AN INVESTIGATION ON THE LOGICAL STRUCTURE OF MATHEMATICS (VI)[0]

CONSISTENT V-SYSTEM T(V) (WITH CORRECTIONS TO PART (XII))

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The V-system T(V) is defined in §2 by using §1, and its consistency is proved in §3. The definition of T(V) is given in such a way that the consistency proof of T(V) in §3 shows a typical way to prove the consistency of some subsystems of UL. Otherwise we could define T(V) more simply by using truth values. After T(V)-sets are treated in §4, it is proved in §5 as a T(V)-theorem that T(V)-sets are all equal to V. In this proof a peculiar T(V)-set $\tilde{R}$, defined similarly as Russell's contradictory set R, is used. The V-system itself is a trivial subsystem of UL, while such a subsystem S of T(V) is important, for which S-unprovability of $\forall x.\ x = V$ is known (§6). As an example for such a subsystem of T(V), the consistent natural number theory $T_1(\tilde{N}) \cap T(V)$ is developed in §7.[0] The sequence V, {V}, {{V}}, . . . is the sequence of natural numbers in $T_1(\tilde{N})$. In §8 the dual 0-system T(0) is defined. As a consequence, the most general duality principle in logic is obtained. In §9 the method of making use of the V-system in proving the consistency of some part of mathematics is briefly described. As an appendix a method to construct an infinitely many number of inconsistent systems of dependent variables is added.

1. Decomposition of a formula in UL

Let G be any formula in UL. The trees T_0, T_1, . . . are constructed successively from G as follows. T_0 is the tree consisting only of G. Assume that $T_k (k \geq 0)$ has been constructed, and that there is a bottom formula, say

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[0] Continuation of the author's previous work with the same major title. This Part (VI) presupposes in particular the terminologies and the knowledge in Parts (I) and (II), forthcoming in Hamburger Abhandlungen, and §§7, 9 of this Part (VI) can only be understandable after reading §§7, 10, 11 of Part (VII), appearing in this volume. The definition of concept and set is given in Part (X) forthcoming elsewhere. The references indicated by upper suffixes (I), (II), . . . refer to the Part (I), (II), . . . respectively.

K, of T_k, over which there is an imprimitive formula, say L, such that in the T_k-string through K there is under L no proof constituent which can be associated to L. T_{k+1} is constructed by placing a proof constituent associated to L directly under K. Any tree T_0, T_1, . . . obtained in this way from G is called a *decomposition of the formula G*. There is a finite number of decompositions of a formula, if the difference of the variables used in the decompositions is ignored.

A string S of a tree is called *affirmative* (*negative*), if all the primitive S-formulas, if any, are affirmative (negative). If in any decomposition of a formula G there is an affirmative (negative) string, G is called to be of *affirmative* (*negative*) *type*. Let T_m be a decomposition of a formula G of affirmative (negative) type and T_k be a decomposition of G which is a prolongation of T_m. Then there is an affirmative (negative) T_m-string S such that for any prolongation T_k of T_m there is an affirmative (negative) T_k-string which is a prolongation of S. Such a T_m-string is called an *affirmative* (*negative*) *direction* of the decomposition T_m of the formula G of affirmative (negative) type. A formula can at the same time be of affirmative and of negative type. For instance, the negation of the formula

(I) $$\forall xyz.\ x=y \wedge x \in z \rightarrow y \in z$$

is of affirmative as well as of negative type.[1]

2. Definition of V-system T(V)

The V-system is a subsystem of UL, which we shall denote simply by T(V). To define T(V) we have only to define the species of sets[2] of T(V). Let, first, p be a dependent variable in UL of order 0, which is defined by

(1) $$\forall u.\ u \in p \equiv F^u.$$

Then p is a T(V)-set, if the negation $\neg F$ of the definiens of p is of negative type. Let, second, p be a dependent variable in UL of order $n (n \geq 1)$, defined by (1). Then p is a T(V)-set, if all the dependent variables (of order less than n) occurring in F are T(V)-sets and further if $\neg F$ is of negative type. There is no T(V)-set other than defined recursively in this way. Thus

[1] The formula (I) is neither of affirmative nor of negative type.

[2] Roughly speaking, the sets[(X)] of a theory are those dependent variables which are allowed to substitute for bound variables in any proof of the theory.

the species of T(V)-sets is defined. The universal constant V is a T(V)-set and the null constant is not.

3. Consistency of T(V)

Assume that T(V) is inconsistent. Then there would be a T(V)-proof P of the second normal form without cut for a contradiction $\sigma\vdash$ with premises σ, where all the variables defined in σ are T(V)-sets. By the definition of T(V)-set we can conclude that there should be a negative P-string, contradicting the primitive cancelling property of P, and consequently T(V) is consistent.

