GENERALIZED JACOBIAN VARIETIES AND SEPARABLE ABELIAN EXTENSIONS OF FUNCTION FIELDS

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Using Frobenius automorphisms ingeniouslly, S. Lang has established an elegant theory of unramified class fields of function fields in several variables over finite fields [2]. As an application of class field theory and theory of reduction he has proved that any separable unramified abelian extension of a function field of one variable comes from *a pull back* of a separable ingeny of its jacobian variety [3].

In the present paper, first we shall prove that any separable abelian extension of a function field of one variable over a perfect field comes from a pull back of a separable homomorphism onto a suitable generalized jacobian variety of the ground field. Secondly, on the base of the pull back theory, we shall show a theory of class fields of function fields of one variable over a perfect field. Especially the class field theory of function fields over finite fields will be treated completely.

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NOTATIONS

Throughout this paper we use following notations:

k : a perfect field of characteristic p, where p may be zero,

K/k : a regular extension of dimension one,

L/K: a separable abelian extension of degree n which is also regular over k,

$G(L/K)=\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$: the galois group of L/K,

$\{\overline{M}_1, \dots, \overline{M}_r\}$: a set of places of K/k containing all the places ramified in L/K, where it may be empty,

e_i : the index of ramification of $\overline{M}_i$ in L/K,

$M_{i,1}, \dots, M_{i,h_i}$: all the places of L/k on $\overline{M}_i$,

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$\overline{\mathfrak{m}}_i$: the maximal ideal of the valuation ring of $\overline{M}_i$ in K/k,

$\mathfrak{m}_{ij}$: the maximal ideal of the valuation ring of M_{ij} in L/k.

§ 1. Generalized jacobian varieties of extension fields

1. 1. We start with local rings

$$\mathfrak{o} = k + \bigcap_{i=1}^{r} \bigcap_{j=1}^{k_i} \mathfrak{m}_{ij}^{\nu_i} \qquad (\nu_i \geqq 1) \tag{1}$$

$$\bar{\mathfrak{o}} = k + \bigcap_{i=1}^{r} \overline{\mathfrak{m}}_i^{\bar{\nu}_i} \qquad (\bar{\nu}_i \geqq 1), \tag{2}$$

where we assume that $N_{L/K}\mathfrak{o} \subset \bar{\mathfrak{o}}$. We say that a divisor $\mathfrak{a}(\bar{\mathfrak{a}})$ of $L/k(K/k)$ is $\mathfrak{o}(\bar{\mathfrak{o}})$-equivalent to zero, if $\mathfrak{a} = (f)$ $(\bar{\mathfrak{a}} = (\bar{f}))$ with $f(\bar{f})$ in $\mathfrak{o}(\bar{\mathfrak{o}})$. We mean also by $\mathfrak{o}(\bar{\mathfrak{o}})$-equivalence relation its prolongation in any constant extension of $L/k(K/k)$. Let $C_{\mathfrak{o}}(\overline{C}_{\bar{\mathfrak{o}}})$ be a projective model of $L/k(K/k)$ which has a point $M(\overline{M})$ such that i) the local ring at $M(\overline{M})$ is $\mathfrak{o}(\bar{\mathfrak{o}})$ and ii) $C_{\mathfrak{o}}^* = C_{\mathfrak{o}} - M$ $(\overline{C}_{\bar{\mathfrak{a}}}^* = \overline{C}_{\bar{\mathfrak{o}}} - \overline{M})$ is everywhere regular.[1] Let $J_{\mathfrak{o}}(\bar{J}_{\bar{\mathfrak{o}}})$ be the generalized jacobian variety associated with $\mathfrak{o}(\bar{\mathfrak{o}})$-equivalence relation on $C_{\mathfrak{o}}^*(\overline{C}_{\bar{\mathfrak{o}}}^*)$ and $\varphi_{\mathfrak{o}}(\bar{\varphi}_{\bar{\mathfrak{o}}})$ be a canonical mapping of $C_{\mathfrak{o}}^*(\overline{C}_{\bar{\mathfrak{o}}}^*)$ into $J_{\mathfrak{o}}(\bar{J}_{\bar{\mathfrak{o}}})$ where we assume that $J_{\mathfrak{o}}(\bar{J}_{\bar{\mathfrak{o}}})$ is defined over k by Chow's method.[2]

In section 1 and 2 we shall use only the following properties of $J_{\mathfrak{o}}$: 1) the subgroup consisting of all the k-rational points of $J_{\mathfrak{o}}$ is isomorphic to the group of $\mathfrak{o}$-equivalence classes of degree zero of L/k, 2) if g is the dimension of $J_{\mathfrak{o}}$ and $P_1, \dots, P_g$ are independent generic points of $C_{\mathfrak{o}}^*$ over k, then $\varphi_{\mathfrak{o}}(P_1 + \dots + P_g)$[3] is a generic point of $J_{\mathfrak{o}}$ over k and $k(\varphi_{\mathfrak{o}}(P_1 + \dots + P_g)) = k(\varphi_{\mathfrak{o}}(P_1), \dots, \varphi_{\mathfrak{o}}(P_g))_s$,[4] 3) if $m \geqq 2g$, for any point y of $J_{\mathfrak{o}}$ there exist points $P_1, \dots, P_m$ of $C_{\mathfrak{o}}^*$ such that $y = \varphi_{\mathfrak{o}}(P_1 + \dots + P_m)$, and 4) $\varphi_{\mathfrak{o}}$ is biregular mapping between $C_{\mathfrak{o}}^*$ and $\varphi_{\mathfrak{o}}(C_{\mathfrak{o}}^*)$.

1. 2. $N_{L/K}\mathfrak{o} \subset \bar{\mathfrak{o}}$ implies that the trace mapping $\hat{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}$ of $C_{\mathfrak{o}}^*$ onto $\overline{C}_{\bar{\mathfrak{o}}}^*$ induces the trace mapping (homomorphism) $\pi_{\mathfrak{o},\bar{\mathfrak{o}}}$ of $J_{\mathfrak{o}}$ onto $\bar{J}_{\bar{\mathfrak{o}}}$:

$$\pi_{\mathfrak{o},\bar{\mathfrak{o}}}\varphi_{\mathfrak{o}}(\mathfrak{a}) = \bar{\varphi}_{\bar{\mathfrak{o}}}(\hat{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}(\mathfrak{a})), \tag{3}$$

where $\mathfrak{a}$ runs over divisors of degree zero of $C_{\mathfrak{o}}^*$. The galois automorphisms

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[1] Such a model always exists. Cf. Theorem 5, pp. 174 [4].

[2] Cf. Theorem 2, pp. 185 [1].

[3] $\varphi_{\mathfrak{o}}(P_1 + \dots + P_g)$ means $\varphi_{\mathfrak{o}}(P_1) + \dots + \varphi_{\mathfrak{o}}(P_g)$.

[4] $k(t_1, \dots, t_g)_s$ means the subfield of $k(t_1, \dots, t_g)$ consisting of elements fixed by any permutation of suffices.

$\varepsilon_1, \ldots, \varepsilon_n$ also induce the automorphisms $\eta(\varepsilon_1), \ldots, \eta(\varepsilon_n)$ of J_D:

$$\eta(\varepsilon_\nu)\varphi(\mathfrak{a}) = \varphi(\mathfrak{a}^{\varepsilon_\nu^{-1}}) \qquad (\nu = 1, 2, \ldots, n), \tag{4}$$

where $\mathfrak{a}$ runs over divisors of degree zero of C_D^*.

1. 3. Let $x_1, \ldots, x_n$ be independent generic points of J_D over k and $\hat{B}_D$ be the locus of $\sum_{\nu=1}^{n}(\delta_{J_D} - \eta(\varepsilon_\nu))x_\nu$ over k.[5] Then $\hat{B}_D$ is a subgroup variety defined over k. Let $\hat{A}_D$ be the quotient group variety of J_D by $\hat{B}_D$ and β_D be the natural separable homomorphism of J_D onto $\hat{A}_D$.

