$ for the case of the transformation (a)→(a′), since the case (b)→(b′) can be treated similarly.

Since the formula K in (a) is at the place where the heights of formulas decrease from r to t, the heights of the formulas in (a) which are under K and over the particular cut (including) are all equal to r.

Now, we observe the heights of formulas in (a′). The figures over K are the same in (a) and in (a′), so that the height of K in (a′) is also t. Since the degree of the inserted upper cut in (a′) is exactly one smaller than that of the lower cut explicitly written in (a′), the heights of the cut formulas $\forall x F^x$ and

[20] Note that owing to the definition of the height of a formula of a $T_1(N)$-proof (§ 14) it follows from $\lambda'<\lambda$ that the P-number of the formula directly upon the cut $\overline{F^0 \quad \neg F^0}$ in (4) is smaller than the P-number of the formula directly upon the [NN**] in (1).

$\neg\forall x F^x$ in (a′) are also r.

Let s be the height in (a′) of the formulas F^m and $\neg F^m$. We shall prove $t \leq s < r$. Since $t \leq s \leq r$ is clear, we have only to prove $s < r$. Let E be the constituent in (a) situated directly under K, accordingly in (a′) directly under F^m and $\neg F^m$. Since the heights in (a) of the formulas carried by E are r, and since the height t of K is smaller than r, E must be a cut[21] of which the degree is r. On the other hand, the degree of the lower cut in (a′) can not be greater than r, or the degree of the upper cut in (a′) can not be greater than $r-1$. Therefore, in virtue of $t < r$, we have $s \leq r-1 < r$.

From the above consideration we see also that the height of a formula in (a′) under F^m and $\neg F^m$ is equal to or smaller than the height of the formula in (a) which is at the corresponding position in (a).

Let, now, α be the ordinal number of the constituent in (a) directly under K, and α_1 and α_2 the ordinal numbers of the constituents in (a′) directly under F^m and $\neg F^m$, respectively. Since the figure in (a′) under the cut formula $\neg F^m$ differs from that in (a) under K only in erasing the constituent $\overline{\neg F^m}$ up to the substitution of m for w, we have by lemma 3

(6) $$\alpha_2 < \alpha.$$

On the other hand, we have again by lemma 3

(7) $$\alpha_1 < \alpha,$$

since the figure in (a′) under the cut formula F^m is obtained by erasing from the figure in (a) under K the constituent $\overline{F^w}$ and the cut $\overline{\forall x F^x \ \neg\forall x F^x}$, together with the part under the cut formula $\neg\forall x F^x$; up to the substitution of m for w.

The ordinal number λ of K in (a) is by our definition (iii) in § 17

(8) $$\lambda = \varphi(r-t;\ \alpha) = \varphi(s-t;\ \varphi(r-s;\ \alpha))$$

Let μ_1 and μ_2 be the ordinal numbers of the cut formulas F^m and $\neg F^m$ in (a′), respectively. We have

(9) $$\mu_1 = \varphi(r-s;\ \alpha_1),$$

(10) $$\mu_2 = \varphi(r-s;\ \alpha_2).$$

21) Since E is in the intuitive part, E is not an [NN**].

Hence, the ordinal number, say ν, of the upper cut in (a′) is

$$\nu=\mu_1+\mu_2+1 \tag{11}$$

and the ordinal number λ' of K in (a′) is $\lambda'=\varphi(s-t;\ \nu)$. Hence, in order to prove $\lambda'<\lambda$, we have, in virtue of (8), only to prove $\nu<\varphi(r-s;\ \alpha)$. But this follows from (6), (7), (9), (10) and (11).

Mathematical Institute
Nagoya University