To determine a negative P-string we descend, as follows, along a P-string from the P-top sequence $\neg\sigma$. Since all the P-top formulas are negative there is at least one P-constituent. Let E_1 be the uppermost P-constituent. If E_1 is associated with a defining formula, say (1), in σ, then E_1 is either [DA] or [DN]:

[DA] $$\overline{m\notin p \qquad F^m},$$

[DN] $$\overline{m\in p \qquad \neg F^m}.$$

In the former case we descend to $m\notin p$ and in the latter to $\neg F^m$. If E_1 is associated with the premise (I), namely if E_1 is of the form

[I] $$\overline{m=n \quad m\in l \quad n\notin l}$$

we descend to $n\notin l$, i.e. to the negative direction of the decomposition of (I).

Assume now that we have descended to a P-formula directly under which there is a P-constituent E_k ($k\geq 1$). When E_k is associated with a defining formula we descend to a formula of E_k in the same way as above. Otherwise, assume that E_k belongs to a decomposition of the formula (I) or of the negation of a definiens attached to a [DN] over E_k, and further that all the formulas, occurring over E_k and carried by the decomposition belong to a negative direction of the decomposition. This assumption is fulfilled trivially for $k=1$. We descend to a formula of E_k, which belongs to a negative direction of the decomposition. This is possible, since the formula (I) as well as the negation of the definiens of a T(V)-set is of negative type and since, as is stated at the end of §1, there is a negative direction for any decomposition of a formula of negative type. Thus we proceed an inductive step, and the P-string we are descending must be a negative P-string.

4. T(V)-sets

First of all, the elementary sets $\{m_1, \ldots, m_k\}$ and accordingly the ordered k-tuple $\langle m_1, \ldots, m_k\rangle$ are T(V)-sets, provided that $m_1, \ldots, m_k$ are T(V)-sets or independent variables. All the other dependent variables used in the fundamental deductions in Part (III) are also T(V)-sets. Therefore, all the formulas proved in Part (III) are theorems of the consistent V-system T(V). However, the complement $C^E a$ of a in E, defined by

$$(2) \qquad \forall u.\ u \in C^E a \equiv u \in E \wedge u \notin a$$

is not a T(V)-set; nor the complement $Ca = C^V a$ of a in V, which is defined by the specialization[II] of (2):

$$(3) \qquad \forall u.\ u \in Ca \equiv u \notin a.$$

5. Proof of $\forall x.\ x = \mathrm{V}$ in T(V)

Since V is a T(V)-set, the constant $\tilde{\mathrm{R}}$ defined by

$$(4) \qquad \forall u.\ u \in \tilde{\mathrm{R}} \equiv u \notin u \vee u = \mathrm{V}$$

is a T(V)-set. A similar proof as that of Russell's contradiction, when applied to $\tilde{\mathrm{R}}$, yields the following T(V)-theorem.

$\tilde{\mathrm{V}}*1 \qquad \tilde{\mathrm{R}} = \mathrm{V}$

Proof

$[\tilde{\mathrm{R}}]^1$ ——— $\tilde{\mathrm{V}}*1$ ——— Cut

2 $\quad \tilde{\mathrm{R}} \in \tilde{\mathrm{R}}$	2 $\quad \tilde{\mathrm{R}} \notin \tilde{\mathrm{R}}$
$-\quad \tilde{\mathrm{R}} \notin \tilde{\mathrm{R}} \vee \tilde{\mathrm{R}} = \mathrm{V}$	$-\quad \neg.\ \tilde{\mathrm{R}} \notin \tilde{\mathrm{R}} \vee \tilde{\mathrm{R}} = \mathrm{V}$
$\tilde{\mathrm{R}} \notin \tilde{\mathrm{R}}$	$\underset{(2)}{\tilde{\mathrm{R}} \in \tilde{\mathrm{R}}} \qquad \underset{(1)}{\tilde{\mathrm{R}} \neq \mathrm{V}}$
Spf. $\underset{(2)}{\tilde{\mathrm{R}} = \mathrm{V}}$	

By using $\tilde{\mathrm{V}}*1$ we have the T(V)-theorem:

$\tilde{\mathrm{V}}*2 \qquad a \neq \mathrm{V} \rightarrow a \notin a$

Proof

$-\quad \tilde{\mathrm{V}}*2$

$1\quad a = \mathrm{V}$

$2\quad a \notin a \quad$ Cut $\tilde{\mathrm{V}}*1$

$-\quad \tilde{\mathrm{R}} \neq \mathrm{V}$

$-\quad \neg \forall x.\ x \in \tilde{\mathrm{R}}$

$-\quad a \notin \tilde{\mathrm{R}}$

$-\quad \neg.\ a \notin a \vee a = \mathrm{V}$

$\underset{(2)}{a \in a} \qquad \underset{(1)}{a \neq \mathrm{V}}$

Corresponding to Ca, defined by (3), we define *a* T(V)-set $\c{C}a$ by

$$\forall' u.\ u \in \tilde{C}a \equiv u \notin a \vee a = \mathrm{V}.$$

Then we have the following T(V)-theorems: $\tilde{\mathrm{V}}*3$ by using $\tilde{\mathrm{V}}*2$; $\tilde{\mathrm{V}}*4$ by using $\tilde{\mathrm{V}}*3$; $\tilde{\mathrm{V}}*5$ directly from $\tilde{\mathrm{V}}*4$; and $\tilde{\mathrm{V}}*6$ directly from $\tilde{\mathrm{V}}*5$.