LEMMA 1. *If P is a point of C_D^*, then the points $\varphi_D(P^{\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\nu)\varphi_D(P)$ $(\nu = 1, 2, \ldots, n)$ do not depend on the choice of P.*

Proof. Let Q be another point of C_D^*. Then we observe that $\varphi_D(P^{\varepsilon_\nu^{-1}} - Q^{\varepsilon_\nu^{-1}}) = \varphi_D((P-Q)^{\varepsilon_\nu^{-1}}) = \eta(\varepsilon_\nu)\varphi_D(P-Q)$. This proves the lemma.

We put

$$b_D(\varepsilon_\nu) = \varphi_D(P^{\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\nu)\varphi_D(P) \qquad (\nu = 1, 2, \ldots, n). \tag{5}$$

LEMMA 2. $\beta_D\eta(\varepsilon_\mu)b_D(\varepsilon_\nu) = \beta_D b_D(\varepsilon_\nu) \qquad (\mu, \nu = 1, 2, \ldots, n).$

Proof. Let P be a point of C_D^*. Then, since $G(L/K)$ is abelian, we observe that

$$\begin{aligned}
\beta_D\eta(\varepsilon_\mu)b_D(\varepsilon_\nu) &= \beta_D\eta(\varepsilon_\mu)(\varphi_D(P^{\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\nu)\varphi_D(P)) \\
&= \beta_D\eta(\varepsilon_\mu)(\varphi_D(P^{\varepsilon_\nu^{-1}}) - \varphi_D(P)) + \beta_D\eta(\varepsilon_\mu)(\varphi_D(P) - \eta(\varepsilon_\nu)\varphi_D(P)) \\
&= \beta_D\eta(\varepsilon_\mu)(\varphi_D(P^{\varepsilon_\nu^{-1}} - P)) \\
&= \beta_D\varphi_D(P^{\varepsilon_\mu^{-1}\varepsilon_\nu^{-1}} - P^{\varepsilon_\mu^{-1}}) \\
&= \beta_D(\varphi_D)P^{\varepsilon_\mu^{-1}\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\nu)\varphi_D(P^{\varepsilon_\mu^{-1}})) - \beta_D(\varphi_D(P^{\varepsilon_\mu^{-1}}) - \eta(\varepsilon_\nu)\varphi_D(P^{\varepsilon_\mu^{-1}})) \\
&= \beta_D b_D(\varepsilon_\nu).
\end{aligned}$$

PROPOSITION 1. $\beta_D b_D(\varepsilon_\mu\varepsilon_\nu) = \beta_D b_D(\varepsilon_\mu) + \beta_D b_D(\varepsilon_\nu) \qquad (\mu, \nu = 1, 2, \ldots, n).$

Proof. Let P be a point of C_D^*. Then, since $G(L/K)$ is abelian, by virtue of Lemma 1 and 2, we get

$$\begin{aligned}
\beta_D b_D(\varepsilon_\mu\varepsilon_\nu) &= \beta_D(\varphi_D(P^{\varepsilon_\mu^{-1}\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\mu\varepsilon_\nu)\varphi_D(P)) \\
&= \beta_D(\varphi_D(P^{\varepsilon_\mu^{-1}\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\mu)\varphi_D(P^{\varepsilon_\nu^{-1}})) + \beta_D\eta(\varepsilon_\mu)(\varphi_D(P^{\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\nu)\varphi_D(P)) \\
&= \beta_D(\varphi_D(P^{\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\mu)\varphi_D(P)) + \beta_D(\varphi_D(P^{\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\nu)\varphi_D(P)).
\end{aligned}$$

[5] δ_{J_D} means the identity automorphism of J_D.

Hence we have

$$\beta_0 b_0(\varepsilon_\mu \varepsilon_\nu) = \beta_0 b_0(\varepsilon_\mu) + \beta_0 b_0(\varepsilon_\nu).$$

1.4. We say that a local ring $\mathfrak{o}(\bar{\mathfrak{o}})$ is co-ample relative to L/K, if there exists an integral element ξ in L/k over $\mathfrak{o}(\bar{\mathfrak{o}})$ such that $\mathrm{tr}_{L/K}\xi \notin \mathfrak{m}(\mathfrak{o})(\bar{\mathfrak{m}}(\bar{\mathfrak{o}}))$, where $\mathfrak{m}(\mathfrak{o})(\bar{\mathfrak{m}}(\bar{\mathfrak{o}}))$ means the maximal ideal of $\mathfrak{o}(\bar{\mathfrak{o}})$.

LEMMA 3. *Let h be a positive integer satisfying $0 < h < n$. Then, if $\mathfrak{o}$ is co-ample relative to L/K and L/K is cyclic, there exists no set of points $\{P_1, \ldots, P_{nm}, R\}$ of C_0^* such that*

$$\varphi_0(P_1^{\varepsilon^{-1}} + \ldots + P_{um}^{\varepsilon^{-1}} + R^{\varepsilon^{-h}} - P_1 - \ldots - P_{nm} - R) = 0,$$

where ε is a generator of $G(L/K)$.

Proof. We may assume that k is algebraically closed. Suppose that $\{P_1, \ldots, P_{mn}, R\}$ is a set of points of C_0^* satisfying the above condition. Let f be the function in $\mathfrak{o}$ such that

$$(f) = P_1^{\varepsilon^{-1}} + \ldots + P_{nm}^{\varepsilon^{-1}} + R^{\varepsilon^{-h}} - P_1 - \ldots - P_{nm} - R.$$

Since $N_{L/K}f$ is a constant, we may assume that $N_{L/M}f = 1$ and $f \equiv 1 \bmod \mathfrak{m}(\mathfrak{o})$. Let ξ be an integral element over $\mathfrak{o}$ such that $\mathrm{tr}_{L/K}\xi \notin \mathfrak{m}(\mathfrak{o})$. Put

$$\psi = \xi + \xi^{\varepsilon^{-1}} f + \xi^{\varepsilon^{-2}} f^{1+\varepsilon^{-1}} + \ldots + \xi^{\varepsilon^{-(n+1)}} f^{1+\varepsilon^{-1}+\ldots+\varepsilon^{-(n-2)}}.$$

Then we have

$$\psi \equiv \mathrm{tr}_{L/K}\xi \qquad \bmod \mathfrak{m}(\mathfrak{o})$$

and $f\psi^{\varepsilon^{-1}} = \psi$. Hence, putting $\psi_1 = (\mathrm{tr}\,\xi)^{-1}\psi$, we have $\psi_1 \equiv 1 \bmod \mathfrak{m}(\mathfrak{o})$. This shows that all the places contained in $(\psi_1)_0$ and $(\psi_1)_\infty$ are unramified in L/K. Putting $\mathfrak{a} = P_1 + \ldots + P_{mn}$, we observe that

$$(f) = \mathfrak{a}^{\varepsilon^{-1}} - \mathfrak{a} + R^{\varepsilon^{-h}} - R = (\psi_1) - (\psi_1)^{\varepsilon^{-1}}$$

Namely $R^{\varepsilon^{-h}} - R = (\psi_1) + \mathfrak{a} - ((\psi_1) + \mathfrak{a})^{\varepsilon^{-1}}$. Let m_ν be the multiplicity of $R^{\varepsilon^{-\nu}}$ in $(\psi_1) + \mathfrak{a}$ and $\mathfrak{b}$ be the divisor $(\psi_1) + \mathfrak{a} - \sum_{\nu=0}^{n-1} m_\nu R^{\varepsilon^{-\nu}}$. Then we observe that $\mathfrak{b}^{\varepsilon^{-1}} - \mathfrak{b} = 0$. Since all the places contained in $\mathfrak{b}$ are unramified, we have $\deg \mathfrak{b} \equiv 0 \bmod n$. On the other hand we have

$$R^{\varepsilon^{-h}} - R = \sum_{\nu=0}^{n-1} m_\nu R^{\varepsilon^{-\nu}} - \sum_{\nu=0}^{n-1} m_\nu R^{\varepsilon^{-(\nu+1)}}.$$

Therefore we observe that

$$m_\nu - m_{\nu-1} = \begin{cases} 0 & \text{for} \quad \nu \neq h,\ 0 \\ 1 & \text{for} \quad \nu = h \\ -1 & \text{for} \quad \nu = 0,\ n, \end{cases}$$

where $m_0 = m_n$. This shows that $\sum_{\nu=0}^{n-1} m_\nu = nm_0 + (n-h) \equiv -h \mod n$. This contradictes $\sum_{\nu=1}^{n-1} m_\nu = \deg(\varphi_1) + \mathfrak{a} - \mathfrak{b} \equiv 0 \mod n$.