$\tilde{\mathrm{V}}*3$ $\quad \tilde{C}\{\mathrm{V}\} = \mathrm{V}$

Proof

$$\begin{array}{ll}
[\tilde{C}\{\mathrm{V}\}]^{1} & \tilde{\mathrm{V}}*3 \quad \text{Cut } \tilde{\mathrm{V}}*2 \\
\hline
- & \neg.\ \tilde{C}\{\mathrm{V}\} \neq \mathrm{V} \rightarrow \tilde{C}\{\mathrm{V}\} \notin \tilde{C}\{\mathrm{V}\} \\
\hline
\underset{(1)}{\tilde{C}\{\mathrm{V}\} \neq \mathrm{V}} \quad - & \tilde{C}\{\mathrm{V}\} \in \tilde{C}\{\mathrm{V}\} \\
\hline
- & \tilde{C}\{\mathrm{V}\} \notin \{\mathrm{V}\} \vee \{\mathrm{V}\} = \mathrm{V} \\
\hline
- & \tilde{C}\{\mathrm{V}\} \in \{\mathrm{V}\} \\
\text{Spf.} & \{\mathrm{V}\} = \mathrm{V} \\
\hline
 & \underset{(1)}{\tilde{C}\{\mathrm{V}\} = \mathrm{V}}
\end{array}$$

$\tilde{\mathrm{V}}*4$ $\quad \{\mathrm{V}\} = \mathrm{V}$

Proof

$$\begin{array}{ll}
1 & \tilde{\mathrm{V}}*4 \quad \text{Cut } \tilde{\mathrm{V}}*3 \\
\hline
- & \tilde{C}\{\mathrm{V}\} \neq \mathrm{V} \\
\hline
- & \neg \forall x.\ x \in \tilde{C}\{\mathrm{V}\} \\
\hline
[\mathrm{V}]\ - & \mathrm{V} \notin \tilde{C}\{\mathrm{V}\} \\
\hline
- & \neg.\ \mathrm{V} \notin \{\mathrm{V}\} \vee \{\mathrm{V}\} = \mathrm{V} \\
\hline
- & \mathrm{V} \in \{\mathrm{V}\} \qquad \underset{(1)}{\{\mathrm{V}\} \neq \mathrm{V}} \\
\hline
 & \underset{(=)}{\mathrm{V} = \mathrm{V}}
\end{array}$$

$\tilde{\mathrm{V}}*5$ $\quad a = \mathrm{V}$

$\tilde{\mathrm{V}}*6$ $\quad a \in b$

The T(V)-sets used in the above proofs are indicated in the brackets [ ] at the place of substitution for bound variables. These are $\tilde{\mathrm{R}}$ in $\tilde{\mathrm{V}}*1$, $\tilde{C}\{\mathrm{V}\}$ in $\tilde{\mathrm{V}}*3$, and V in $\tilde{\mathrm{V}}*4$. Taking the closure of the species of these sets, the T(V)-sets $\tilde{\mathrm{R}}$, $\tilde{C}\{\mathrm{V}\}$, $\{\mathrm{V}\}$, and V are used in the above deductions. Denote by τ the sequence of the defining formulas of these sets. (The proof of V$*$1 shows an example to use a singular cut in the deduction of a consistent system.)

6. Subsystem of T(V)

Let Σ be a subspecies of the species of all T(V)-sets, and T_Σ the theory of which Σ is the species of sets. The theory T_Σ, as subsystem of the consistent theory T(V), is consistent. In T_Σ the formula $m \neq \mathrm{V}$ is unprovable for any m belonging to Σ; otherwise $m \neq \mathrm{V}$ would be a T(V)-theorem, contradicting in virtue of $\tilde{\mathrm{V}}*5$ the consistency of T(V). If, on the contrary, it is known that $m_1 = \mathrm{V}, \ldots, m_k = \mathrm{V}$ are unprovable for some $m_1, \ldots, m_k$ belonging to Σ, then a T_Σ-theorem of the form

(5) $$\sigma \vdash.\ m_1 \neq \mathrm{V} \wedge \ldots \wedge m_k \neq \mathrm{V} \wedge a_1 \neq \mathrm{V} \wedge \ldots \wedge a_l \neq \mathrm{V} \rightarrow H,$$

where $a_1, \ldots, a_l$ are independent variables occurring free in H, means more than the fact that the formula obtained from (5) by replacing σ by τ (see at the end of § 5) is a trivial consequence from $\tilde{V}*5$.

7. Natural-number theory $T_1(\tilde{N})$

We define $\tilde{N}$ by replacing the null constant 0 in the defining formula[(VII)] of N by the universal constant V. Namely, $(y'=\{y\})$

$$(\tilde{N}) \qquad \forall u\colon u\in\tilde{N}\equiv\forall x\colon V\in x\wedge\forall y.\ y\in x\rightarrow y'\in x.\rightarrow u\in x.$$

The constant N is not a T(V)-set, while $\tilde{N}$ is a T(V)-set. Let Σ be the species consisting of V, $\tilde{N}$, and any elementary sets generated by V and $\tilde{N}$. Then T_Σ is a subsystem of T(V). We define the theory $T_1(\tilde{N})$ by adjoining to Σ any sets for induction in the same way as in defining[(VII)] $T_1(N)$. $T_1(\tilde{N})$ is not a subsystem of T(V). The subsystem $T(V)\cap T_1(\tilde{N})$ is the part of $T_1(\tilde{N})$ in which only the T(V)-sets are allowed to use as sets for induction. $N\neq V$ is a $T_1(N)$-theorem,[(VII)] while $\tilde{N}\neq V$ is T(V)-unprovable. On the other hand, we can prove not only the consistency of $T_1(\tilde{N})$ but also the $T_1(\tilde{N})$-unprovability of $\tilde{N}=V$ just in the same way as in the consistency proof[3] of $T_1(N)$. Hence we have a parallelism between two consistent theories $T_1(N)$ and $T_1(\tilde{N})$. Namely as follows.[4]

The $T_1(\tilde{N})$-theorem corresponding to $T_1(N)$-theorem $*N*1$ is

$$*\tilde{N}*1 \qquad V\in\tilde{N}.$$

The $T_1(N)$-theorem $N*2$ and $N*3$ are also $T_1(\tilde{N})$-theorems with $\tilde{N}$ instead of N, while, corresponding to $N*4$, we have

$$\tilde{N}*4 \qquad \tilde{N}\neq V\rightarrow a'\neq V.$$

The proof of $\tilde{N}*4$ is the same as that of $V*2$ by using $\tilde{N}$ instead of 0. In virtue of $\tilde{N}*4$ we can prove under the assumption $\tilde{N}\neq V$ that $\tilde{N}$ is an infinite set. Namely, in the same way as in Part (VII), § 10, we have the $T_1(\tilde{N})$-theorems:

$$\tilde{N}\neq V\rightarrow V^{(m)}\neq V^{(n)}, \qquad (m\neq n,\ V^{(0)}=V,\ V^{(n+1)}=\{V^{(n)}\}),$$

$$\tilde{N}\neq V\rightarrow V\neq\{a_1, \ldots, a_n\}. \qquad (m \text{ and } n \text{ are metalogical numbers.})$$

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3) The transfinite induction up to first ε-number is used in the consistency proof of $T_1(N)$ which is given in Part (VIII). The author can not decide at present whether $\tilde{N}\neq V$ is $T_1(\tilde{N})$-provable or unprovable.

4) In this § 7 the proofs given in §§ 7, 10, 11 of Part (VII) are presupposed. The formula indicated by $N*$ and $V*$ are all found there.

From N*5 and N*6 we get $\tilde{\mathrm{N}}$*5 and $\tilde{\mathrm{N}}$*6, if we replace 0 and N in N*5 and N*6 by V and $\tilde{\mathrm{N}}$, respectively. Hence, Peano's system of axioms holds in $\mathrm{T}_1(\tilde{\mathrm{N}})$ under the assumption $\tilde{\mathrm{N}} \neq \mathrm{V}$. More precisely, corresponding to a $\mathrm{T}_1(\mathrm{N})$-theorem, say $\sigma \vdash H$, which is deduced from Peano's system of axioms, either $\tilde{\sigma} \vdash \tilde{H}$ or $\tilde{\sigma} \vdash .\ \tilde{\mathrm{N}} \neq \mathrm{V} \rightarrow \tilde{H}$ is provable as $\mathrm{T}_1(\tilde{\mathrm{N}})$-theorem, where $\tilde{\sigma}$ and $\tilde{H}$ are results of replacing 0 and N in σ and H by V and $\tilde{\mathrm{N}}$, respectively. For instance, instead of N*9 we have

$\tilde{\mathrm{N}}$*9 $$\tilde{\mathrm{N}} \neq \mathrm{V} \wedge a \in \tilde{\mathrm{N}} \rightarrow a' \neq a.$$

This is proved by the induction with the set for induction P defined by $u \in \mathrm{P} \equiv u' \neq u$ and by using Cut $\tilde{\mathrm{N}}$*4 with V for a in $\tilde{\mathrm{N}}$*4.

In Part (VII) N*9 is deduced from N*7 and N*8 by making use of the speciality of the definition of N instead of Peano's system of axioms. This method can also be applicable for $\mathrm{T}_1(\tilde{\mathrm{N}})$. Namely, we can prove

$\tilde{\mathrm{N}}$*7 $$\tilde{\mathrm{N}} \neq \mathrm{V} \wedge a \neq \mathrm{V} \wedge a \in \tilde{\mathrm{N}} \rightarrow a \notin a,$$

$\tilde{\mathrm{N}}$*8 $$\tilde{\mathrm{N}} \neq \mathrm{V} \wedge a \neq \mathrm{V} \wedge a \in \tilde{\mathrm{N}} \rightarrow a' \subseteq a.$$

To prove $\tilde{\mathrm{N}}$*7 we use the T(V)-set $\tilde{\mathrm{R}}$ defined by (4) as set for induction instead of Russell's set R in the proof of N*7, and use a cut by $\tilde{\mathrm{N}}$*4 with V for a. The proof of $\tilde{\mathrm{N}}$*8 is the same as that of N*8 by using cut by $\tilde{\mathrm{N}}$*7.