PROPOSITION 2. *If L/K is cyclic and $\mathfrak{d}$ is co-ample relative to L/K, then* $\varepsilon^\nu \to \beta_\mathfrak{d} b_\mathfrak{d}(\varepsilon^\nu)$ $(\nu = 1, 2, \ldots, n)$ is an isomorphism.

Proof. Let m be a positive integer such that $mn \geqq 2\dim J_\mathfrak{d}$ and R be a point of $C_\mathfrak{d}^*$. Then we have

$$\varphi_\mathfrak{d}(R^{\varepsilon^{-\nu}} - R) = (\eta(\varepsilon^\nu) - \delta_{J_\mathfrak{d}})\varphi_\mathfrak{d}(R) + (\varphi_\mathfrak{d}(R^{\varepsilon^{-\nu}}) - \eta(\varepsilon^\nu)\varphi_\mathfrak{d}(R)).$$

This shows that

$$\varphi_\mathfrak{d}(R^{\varepsilon^{-\nu}} - R) - b_\mathfrak{d}(\varepsilon^\nu)$$

belongs te $\hat{B}_\mathfrak{d}$. Now we suppose that $\beta_\mathfrak{d} b_\mathfrak{d}(\varepsilon^h) = 0$ for an h satisfying $0 < h < n$. Then we observe that

$$\beta_\mathfrak{d}(\varphi_\mathfrak{d}(R^{\varepsilon^{-h}} - R) - \varphi_\mathfrak{d}(R^{\varepsilon^{-1}} - R) + b_\mathfrak{d}(\varepsilon)) = 0.$$

On the other hand, by virtue of Proposition 1, we have $\beta_\mathfrak{d} b_\mathfrak{d}(\varepsilon^n) = n\beta_\mathfrak{d} b_\mathfrak{d}(\varepsilon) = 0$, hence we get

$$\beta_\mathfrak{d} b_\mathfrak{d}(R^{\varepsilon^{-h}} - R^{\varepsilon^{-1}}) + (mn+1)\beta_\mathfrak{d} b_\mathfrak{d}(\varepsilon) = 0.$$

Namely $\varphi_\mathfrak{d}(R^{\varepsilon^{-h}} - R^{\varepsilon^{-1}}) + (mn+1)b_\mathfrak{d}(\varepsilon)$ belongs to $\hat{B}_\mathfrak{d}$. Since $nm \geqq 2\dim J_\mathfrak{d}$ and $(\delta_{J_\mathfrak{d}} - \eta(\varepsilon))(J_\mathfrak{d}) = \hat{B}_\mathfrak{d}$, there exists a set of points $\{P_1, \ldots, P_{mn}\}$ of $C_\mathfrak{d}^*$ such that

$$\begin{aligned}(\delta_{J_\mathfrak{d}} - \eta(\varepsilon))\varphi_\mathfrak{d}(P_1 + \ldots + P_{nm}) = &-(\delta_{J_\mathfrak{d}} - \eta(\varepsilon))\varphi_\mathfrak{d}(R) \\ &+ \varphi_\mathfrak{d}(R^{\varepsilon^{-h}} - R^{\varepsilon^{-1}}) + (mn+1)b_\mathfrak{d}(\varepsilon).\end{aligned}$$

Therefore we have

$$\begin{aligned}\varphi_\mathfrak{d}(P_1 + \ldots + P_{nm} + R) &- \varphi_\mathfrak{d}(P_1^{\varepsilon^{-1}} + \ldots + P_{mn}^{\varepsilon^{-1}} + R^{\varepsilon^{-1}}) \\ &+ \sum_{i=1}^{nm}(\varphi_\mathfrak{d}(P_i^{\varepsilon^{-1}}) - \eta(\varepsilon)\varphi_\mathfrak{d}(P_i)) + (\varphi_\mathfrak{d}(R^{\varepsilon^{-1}}) - \eta(\varepsilon)\varphi_\mathfrak{d}(R)) \\ &= (mn+1)b_\mathfrak{d}(\varepsilon) + \varphi_\mathfrak{d}(R^{\varepsilon^{-h}} - R^{\varepsilon^{-1}}).\end{aligned}$$

On the other hand $\varphi_\mathfrak{d}(P_i^{\varepsilon^{-1}}) - \eta(\varepsilon)\varphi_\mathfrak{d}(P_i) = \varphi_\mathfrak{d}(R^{\varepsilon^{-1}}) - \eta(\varepsilon)\varphi_\mathfrak{d}(R) = b_\mathfrak{d}(\varepsilon)$ $(i = 1, 2,$

. . . , nm), hence we have

$$\varphi_{\mathfrak{D}}(P_1 + \dots + P_{nm} + R - P_1^{\varepsilon^{-1}} - \dots - P_{mn}^{\varepsilon^{-1}} - R^{\varepsilon^{-h}}) = 0.$$

This contradicts Lemma 3.

§ 2. A proof of the pull back theorem

We use following notations:

$B_{\mathfrak{D},\bar{\mathfrak{D}}}$: the irreducible component of $\pi_{\mathfrak{D},\bar{\mathfrak{D}}}^{-1}(0)$ containning $\{0\}$,

$\ddot{A}_{\mathfrak{D},\bar{\mathfrak{D}}}$: the quotient group variety of $J_{\mathfrak{D}}$ by $B_{\mathfrak{D},\bar{\mathfrak{D}}}$,

$\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}$: the natural separable homomorphism $J_{\mathfrak{D}}$ onto $\ddot{A}_{\mathfrak{D},\bar{\mathfrak{D}}}$,

$\bar{\pi}_{\mathfrak{D},\bar{\mathfrak{D}}}$: the homomorphism of $\bar{A}_{\mathfrak{D},\bar{\mathfrak{D}}}$ onto $\bar{J}_{\bar{\mathfrak{D}}}$ such that $\pi_{\mathfrak{D},\bar{\mathfrak{D}}} = \bar{\pi}_{\mathfrak{D},\bar{\mathfrak{D}}}\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}$,

$a_{\mathfrak{D},\bar{\mathfrak{D}}}(\varepsilon_\nu)$: $\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}b_{\mathfrak{D}}(\varepsilon_\nu)$ $\quad(\nu = 1, 2, \dots, n)$.

2. 1. Let k' be a finitely normal extension of k over which $B_{\mathfrak{D},\bar{\mathfrak{D}}}$ is defined and σ be any antomorphism of k'/k. Then $B_{\mathfrak{D},\bar{\mathfrak{D}}}^{\sigma}$ is also a component of $\pi_{\mathfrak{D},\bar{\mathfrak{D}}}^{-1}(0)$ and is also a subgroup of $\bar{\pi}_{\mathfrak{D},\bar{\mathfrak{D}}}^{-1}(0)$. Since k is perfect, the irreducible group $B_{\mathfrak{D},\bar{\mathfrak{D}}}$ is defined over k. Therefore $B_{\mathfrak{D},\bar{\mathfrak{D}}}$, $A_{\mathfrak{D},\bar{\mathfrak{D}}}$, $\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}$ and $\bar{\pi}_{\mathfrak{D},\bar{\mathfrak{D}}}$ are also defined over k.

LEMMA 1. *Let Λ be a commutative group variety of dimension g and λ be a homomorphism of Λ onto a generalized jacobian variety $\bar{J}_{\bar{\mathfrak{D}}}$ of dimension g, where Λ, λ and $\bar{J}_{\bar{\mathfrak{D}}}$ are defined over k. Let y be a point of Λ such that λy is a generic point of $\bar{\varphi}_{\bar{\mathfrak{D}}}(\bar{C}_{\bar{\mathfrak{D}}}^*)$ over k. Then, if $\bar{k}(y)/\bar{k}(\lambda y)$ is purely inseparable, λ is purely inseparable.*

Proof. Let $\tilde{C}$ be the locus of y over $\bar{k}$ and $y_1, y_2, \dots, y_g$ are independent generic points of $\tilde{C}$ over k. Then $\lambda y_1, \lambda y_2, \dots, \lambda y_g$ are independent generic points of $\bar{\varphi}_{\bar{\mathfrak{D}}}(\bar{C}_{\bar{\mathfrak{D}}}^*)$ over k and $\sum_{i=1}^{g}\lambda y_i$ is a generic point of $\bar{J}_{\bar{\mathfrak{D}}}$ over k. This shows that $\sum_{i=1}^{g} y_i$ is a generic point of Λ over k. On the other hand $k(\sum_{i=1}^{g}\lambda y_i) = k(\lambda y_1, \dots, \lambda y_g)_s$,[6] hence $\bar{k}(\sum_{i=1}^{g}\lambda y_i) = \bar{k}(\lambda y_1, \dots, \lambda y_g)_s = \bar{k}(y_1, \dots, y_g)_s^* = \bar{k}(\sum_{i=1}^{g} y_i)^*$ where $k(\ )^*$ means the maximal separable subfield of $k(\ )$ aver $k(\sum\lambda y_i)$. This shows that λ is purely inseparable.

LEMMA 2. $\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}\eta(\varepsilon_\nu) = \alpha_{\mathfrak{D},\bar{\mathfrak{D}}}$ $\quad(\nu = 1, 2, \dots, n)$.

Proof. Since $\pi_{\mathfrak{D},\bar{\mathfrak{D}}}(\hat{B}_{\mathfrak{D}}) = \pi_{\mathfrak{D},\bar{\mathfrak{D}}}(\beta_{\mathfrak{D}}^{-1}(0)) = 0$ and $\hat{B}_{\mathfrak{D}}$ is irreducible, we have

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6) See 4).

$B_{\mathfrak{D},\bar{\mathfrak{D}}} \supset \hat{B}_{\mathfrak{D}}$. Hence $\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}(\delta_{J_{\mathfrak{D}}} - \eta(\varepsilon_\nu)) = 0$ $(\nu = 1, 2, \ldots, n)$.

PROPOSITION 3. $\bar{\pi}_{\mathfrak{D},\bar{\mathfrak{D}}}$ *is separable and*

$$\pi^{-1}_{\mathfrak{D},\bar{\mathfrak{D}}}(0) = \{a_{\mathfrak{D},\bar{\mathfrak{D}}}(\varepsilon_\nu) : \nu = 1, 2, \ldots, n\}.$$

Proof. Let A^* be the quotient group variety of $\bar{A}_{\mathfrak{D},\bar{\mathfrak{D}}}$ by $\{a_{\mathfrak{D},\bar{\mathfrak{D}}}(\varepsilon_\nu) : \nu = 1, 2, \ldots, n\}$ and γ be the natural separable homorphism of $\bar{A}_{\mathfrak{D},\bar{\mathfrak{D}}}$ onto A^*. Let λ be the homomorphism of A^* onto $\bar{J}_{\bar{\mathfrak{D}}}$ such that $\bar{\pi}_{\mathfrak{D},\bar{\mathfrak{D}}} = \lambda\gamma$. Let P be a generic point of $C^*_{\mathfrak{D}}$ over k. Then we have

$$\gamma\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}\varphi_{\mathfrak{D}}(P^{\varepsilon_\nu^{-1}}) = \gamma\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}(\eta(\varepsilon_\nu)\varphi_{\mathfrak{D}}(P) + b_{\mathfrak{D}}(\varepsilon_\nu)) = \gamma\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}\eta(\varepsilon_\nu)\varphi_{\mathfrak{D}}(P) = \gamma\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}\varphi_{\mathfrak{D}}(P).$$

Since $\bar{k}(\varphi_{\mathfrak{D}}(P))/\bar{k}(\pi_{\mathfrak{D},\bar{\mathfrak{D}}}\varphi_{\mathfrak{D}}(P))$ is separable, we have

$$\bar{k}(\gamma\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}\varphi_{\mathfrak{D}}(P)) = \bar{k}(\pi_{\mathfrak{D},\bar{\mathfrak{D}}}\varphi_{\mathfrak{D}}(P)) = \bar{k}(\lambda\gamma\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}\varphi_{\mathfrak{D}}(P)).$$

Hence, by virtue of Lemma 1, λ is an isomorphism. This proves the proposition.

2.2. Let $\hat{A}^*_{\mathfrak{D}}$ be the quotient group variety of $\hat{A}_{\mathfrak{D}}$ by $\{\beta_{\mathfrak{D}}b_{\mathfrak{D}}(\varepsilon_\nu) : \nu = 1, 2, \ldots, n\}$ and $\gamma_{\mathfrak{D}}$ be the natural separable homorphism of $\hat{A}_{\mathfrak{D}}$ onto $\hat{A}^*_{\mathfrak{D}}$. Then we have

$$\gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}\varphi_{\mathfrak{D}}(P^{\varepsilon_\nu^{-1}}) = \gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}\eta(\varepsilon_\nu)\varphi_{\mathfrak{D}}(P) + \gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}b_{\mathfrak{D}}(\varepsilon_\nu) = \gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}\varphi_{\mathfrak{D}}(P).$$

On the other hand, since $\pi_{\mathfrak{D},\bar{\mathfrak{D}}} = \mu_{\mathfrak{D}}\gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}$ with a homomorphism $\mu_{\mathfrak{D}}$ of $\hat{A}_{\mathfrak{D}}$ onto $\bar{J}_{\bar{\mathfrak{D}}}$, we have $\bar{k}(\gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}\varphi_{\mathfrak{D}}(P)) \supset \bar{k}(\pi_{\mathfrak{D},\bar{\mathfrak{D}}}\varphi_{\mathfrak{D}}(P))$. Hence $\bar{k}(\gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}\varphi_{\mathfrak{D}}(P)) = \bar{k}(\pi_{\mathfrak{D},\bar{\mathfrak{D}}}\varphi_{\mathfrak{D}}(P)) = \bar{k}(\bar{\varphi}_{\bar{\mathfrak{D}}}(\hat{\pi}_{\mathfrak{D},\bar{\mathfrak{D}}}(P))$. We denote by $\psi_{\mathfrak{D}}$ the biregular mapping of $C^*_{\bar{\mathfrak{D}}}$ onto $\gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}\varphi_{\mathfrak{D}}(C^*_{\mathfrak{D}})$ such that:

$$\psi_{\mathfrak{D}}(\hat{\pi}_{\mathfrak{D},\bar{\mathfrak{D}}}(P)) = \gamma_{\mathfrak{D}}\beta_{\mathfrak{D}}\varphi_{\mathfrak{D}}(P).$$

LEMMA 3. *Let* (a_{ij}) *be an* n*-square matrix with integral elements respect with a valuation ring* $k + \mathfrak{m}$. *If* $(g_1, \ldots, g_n)$ *is a vector which is non-zero modulo* $\mathfrak{m}$ *and det* $(a_{ij}) \not\equiv 0 \bmod \mathfrak{m}^{d+1}$, *then*

$$(g_1, \ldots, g_n)\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ & \ldots & \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \not\equiv (0, \ldots, 0) \qquad \bmod \mathfrak{m}^{d+1}.$$

Proof. Since $k + \mathfrak{m}$ is a valuation ring, we have unimodular matrices U and V such that

$$U(a_{ij})V$$

is a diagonal matrix with $(k + \mathfrak{m})$-integral elements which do not belong to

$\mathfrak{m}^{d+1}$. This shows that

$$(g_1, \ldots, g_n)U^{-1}(U(a_{ij})V)V^{-1} \not\equiv (0, \ldots, 0) \qquad \mathrm{mod}\ \mathfrak{m}^{d+1}.$$