$\tilde{\mathrm{N}}$*10 $$a \in \tilde{\mathrm{N}} \wedge a \neq \mathrm{V} \rightarrow \overset{\tilde{\mathrm{N}}}{\exists} x.\ a = x',$$

$\tilde{\mathrm{N}}$*11 $$\tilde{\mathrm{N}} \neq \mathrm{V} \wedge a' \in \tilde{\mathrm{N}} \rightarrow a \in \tilde{\mathrm{N}}$$

are the $\mathrm{T}_1(\tilde{\mathrm{N}})$-theorems corresponding to N*10 and N*11. The set for induction to prove $\tilde{\mathrm{N}}$*10 is defined as in the proof of N*10 by using V instead of 0. The $\mathrm{T}_1(\mathrm{N})$-theorems V*5-V*7 are also $\mathrm{T}_1(\tilde{\mathrm{N}})$-theorems.

As for N*N i.e. the $\mathrm{T}_1(\mathrm{N})$-theorem $\mathrm{N} \notin \mathrm{N}$, we have the $\mathrm{T}_1(\tilde{\mathrm{N}})$-theorem:

$\tilde{\mathrm{N}}$*$\tilde{\mathrm{N}}$ $$\tilde{\mathrm{N}} \neq \mathrm{V} \equiv \tilde{\mathrm{N}} \notin \tilde{\mathrm{N}}.$$

The proof of $\tilde{\mathrm{N}} \neq \mathrm{V} \rightarrow \tilde{\mathrm{N}} \notin \tilde{\mathrm{N}}$ can be done in the same way as in the proof of N*N and its converse is clear.

Corresponding to N*$\mathfrak{P}$ we have the $\mathrm{T}_1(\tilde{\mathrm{N}})$-theorem

N*$\mathfrak{P}$ $$\tilde{\mathrm{N}} \neq \mathrm{V} \rightarrow \mathfrak{P}^{(n)}(\tilde{\mathrm{N}}) \neq \mathrm{V}, \qquad (n = 0, 1, 2, \ldots).$$

In the above deduction, we used two sets for induction which are not T(V)-

sets: namely the set P with the definiens $u' \neq u$ in the proof of $\tilde{N}*9$ and the set P with the definiens $u \neq \tilde{N}$ in the proof of $\tilde{N} \neq V \rightarrow \tilde{N} \notin \tilde{N}$. However, in the second proof of $\tilde{N}*9$ we used the T(V)-set $\tilde{R}$. Therefore all the $T_1(\tilde{N})$-theorems proved above, except $\tilde{N}*\tilde{N}$, are $T_1(\tilde{N}) \cap T(V)$-theorems.

8. Consistent 0-system T(0)

Dually to T(V) we define the 0-system T(0). Namely, a dependent variable p defined by (1) is called a T(0)-set exactly if all the dependent variables, if any, occurring in the definiens F^u of p, are T(0)-sets and further if F^u is of affirmative type. The null constant 0 is a T(0)-set, while V is not. The 0-system is consistent.

The elementary sets $\{a\}$, $\{a, b\}$, . . . and ordered tuples $\langle a, b\rangle$, . . . are not T(0)-sets. However, if we define $^*\{a\}$, $^*\{a, b\}$, . . . $^*\langle a, b\rangle$, . . . by

$$\forall u.\ u \in {}^*\{a\} \equiv u \neq a, \qquad \forall u.\ u \in {}^*\{a, b\} \equiv u \neq a \wedge u \neq b$$
$$\forall u.\ u \in {}^*\langle a, b\rangle \equiv u \neq {}^*\{a\} \wedge u \neq {}^*\{a, b\}$$

and so on, these dependent variables are T(0)-sets and we can prove

$$^*\langle a_1, \ldots, a_n\rangle = {}^*\langle b_1, \ldots, b_n\rangle \equiv .\ a_1 = b_1 \wedge \ldots \wedge a_n = b_n.$$

So we sometimes look upon $\{\cdot\cdot\}$, $\langle\cdot\cdot\rangle$ as $^*\{\cdot\cdot\}$, $^*\langle\cdot\cdot\rangle$, respectively, when we are speaking about 0-system. By using the T(0)-set $^*\tilde{R}$ defined by $\forall u.\ u \in {}^*\tilde{R} \equiv u \notin u \wedge u \neq 0$ we can prove the T(0)-theorem $^*\tilde{R} = 0$ just in the same way as in the proof of $\tilde{V}*1$. A similar proof to that of $\tilde{V}*2$ yields the T(0)-theorem $a \neq 0 \rightarrow a \in a$, which is the dual theorem of $\tilde{V}*2$. Using $^*\tilde{C}a$ defined by $\forall u.\ u \in {}^*\tilde{C}a \equiv u \notin a \wedge a \neq 0$ instead of $\tilde{C}a$ we have the T(0)-theorem $^*\tilde{C}{}^*\{0\} = 0$, dually to $\tilde{V}*3$. Further, dually to $\tilde{V}*4$ we have $^*\{0\} = 0$ from which we have the T(0)-theorems $a = 0$ and $a \notin b$, dually to $\tilde{V}*5$ and $\tilde{V}*6$, respectively. Thus, the null set 0 is the unique dependent variable in the theory T(0), which is the dual theory of T(V). The above augument is a special case of the duality principle of UL we shall formulate in the following way.