LEMMA 4. *Let L/K be cyclic and $\sum_{i=1}^{r}\sum_{j=1}^{h_i} d_i M_{ij}$ be the discriminant $d_{L/K}$ of L/K. Put $\mathfrak{o} = k + \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i} \mathfrak{m}_{ij}^{d_i+1}$. Then for any $\mathfrak{o}$-integral element g satisfying $N_{L/K} g \equiv 1 \mod \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i} \mathfrak{m}_{ij}^{d_i+1}$ there exists an $\mathfrak{o}$-integral element ξ such that*

$$\xi + \xi^{\varepsilon^{-1}} g + \xi^{\varepsilon^{-2}} g^{1+\varepsilon^{-1}} + \ldots + \xi^{\varepsilon^{-(n-1)}} g^{1+\varepsilon^{-1}+\ldots+\varepsilon^{-(n-2)}}$$

does not belong to $\mathfrak{m}_{ij}^{d_i+1}$ $(i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, h)$

Proof. After changing the second suffices, we may assume that $M_{i,j} = M_{i,h_i}^{\varepsilon^{-j}}$ $(i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, h_i)$. We denote by $L_{(i)}$ the subfield of L/K corresponding to (ε^{-h_i}). Then $L/L_{(i)}$ completely ramifies at $M_{i,1}, \ldots, M_{i,h_i}$. Let $\xi_{(i)}^*$ be an $\mathfrak{o}$-integral element of L such that the multiplicity of M_{i,h_i} in the discriminant $d_{L/K}(\xi_{(i)}^*)$ of $\xi_{(i)}^*$ is d_i and $\eta_{(i)}$ be an $\mathfrak{o}$-integral element of L such that $\eta_{(i)} \equiv 1 \mod \mathfrak{m}_{i,h_i}^{d_i+1}$ and $\eta_{(i)} \equiv 0 \mod \mathfrak{m}_{i',j'}^{d_{i'}+1}$ for $\mathfrak{m}_{i,j} \neq \mathfrak{m}_{i',j'}$.

We put $\xi_{(j)} = \eta_{(i)}\xi_{(i)}^*$ and $\xi_{(i,j)} = \xi_{(i)}^{\varepsilon^{-j}}$ $(j = 0, 1, \ldots, h_i - 1)$. Since the multiplicity $M_{i,j}$ in the discriminant $d_{L/L_{(i)}}(\xi_{(i,j)})$ of $\xi_{(i,j)}$ is not greater than d_i, if we put

$$a_{ij} = (\xi_{(i',j')}^{j})^{\varepsilon^{-h_{i'}(i-1)}} \qquad (i,\ j = 1, 2, \ldots, e_i),$$

we have a matrix (a_{ij}) such that $\det(a_{ij}) = N_{L/L_{(i')}}(\xi_{(i',j')})(d_{L/L_{(i')}}(\xi_{(i',j')}))^{1/2} \not\equiv 0$ mod $\mathfrak{m}_{i',j'}^{d_{i'}+1}$.

Since $g \not\equiv 0 \mod \mathfrak{m}_{i',h_{i'}}$, putting $g_i = g^{1+\varepsilon^{-1}+\ldots+\varepsilon^{-(h_{i'}(i-1)-1)}}$ we have $(g_1, \ldots, g_{e_{i'}}) \not\equiv (0, \ldots, 0) \mod \mathfrak{m}_{i',h_{i'}}^{d_{i'}+1}$. Hence, by virtue of Lemma 3, we have an integer $\nu_{i'}$ such that i) $0 < \nu_{i'} < e_{i'}$ and ii)

$$\sum_{i=1}^{e_{i'}} g_i a_{i,\nu_{i'}} = \sum_{i=1}^{e_{i'}} (\xi_{(i')})^{\varepsilon^{-h_{i'}(i-1)}} g^{1+\varepsilon^{-1}+\ldots+\varepsilon^{-h_{i'}(i-1)-1}} \not\equiv 0 \qquad \mathrm{mod}\ \mathfrak{m}_{i',h_{i'}}^{d_{i'}+1}.$$

Moreover, we have

$$\begin{aligned} g^{1+\varepsilon^{-1}+\ldots+\varepsilon^{-(j'-1)}}\Big(\sum_{i=1}^{e_{i'}} g_i a_{i,\nu_{i'}}\Big)^{\varepsilon^{-j'}} &= \sum_{i=1}^{e_{i'}} (\xi_{(i')}^{\nu_{i'}})^{\varepsilon^{-h_{i'}(i-1)-j'}} g^{1+\varepsilon^{-1}+\ldots+\varepsilon^{-(h_{i'}(i-1)+j'-1)}} \\ &= \sum_{i=1}^{e_{i'}} (\xi_{(i',j)}^{\nu_{i'}})^{\varepsilon^{-h_{i'}(i-1)}} g^{1+\varepsilon^{-1}+\ldots+\varepsilon^{-(h_{i'}(i-1)+j'-1)}} \\ &\not\equiv 0 \qquad \mathrm{mod}\ \mathfrak{m}_{i',j'}^{d_{i'}+1}. \end{aligned}$$

Put $\xi = \sum_{i=1}^{r} \xi_{(i)}^{\nu_i}$. Then we observe that

$$\begin{aligned}\sum_{j=0}^{n-1}\xi^{\varepsilon^{-j}}g^{1+\varepsilon^{-1}+\dots+\varepsilon^{-(j-1)}} &= \sum_{i=1}^{r}\sum_{j=0}^{n-1}(\xi_{(i)}^{\nu_i})^{\varepsilon^{-j}}g^{1+\varepsilon^{-1}+\dots+\varepsilon^{-(j-1)}} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{h_i}\sum_{l=1}^{e_i}(\xi_{(i,j)}^{\nu_i})^{\varepsilon^{-h_i(l-1)}}g^{1+\varepsilon^{-1}+\dots+\varepsilon^{-(h_i(l-1)+j-1)}} \\ &\equiv \sum_{l=1}^{e_i}(\xi_{(i,j)}^{\nu_i})^{\varepsilon^{-h_i(l-1)}}g^{1+\varepsilon^{-1}+\dots+\varepsilon^{-(h_i(l-1)+j-1)}} \\ &\not\equiv 0 \qquad \mathrm{mod}\ \mathfrak{m}_{i,j'}^{d_i+1}\end{aligned}$$

This proves the lemma.

LEMMA 5. *If L/K is cyclic and e_i $(i=1, 2, \dots, r)$ are coprime to p, then we can choose $e_i - 1$ instead of d_i in Lemma 4.*

Proof. Let τ_{ij} be an $\mathfrak{o}$-integral element such that

$$\tau_{ij}^{\varepsilon^{-h_i}} = \zeta\tau_{ij},$$

where ζ is a primitive root of unity.

On the other hand, since $N_{L/K}g \equiv 1 \bmod \mathfrak{m}_{ij}$, there exists an integer ν_{ij} such that

$$g \equiv e^{2\pi i\nu_{ij}/e_i} \qquad \mathrm{mod}\ \mathfrak{m}_{ij}.$$

Putting $\xi_{ij} = \eta_{ij}\cdot\tau_{ij}{}^{(e_i-\nu_{ij})}$ and $\xi = \sum_{i=1}^{r}\sum_{j=1}^{h_i}\xi_{ij}$, we have

$$\xi + \xi^{\varepsilon^{-1}}g + \dots + \xi^{\varepsilon^{-(n-1)}}g^{1+\varepsilon^{-1}+\dots+\varepsilon^{-(n-2)}}$$
$$\mathrm{mod}\ \mathfrak{m}_{ij}^{(e_i-\nu_{ij})+1} \qquad (i=1, 2, \dots, r;\ j=1, 2, \dots, h_i).$$

This proves the lemma.