Concerning the dual operator $*$ we are going to define, we remark in advance that $*$ is applicable to any logical operators, including $\in$, to any variables, to formulas, to assertions, and to proofs in UL; that $*$ has the property that $^*(^*Q) = Q$ for any object Q in UL; and that all the auxiliary symbols remain unchanged when they are contained in the object to which $*$ is applied.

First, we define the dual logical symbols as follows. The negation is selfdual, i.e. $*\neg$ is $\neg$ at any places wherever $\neg$ occurs. Other logical symbols, including $\in$, are affected by $*$ in different ways according as they occur inside or outside the definientes of dependent variables. Namely we define

		Inside definiens		Outside definiens
$*\in$	as	$\in$ (selfdual)	and as	$\notin$;
$*\wedge$	"	$\vee$	"	$\wedge$ (selfdual);
$*\forall$	"	$\exists$	"	$\forall$ (selfdual).

The $\in$ on the left-hand side of $\equiv'$ in any arbitrary defining formula (1) and the right formulas F^m and $\neg F^m$ of the proof constituents [DA] and [DN] associated with (1), respectively, are looked upon, when operated by $*$, as they are inside definiens.

Second, if Q is a formula of the form $m\in l$, $\neg A$, $A\wedge B$ or $\forall xF^x$, we define the dual formula $*Q$ of Q as $*m\,*\!\in *l$, $*\neg(*A)$, $(*A)\,*\!\wedge\,(*B)$, or $*\forall\,*x.\,*F^x$, respectively.

By the above definition of the dual operator $*$ we can set

		Inside definiens		Outside definiens
$\rightarrow$	as	$\nrightarrow$	and as	$\rightarrow$ (selfdual)
$*\equiv$	"	$\not\equiv$	"	$\equiv$ (selfdual)
$*=$	"	$\neq$	"	$=$ (selfdual)
$*\subseteq$	"	$\not\supseteq$	"	$\supseteq$

Third, we define the dual variable of a variable. Any independent variables are selfdual, i.e. $*a$ is a at any places wherever a occurs, a being an independent variable. The dual variable $*p$ of a dependent variable p with defining formula D is the dependent variable with the defining formula $*$D (proceeding recursively from the dependent variables of order 0 to those of higher order).

Lastly, if Q consists of formulas and auxiliary symbols, then $*Q$ is the figure obtained from Q by replacing each Q-formula by its dual formula.[5]

[5] It will be clear what is meant by the dual theory $*T$ of T. The duality theorem C in § 2, Part (V) (forthcoming in J. Symb. Logic), is a special case of the duality principle.

Thus, for instance, *V is 0 and *0 is V; the variables $^*\{a, b\}$, $^*\langle a, b\rangle$, $^*\tilde{\mathrm{R}}$, and $^*\tilde{\mathrm{C}}$a defined above are the dual variables of $\{a, b\}$, $\langle a, b\rangle$, $\tilde{\mathrm{R}}$, and $\tilde{\mathrm{C}}a$, respectively: $^*\cap$ is $\cup$ and $^*\cup$ is $\cap$ so that $^*(a\subseteq b\rightarrow a\cap b=a)$ is $a\supseteq b\rightarrow a\cup b=a$; and so on.

It is easily seen that the dual operator $*$ has the same effect if we define $*$ as follows. Any object Q, namely formulas, proofs, subsystems etc. of UL, operated by $*$, is changed into an object $^*Q$ which we obtain when we replace every $\in$ wherever it occurs in Q by its negation $\notin$ and all the dependent variables p by $^*p$, leaving all the other symbols in Q unchanged.

Now, we formulate the *duality principle* as follows: *If P is a proof in* UL *for an assertion $\sigma\vdash H$, then $^*P$ is a proof for the dual assertion $^*\sigma\vdash{}^*H$.* This principle can be easily proved in virtue of the definitions of the proof in UL and of the dual operator $*$, and may be said to be a generalization of the duality prevailing in logic since de Morgan. As an immediate consequence of the duality principle we have the following: *If a theory T is a consistent subsystem of* UL, *then the dual theory $^*T$ is also a consistent subsystem of* UL, *and if A is a T-theorem, then $^*A$ is the dual $^*T$-theorem; in particular, if $\sigma\vdash H$ is a* UL-*theorem, so is also $^*\sigma\vdash{}^*H$.*

9. Further deduction in T(V)

It may be allowed to report here, preliminarily without detailed formulation, the method of deducing some part of mathematics consistently in a subsystem of T(V). For instance, we have the following T(V)-theorem.

(i) If E is different from V, then $a\cup b=E$ and $a\cap b=0^E$ imply[6] $b=\mathrm{C}^E a$.

(ii) If a is different from V there is no one-to-one mapping between a and its power set $\mathfrak{P}(a)$.

(iii) Two well-ordered sets are either similar to each other or one is similar to a section of the other, provided that they are different from V.

The proofs of these propositions are performed as in the so-called "naive set theory," and, for instance, the proposition (ii) acquires meaning, if there is a subsystem T of T(V) such that T-unprovability of $a=\mathrm{V}$ for some T-set a is known and the dependent variables used as sets in the proof of (ii) are T-sets.

We first examine the usual proof of the inequivalence of a and $\mathfrak{P}(a)$. To

[6] The T(V)-set 0^E is defined by $\forall u.\ u\in 0^E\equiv u\neq u\vee u=\mathrm{V}$.

prove this we have only to prove

$$(6) \qquad \neg .\ \sigma \in \mathrm{Un} \wedge \mathrm{D}_\sigma = a \wedge \mathrm{W}_\sigma = \mathfrak{P}(a)$$

where Un, D_σ, and W_σ are defined by

$$\forall u.\ u \in \mathrm{Un} \equiv \forall xyz.\ \langle xy \rangle \in u \wedge \langle xz \rangle \in u \rightarrow y = z,$$

$$\forall u.\ u \in \mathrm{D}_\sigma \equiv \exists x.\ \langle xu \rangle \in \sigma, \qquad \forall u.\ u \in \mathrm{W}_\sigma \equiv \exists x.\ \langle ux \rangle \in \sigma.$$

These variables are all V-sets. However, in the proof of (6) we need a variable $\mathrm{L}^{a,\sigma}$ used as set and defined by

$$(7) \qquad \forall u.\ u \in \mathrm{L}^{a,\sigma} \equiv u \in a \wedge \neg \exists x.\ u \in x \wedge \langle ux \rangle \in \sigma.$$

$\mathrm{L}^{a,\sigma}$ is the set of such elements of a that the image by σ of u does not contain u itself. $\mathrm{L}^{a,\sigma}$ is not a T(V)-set so that (6) is not a T(V)-theorem. Moreover, if a and σ are specialized to V and ι, where ι is the identical mapping defined by

$$\forall u.\ u \in \iota \equiv \exists x.\ u = \langle xx \rangle,$$

we see that the definiens of $\mathrm{L}^{\mathrm{V},\iota}$ is equivalent to $u \notin u$, so that $\mathrm{L}^{\mathrm{V},\iota}$ is equal to Russell's contradictory set R. Indeed, from (6) we have

$$(8) \qquad \neg .\ \iota \in \mathrm{Un} \wedge \mathrm{D}_\iota = \mathrm{V} \wedge \mathrm{W}_\iota = \mathfrak{P}(\mathrm{V})$$

and, since $\iota \in \mathrm{Un}$ and $\mathrm{D}_\iota = \mathrm{V}$ are UL-theorems (or more strongly they are T(V)-theorems), we have $\mathrm{W}_\iota \quad \mathfrak{P}(\mathrm{V})$ from (8). On the other hand, we have T(V)-theorems[(III)] $\mathrm{W}_\iota = \mathrm{V}$ and $\mathfrak{P}(\mathrm{V}) = \mathrm{V}$, so we have come to the contradiction $\mathrm{V} \neq \mathrm{V}$. Eliminating all the concepts and auxiliary sets[(X)] in the proof of this contradiction, we see that the contradiction is deduced only by using[7)] $\mathrm{L}^{\iota,\mathrm{V}}$, ι, V as sets. Namely, we have the proof of contradiction for $\sigma \vdash$ where σ is the sequence of defining formulas of $\mathrm{L}^{\iota,\mathrm{V}}$, ι, and V. We can easily see that $\mathrm{L}^{\sigma,a}$ is a T(0)-set, so that the two T(V)-sets V and ι and one T(0)-set $\mathrm{L}^{\sigma,a}$ combined give rise to the above contradiction. Still (6) is a UL-theorem. In fact, after eliminating all the concepts and auxiliary sets[(X)] from the proof of (6) we see that only $\mathrm{L}^{a,\sigma}$ is used as proper set[(X)] in the proof of (6). Since $\mathrm{L}^{a,\sigma}$ is a set of consistent T(0)-system, (6) is a UL- or more strongly T(0)-theorem, but not T(V)-theorem.

In order to get a T(V)-theorem corresponding to (6), we define $\tilde{\mathrm{L}}^{a,\sigma}$ by

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7) Here the unspecialized definition[(II)] of $\mathrm{L}^{\iota,\mathrm{V}}$ is meant. As is said above, the specialized definition[(II)] of $\mathrm{L}^{\iota,\mathrm{V}}$ is the same as that of Russell's set R.