LEMMA 6. *Let L/K be cyclic and $\sum_{i=1}^{r}\sum_{j=1}^{h_i}d_{ij}M_{ij}$ be the diskriminant of L/K. Put $\mathfrak{o} = k + \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}^{\nu_{ij}}$ and $\mathfrak{o}' = k + \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}^{\nu_{ij}+d_{ij}+1}$. Then for any element $\bar{f}$ in $\mathfrak{o}' \cap K$ we have $\psi_{\mathfrak{o}}((\bar{f})) = 0$.*

Proof. Since $\psi_{\mathfrak{o}}$ does not depend on k, we may assume that k is algebraically closed. Let H be the kernel of natural homomorphism of $J_{\mathfrak{o}}$ onto the ordinary jacobian variety. Then the quotient variety of $\hat{A}_{\mathfrak{o}}^{*}$ by $(\gamma_{\mathfrak{o}}\beta_{\mathfrak{o}})(H)$ is an abelian variety. Hence, by virtue of the universal property of ordinary jacobian varieties, we observe that $\psi_{\mathfrak{o}}((\bar{f}))$ belongs to $(\gamma_{\mathfrak{o}}\beta_{\mathfrak{o}})(H)$. Namely there exists an $\mathfrak{o}$-integral element g such that $\psi_{\mathfrak{o}}((\bar{f})) = \gamma_{\mathfrak{o}}\beta_{\mathfrak{o}}\varphi_{\mathfrak{o}}(g)) = \psi_{\mathfrak{o}}((N_{L/K}g))$. Therefore it is sufficient to prove $\psi_{\mathfrak{o}}((N_{L/K}g)) = 0$ for any g satisfying $N_{L/K}g \equiv 1$

$\mod \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i} \mathfrak{m}_{ij}^{\nu_{ij}+d_{ij}+1}$. Let g be such an element. Then, by virtue of Lemma 4, there exists an $\mathfrak{o}$-integral element ξ such that $h=\xi+\xi^{\varepsilon^{-1}}g+\xi^{\varepsilon^{-2}}g^{1+\varepsilon^{-1}}+\ldots+\xi^{\varepsilon^{-(n-1)}}g^{1+\varepsilon^{-1}+\ldots+\varepsilon^{-(n-2)}}$ does not belong to $\mathfrak{m}_{ij}^{d_{ij}+1}$ ($i=1, 2, \ldots, r$; $j=1, 2, \ldots, h_i$). Since $yh^{\varepsilon^{-1}}=h+\xi(N_{L/K}g-1)$ and $\xi(N_{L/K}g-1)\equiv 0 \mod \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}^{\nu_{ij}+d_{ij}+1}$, we have

$$gh^{\varepsilon^{-1}}/h\equiv 1 \qquad \mod \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}^{\nu_{ij}}.$$

Let $\sum_{i=1}^{r}\sum_{j=1}^{h_i}a_{ij}M_{ij}$ be the positive divisor such that

$$\mathfrak{a}=(h)-\sum_{i=1}^{r}\sum_{j=1}^{h_i}a_{ij}M_{ij}$$

has no M_{ij} with non-zero multiplicity. Then, since $h^{\varepsilon^{-1}}/h\equiv g^{-1}\not\equiv 0 \mod \mathfrak{m}_{ij}$ ($i=1, 2, \ldots, r$; $j=1, 2, \ldots, h_i$), we have $(\sum_{i=1}^{r}\sum_{j=1}^{h_i}a_{ij}M_{ij})^{\varepsilon^{-1}}=\sum_{i=1}^{r}\sum_{j=1}^{h_i}a_{ij}M_{ij}$. This shows that

$$\psi_{\mathfrak{o}}((N_{L/K}g))=\gamma_{\mathfrak{o}}\beta_{\mathfrak{o}}\varphi_{\mathfrak{o}}(\mathfrak{a}-\mathfrak{a}^{\varepsilon^{-1}})=\gamma_{\mathfrak{o}}\beta_{\mathfrak{o}}(\varphi_{\mathfrak{o}}(\mathfrak{a})-\eta(\varepsilon)\varphi_{\mathfrak{o}}(\mathfrak{a})-(\deg\mathfrak{a})b_{\mathfrak{o}}(\varepsilon))=0.$$

This proves the lemma.

PROPOSITION 4. *Let L/K be cyclic and $\mathfrak{o}=k+\bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}^{\nu_{ij}}$ be co-ample relative to L/K. Let $\sum_{i=1}^{r}\sum_{j=1}^{h_i}d_{ij}M_{ij}$ be the diskriminant of L/K and put $\mathfrak{o}'=k+\bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}^{\nu_{ij}+d_{ij}+1}$ and $\bar{\mathfrak{o}}'=\mathfrak{o}'\cap K$. Then $\pi_{\mathfrak{o}',\bar{\mathfrak{o}}}^{-1}(0)\cong G(L/K)$.*

Proof. By virtue of Lemma 6, the mapping $\psi_{\mathfrak{o}}$ of $\overline{C}_{\bar{\mathfrak{o}}}^{*}$ onto $\gamma_{\mathfrak{o}}\beta_{\mathfrak{o}}\varphi_{\mathfrak{o}}(C_{\mathfrak{o}}^{*})$ can be extended to a homomorphism μ of $\bar{J}_{\bar{\mathfrak{o}}}$ onto $\hat{A}_{\mathfrak{o}}^{*}$. We denote by $\tau_{\mathfrak{o}',\mathfrak{o}}$ the natural homomorphism of $J_{\mathfrak{o}'}$ onto $J_{\mathfrak{o}}$. Then from the definition of μ we have $\gamma_{\mathfrak{o}}\beta_{\mathfrak{o}}\tau_{\mathfrak{o}',\mathfrak{o}}=\mu\pi_{\mathfrak{o}',\bar{\mathfrak{o}}'}$. On the other hand $\tau_{\mathfrak{o}',\bar{\mathfrak{o}}'}^{-1}(0)$ is irreducible and $\mathfrak{o}$ is co-ample relative to L/K, the number of irreducible components of $(\gamma_{\mathfrak{o}}\beta_{\mathfrak{o}}\tau_{\mathfrak{o}',\bar{\mathfrak{o}}'})^{-1}(0)$ is exactly n. Let B^{*} be the union of the components containing elements of $\pi_{\mathfrak{o}',\bar{\mathfrak{o}}'}^{-1}(0)$ and A^{*} be the quotient variety of $J_{\mathfrak{o}'}$ by B^{*}. Let γ^{*} be the homomorphism of $\hat{A}_{\mathfrak{o}}$ onto A^{*} such that $\gamma^{*}\beta_{\mathfrak{o}}\tau_{\mathfrak{o}',\mathfrak{o}}$ is the natural homomorphism of $J_{\mathfrak{o}'}$ onto A*. By virtue of Proposition 8, $\bar{J}_{\bar{\mathfrak{o}}'}$ is the quotient variety of $J_{\mathfrak{o}'}$ by $\pi_{\mathfrak{o}',\mathfrak{o}'}^{-1}(0)$, hence there exists a homomorphism λ of $\bar{J}_{\bar{\mathfrak{o}}'}$ onto A^{*} such that $\gamma^{*}\beta_{\mathfrak{o}}\tau_{\mathfrak{o}',\mathfrak{o}}=\lambda\pi_{\mathfrak{o}',\bar{\mathfrak{o}}'}$. Let P be a generic point of $C_{\mathfrak{o}'}^{*}$, over k. Then $\lambda\bar{\varphi}_{\bar{\mathfrak{o}}'}(\hat{\pi}_{\mathfrak{o}',\bar{\mathfrak{o}}'}(P))=\gamma^{*}(\beta_{\mathfrak{o}}\varphi_{\mathfrak{o}}(P))$. This means $\bar{k}(\bar{\varphi}_{\bar{\mathfrak{o}}'}(\hat{\pi}_{\mathfrak{o}',\bar{\mathfrak{o}}'}(P)))=\bar{k}(\gamma^{*}\beta_{\mathfrak{o}}\varphi_{\mathfrak{o}}(P))$. Therefore we have $\gamma^{*-1}(0)=\gamma^{-1}(0)$. Namely $A^{*}=\hat{A}^{*}$. This proves that the number of components of $\pi^{-1}_{\mathfrak{o}',\bar{\mathfrak{o}}'}(0)$ is exactly n. Hence by virtue of Proposition 2, we have

$$\bar{\pi}^{-1}_{\mathfrak{o}',\bar{\mathfrak{o}}'}(0) \cong G(L/K).$$

LEMMA 7. *Let $L = L_1 L_2$ and $\mathfrak{o}_1$, $\mathfrak{o}_2$ and $\mathfrak{o}$ be local rings of L_1, L_2 and L, respectively. Then, if $N_{L/L_i}\mathfrak{o} \subset \mathfrak{o}_i$ and $\pi^{-1}_{\mathfrak{o}_i,\bar{\mathfrak{o}}}(0) \cong G(L_i/K)$ $(i = 1, 2)$, we have*