(8) $$\forall u\colon u\in\tilde{L}^{a,\sigma}\equiv u\in a\wedge\colon\neg\exists x.\ u\in x\wedge\langle ux\rangle\in\sigma.\ \vee a=V,$$

adding the disjunctive term $a=V$ so as to make $\tilde{L}^{a,\sigma}$ a T(V)-set. By using $\tilde{L}^{a,\sigma}$, instead of $L^{a,\sigma}$, the proof for (6) is changed into the proof of

(9) $$a\neq V\rightarrow\neg.\ \sigma\in\mathrm{Un}\wedge D_{\sigma}=a\wedge W_{\sigma}=\mathfrak{P}(a),$$

which is a T(V)-theorem. From (9) follows the T(V)-theorem

(10) $$a\neq V\rightarrow a\neq\mathfrak{P}(a).$$

From (10) and the $T_1(\tilde{N})$-theorem $\tilde{N}*\mathfrak{P}$ follows again the T(V)-theorem

(11) $$\tilde{N}\neq V\rightarrow\mathfrak{P}^{(n)}(\tilde{N})\neq\mathfrak{P}^{(n+1)}(\tilde{N})\qquad(n=0,1,\ldots),$$

which is, however, not a $T_1(\tilde{N})$-theorem, since $\tilde{L}^{a,\sigma}$ used in the proof of (11) is not a $T_1(\tilde{N})$-set.

Let T_0 be the extension of $T_1(\tilde{N})$ which we get from $T_1(N)$ by adjoining $\tilde{L}^{a,\sigma}$ where $a=\mathfrak{P}^{(n)}(\tilde{N})$, as T_0-set. Then, not only (11) is a T_0-theorem but also we can prove the T_0-unprovability of $\tilde{N}=V$. Let T be an extension (including T_0) of T_0, and assume that T-unprovability of $\tilde{N}=V$ is proved. In such a theory T the deduction under the assumption $\tilde{N}\neq V$ is consistent. Therefore, we can use

(12) $$\mathfrak{P}^{(n)}(\tilde{N})\neq\mathfrak{P}^{(n+1)}(\tilde{N})\qquad(n=0,1,\ldots)$$

consistently in the deduction in T under the assumption $\tilde{N}\neq V$. Since the T-unprovability of $\tilde{N}=V$ implies the consistency of T under the assumption $\tilde{N}\neq V$, Gödel's theorem concerning the consistency proof is applicable to the proof of T-unprovability of $\tilde{N}=V$, when the species of T-sets is wide enough.[8]

Appendix. $V\neq 0$ is proved by using V or 0 as set, while $V=a$ by § 4. In the proof of $a=V$ ($\tilde{V}*5$) the sets $\tilde{R}$, $\tilde{C}\{V\}$, and V are used. Hence the system of variables $\tilde{R}$, $\tilde{C}\{V\}$, V, and 0 is inconsistent. For the proof of this contradiction 0 is used as set, since a is used in the proof of $\tilde{V}*2$ as set-variable (note the association of $a\in\tilde{R}$ to $\overline{\neg\forall x.\ x\in\tilde{R}}$ in the proof of $\tilde{V}*2$).

[8] $T_1(N)$ does not yet contain enough dependent variables as sets by which the primitive recursive functions of natural numbers can be treated as sets. It is, however, very likely to be able to prove that (12) is a theorem under the assumption $N\neq V$ in such a minimal subsystem $T/T_1(N)$ of UL which has the simple types with the elements of N as basic type and which has the primitive recursive functions as sets.

Corrections to Part (XII)[9]

With the view to reduce the principle of extensionality to some defining formulas the formula (2)→(1) is proved there.[9] I have overlook the other possibility $a \neq b$ occurring as proof formulas. In order to treat both $a=b$ and $a \neq b$ in the same way, the converse (1)→(2) should have been proved also as a UL-theorem from some *suitably* selected defining formulas as premises. Since no way is found to prove this, the principle of extensionality is not replaced by defining formulas.

Owing to the above error I should make the following corrections to Part (XII): 1. page 400, lines 17-19, the sentence "on the contrary . . . $z \in y$", is to be deleted; 2. page 401, the paragraph directly under the proof figure, "the formula (4) means . . . exclusively of defining formulas", is to be deleted; 3. page 402, line 10, "the principle of choice is" instead of "both principles . . . are"; 4. page 403, line 11, period instead of comma after "formulas", and delete the subsequent phrase "so that . . . of defining formulas"; 5. page 403, line 2, insert "other than (I)" after "UL-premises".

After correcting the above, we mean hereafter by UL, as is so in the previous Parts, the logical system defined in Part (I),[10] including eventually the axiom of choice in the way described in Part (XII).[11]

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9) Proc. Japan Acad. Vol. **34** (1958), pp. 400–403. This "corrections" was written in December, 1958.

10) Hamb. Abh. Vol. **22** (1958).

11) Still three errata in Part (XII): 1. page 401, in the proof figure, a line is to be inserted between the abovemost two formulas; 2. page 401, the last line of the foot-note, "$y \in x$" instead of "$x \in y$"; 3. page 402, line 4 from bottom, insert a comma after "Part (II)".