$$\pi^{-1}_{\mathfrak{o},\bar{\mathfrak{o}}}(0) = G(L/K).$$

Proof. We denote by $[\varepsilon_\nu]_i$ the element of $G(L_i/K)$ induced by ε_ν. Then we have

$$\begin{aligned}\alpha_{\mathfrak{o}_i,\bar{\mathfrak{o}}}\pi_{\mathfrak{o},\mathfrak{o}_i}b_{\mathfrak{o}}(\varepsilon_\nu) &= \alpha_{\mathfrak{o}_i,\bar{\mathfrak{o}}}\pi_{\mathfrak{o},\mathfrak{o}_i}(\varphi_{\mathfrak{o}}(P^{\varepsilon_\nu^{-1}}) - \eta(\varepsilon_\nu)\varphi_{\mathfrak{o}}(P))\\ &= \alpha_{\mathfrak{o}_i,\bar{\mathfrak{o}}}\varphi_{\mathfrak{o}_i}(\hat{\pi}_{\mathfrak{o},\mathfrak{o}_i}(P)^{[\varepsilon_\nu]_i^{-1}})\\ &\quad - \alpha_{\mathfrak{o}_i,\bar{\mathfrak{o}}}\eta([\varepsilon_\nu]_i)\varphi_{\mathfrak{o}_i}(\hat{\pi}_{\mathfrak{o},\mathfrak{o}_i}(P))\\ &= \alpha_{\mathfrak{o}_i,\bar{\mathfrak{o}}}b_{\mathfrak{o}_i}([\varepsilon_\nu]_i) = a_{\mathfrak{o},\bar{\mathfrak{o}}_i}([\varepsilon_\nu]_i).\end{aligned}$$

On the other hand $\alpha^{-1}_{\mathfrak{o},\bar{\mathfrak{o}}}(0) \subseteqq \bigcap_{i=1}^{2}(\alpha_{\mathfrak{o}_i,\bar{\mathfrak{o}}}\pi_{\mathfrak{o},\mathfrak{o}_i})^{-1}(0)$, hence we observe that

$$a_{\mathfrak{o},\bar{\mathfrak{o}}}(\varepsilon_\nu) = \alpha_{\mathfrak{o},\bar{\mathfrak{o}}}b_{\mathfrak{o}}(\varepsilon_\nu) = 0 \quad \text{implies} \quad \alpha_{\mathfrak{o}_i,\bar{\mathfrak{o}}}b_{\mathfrak{o}_i}([\varepsilon_\nu]_i) = 0 \qquad (i = 1, 2).$$

This shows that $a_{\mathfrak{o}\bar{\mathfrak{o}}}(\varepsilon_\nu) \neq 0$ for $\varepsilon_\nu \neq e$. By virtue of Proposition 3, this proves the lemma.

PROPOSITION 5. *Let $\mathfrak{o} = k + \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}^{\nu_{ij}}$ be co-ample relative to L/K and $\sum_{i=1}^{r}\sum_{j=1}^{h_i} d_{ij}M_{ij}$ be the diskriminant of L/K. Put $\mathfrak{o}' = k + \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}^{\nu_{ij}+d_{ij}+1}$ and $\bar{\mathfrak{o}}' = \mathfrak{o}' \cap K$. Then, if L/K is abelian, $\bar{\pi}^{-1}_{\mathfrak{o}',\bar{\mathfrak{o}}'}(0) \cong G(L/K)$.*

Proof. Let L_i be any cyclic extension of K in L. Then $\mathfrak{o}_{L_i} = \mathfrak{o} \cap L_i$ is co-ample and $\mathfrak{o}'_{L_i} = \mathfrak{o}' \cap L_i$ and, by virtue of Proposition 4, the proposition is true for this $\mathfrak{o}_{L_i}$. On the other hand L is composed by cyclic extensions L_i, hence, by virtue of Lemma 7, we get the proposition.

2.3. We say that a separable extension L/k of K/k comes from a pull back of a separable homomorphism λ of a commutative group variety onto a generalized jacobian variety $\bar{J}_{\bar{\mathfrak{o}}}$ of K/k, if there exists a model of L/k which is biregularly equivalent to $\lambda^{-1}(\bar{\varphi}_{\bar{\mathfrak{o}}}(\bar{C}^*_{\bar{\mathfrak{o}}}))$ and $G(L/K) = \lambda^{-1}(0)$.

Putting $A = \bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ and $\lambda = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}$, from Proposition 5, we have

THEOREM 1. *Let K/k be a regular extension of dimension one over a perfect field k and L/k be a separable abelian extension of K/k which is also regular* *over k. Then L/k comes from a pull back of a separable homomorphism of a commutative group variety onto a generalized jacobian variety associated a suitable local ring.*

§ 3. Class field theory

3. 1. We denote by G_a and G_m respectively the affine line with addition of coordinates as group multiplication and the affine line with origin deleted and multiplication of coordinates as group multiplication. We mean by affine groups the group varieties which are biregularlly equivalent to entire affine space. An affine group H has a chain of affine subgroups $H=H_1 \supset H_2 \ldots \supset H_r=\{e\}$ such that H_i/H_{i+1} $(i=1, 2, \ldots, r-1)$ are birationally isomorphic to G_a.

By virtue of the structure theorem of generalized jacobian varieties[7] the kernel of the natural homomorphism of $J_{\mathfrak{o}}$ onto the ordinary jacobian variety is birationally isomorphic to a group variety

$$(G_m)^{(\sum_{i=1}^{r}\sum_{j=1}^{h_i}\deg M_{ij})-1} \times H_1,$$

i.e., the direct product of G_m with itself $((\sum_{i=1}^{r}\sum_{j=1}^{h_i}\deg M_{ij})-1)$ times by an affine group H_1, where $\mathfrak{o}=k+\bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}^{\nu_{ij}}(\nu_{ij}\geqq 1)$.

3. 2. If we put

$$\mathfrak{o}_0=k+\bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij} \tag{6}$$

and

$$\bar{\mathfrak{o}}_0=k+\bigcap_{i=1}^{r}\bar{\mathfrak{m}}_i, \tag{7}$$

then $J_{\mathfrak{o}_0}$ and $\bar{J}_{\bar{\mathfrak{o}}_1}$ have no affine subgroup.

PROPOSITION 6. *$J_{\mathfrak{o}_0}$ and $\bar{J}_{\bar{\mathfrak{o}}_0}$ have only a finite number of points of given order.*

Proof. The kernel of natural homomorphism of $J_{\mathfrak{o}_0}(\bar{J}_{\bar{\mathfrak{o}}_0})$ onto the ordinary jacobian variety is isomorphic to a direct product of G_m. Hence the kernel has only a finite number of points of given order. On the other hand ordinary jacobian varieties have only a finite number of points of given order. Therefore

[7] Cf. Theorem 12, [5].

$J_{\mathfrak{o}_0}(\bar{J}_{\bar{\mathfrak{o}}_0})$ has only a finite number of points of given order.

LEMMA 1. $N_{L/K}\mathfrak{o}_0 \subset \bar{\mathfrak{o}}_0$.

Proof. Let f be any element of $\bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}$. Then we have $N_{L/K}(1+f) = 1+\sum_{\nu} f^{\varepsilon_\nu}+\sum_{\nu>\mu} f^{\varepsilon_\nu}f^{\varepsilon_\mu}+\ldots+N_{L/K}f \equiv 1 \mod \bigcap_{i=1}^{r}\bar{\mathfrak{m}}_i$. This proves the lemma.

This lemma shows the existence of the trace mapping (homomorphism) $\pi_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}$ of $J_{\mathfrak{o}_0}$ onto $\bar{J}_{\bar{\mathfrak{o}}_0}$. On the other hand $\mathfrak{o}_0 \supset \bar{\mathfrak{o}}_0$, hence there exists the injection homomorphism $\rho_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}$ of $\bar{J}_{\bar{\mathfrak{o}}_0}$ into $J_{\mathfrak{o}_0}$ such that

$$\rho_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}\bar{\varphi}_{\bar{\mathfrak{o}}}(\bar{\mathfrak{a}}) = \varphi_{\mathfrak{o}_0}(\hat{\pi}^{-1}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\bar{\mathfrak{a}})),$$

where $\bar{\mathfrak{a}}$ runs over divisors of degree zero on $\bar{C}^*_{\bar{\mathfrak{o}}_0}$.

LEMMA 2. $\pi_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}\rho_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0} = n\delta_{\bar{J}_{\bar{\mathfrak{o}}_0}}$.

Proof. Let $\bar{\mathfrak{a}}$ be a divisor of degree zero on $\bar{C}^*_{\bar{\mathfrak{o}}_0}$. Then we have

$$\begin{aligned}\pi_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}\rho_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}\bar{\varphi}_{\bar{\mathfrak{o}}_0}(\bar{\mathfrak{a}}) &= \pi_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}\varphi_{\mathfrak{o}_0}(\hat{\pi}^{-1}_{\mathfrak{o}_0,\mathfrak{o}_0}(\bar{\mathfrak{a}}))\\ &= \bar{\varphi}_{\bar{\mathfrak{o}}_0}(\hat{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}_0}(\hat{\pi}^{-1}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\bar{\mathfrak{a}}))\\ &= n\delta_{\bar{J}_{\bar{\mathfrak{o}}_0}}\bar{\varphi}_{\bar{\mathfrak{o}}_0}(\bar{\mathfrak{a}}).\end{aligned}$$

This proves the lemma.

PROPOSITION 7. *$\rho_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\bar{J}_{\bar{\mathfrak{o}}_0})$ and $\hat{B}_{\mathfrak{o}_0}$ generate $J_{\mathfrak{o}_0}$ and $\rho_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\bar{J}_{\bar{\mathfrak{o}}_0}) \cap \hat{B}_{\mathfrak{o}_0}$ is a finite group whose elements are of order n. Moreover $\hat{B}_{\mathfrak{o}_0} = B_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}$.*

Proof. Let x be a generic point of $J_{\mathfrak{o}_0}$ over k. Then $(\sum_{\nu=1}^{n}\eta(\varepsilon_\nu))x$ is a generic point of $\rho_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\bar{J}_{\bar{\mathfrak{o}}_0})$ over k. First we shall prove that $(n\delta_{J_{\mathfrak{o}_0}}-\sum_{\nu=1}^{n}\eta(\varepsilon_\nu))x$ is a generic point of $\hat{B}_{\mathfrak{o}_0}$ over k. Denoting by $x_1, \ldots, x_n$ independent generic points of $J_{\mathfrak{o}_0}$ over k, we defined $\hat{B}_{\mathfrak{o}_0}$ as the locus of $\sum_{\nu=1}^{n}(\delta_{J_{\mathfrak{o}_0}}-\eta(\varepsilon_\nu))x_\nu$ over k. Since $J_{\mathfrak{o}_0}$ has only a finite number of given order, any point of $J_{\mathfrak{o}_0}$ is divisible. Hence we can put $x_\nu = ny_\nu$ $(\nu = 1, 2, \ldots, n)$. We observe that

$$\begin{aligned}\sum_{\nu=1}^{n}(\delta_{J_{\mathfrak{o}_0}}-\eta(\varepsilon_\nu))x_\nu &= \sum_{\nu=1}^{n}(\delta_{J_{\mathfrak{o}_0}}-\eta(\varepsilon_\nu))(\sum_{l=1}^{n}\eta(\varepsilon_l)y_\nu+(n\delta_{J_{\mathfrak{o}_0}}-\sum_{l=1}^{n}\eta(\varepsilon_l))y_\nu)\\ &= (n\delta_{J_{\mathfrak{o}_0}}-\sum_{l=1}^{n}\eta(\varepsilon_l))(\sum_{\nu=1}^{n}(\delta_{J_{\mathfrak{o}_0}}-\eta(\varepsilon_\nu))y_\nu.\end{aligned}$$

This shows that $(n\delta_{J_{\mathfrak{o}_0}}-\sum_{l=1}^{n}\eta(\varepsilon_l))x$ is a generic point of $\hat{B}_{\mathfrak{o}_0}$ over k. On the other hand

$$(\sum_{\nu=1}^{n}\eta(\varepsilon_\nu))x+(n\delta_{J_{\mathfrak{D}_0}}-\sum_{\nu=1}^{n}\eta(\varepsilon_\nu))x=nx$$

is a generic point of $J_{\mathfrak{D}_0}$. This proves the first assertion. Let y be a point of $\rho_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(\bar{J}_{\bar{\mathfrak{D}}_0})\cap\hat{B}_{\mathfrak{D}_0}$. Then we can put $y=(n\delta_{J_{\mathfrak{D}_0}}-\sum_{\nu=1}^{n}\eta(\varepsilon_\nu))z$. Since $\sum_{\nu=1}^{n}\eta(\varepsilon_\nu)z \in \rho_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(\bar{J}_{\bar{\mathfrak{D}}_0})$, nz belongs to $\rho_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(\bar{J}_{\bar{\mathfrak{D}}_0})$. This shows that $ny=0$. This proves the second assertion. By virtue of the second assertion there exists a homomorphism μ of $\hat{A}_{\mathfrak{D}_0}$ onto $\bar{J}_{\bar{\mathfrak{D}}_0}$ such that $\pi_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}=\mu\beta_{\mathfrak{D}_0}$ and $\mu^{-1}(0)$ is a finite group. This shows that $\hat{B}_{\mathfrak{D}_0}=B_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}$.

3. 3.

LEMMA 3. *$a_{\mathfrak{D},\bar{\mathfrak{D}}}(\varepsilon_\nu)$ $(\nu=1, 2, \ldots, n)$ are k-rational.*

Proof. Let k' be a finitely normal extension of k such that there exists a k'-rational point P on $C_{\mathfrak{D}}^*$. Then we have a canonical mapping $\varphi_{\mathfrak{D}}$ of $C_{\mathfrak{D}}^*$ into $J_{\mathfrak{D}}$ defined over k' such that $Q_1\times Q_2\rightarrow\varphi_{\mathfrak{D}}(Q_1-Q_2)$ is a mapping of $C_{\mathfrak{D}}^*\times C_{\mathfrak{D}}^*$ into $J_{\mathfrak{D}}$ defined over k. Let σ be any automorphism of k'/k and Γ be the graph of $\varphi_{\mathfrak{D}}$ on $C_{\mathfrak{D}}^*\times J_{\mathfrak{D}}$. Then Γ^σ is the graph of the canonical mapping $\varphi_{\mathfrak{D}}^\sigma$ and $\varphi_{\mathfrak{D}}^\sigma-\varphi_{\mathfrak{D}}$ is a constant mapping of $C_{\mathfrak{D}}^*$ onto a point c of $J_{\mathfrak{D}}$. On the other hand $Q\rightarrow Q^{\varepsilon_\nu^{-1}}$ $(\nu=1, 2, \ldots, n)$ are mappings defined over k, hence we have $P^{\varepsilon_\nu^{-1}\sigma}=P^{\sigma\varepsilon_\nu^{-1}}$ $(\nu=1, 2, \ldots, n)$. Therefore we observe that

$$\begin{aligned}a_{\mathfrak{D},\bar{\mathfrak{D}}}(\varepsilon_\nu)^\sigma&=(\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}(\varphi_{\mathfrak{D}}(P^{\varepsilon_\nu^{-1}})-\eta(\varepsilon_\nu)\varphi_{\mathfrak{D}}(P))^\sigma\\&=\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}(\varphi_{\mathfrak{D}}^\sigma(P^{\varepsilon_\nu^{-1}\sigma})-\eta(\varepsilon_\nu)\varphi_{\mathfrak{D}}^\sigma(P^\sigma))\\&=\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}(\varphi_{\mathfrak{D}}(P^{\sigma\varepsilon_\nu^{-1}})-\eta(\varepsilon_\nu)\varphi_{\mathfrak{D}}(P^\sigma)+(c-\eta(\varepsilon_\nu)c))\\&=\alpha_{\mathfrak{D},\bar{\mathfrak{D}}}b_{\mathfrak{D}}(\varepsilon_\nu)=a_{\mathfrak{D},\bar{\mathfrak{D}}}(\varepsilon_\nu)\qquad(\nu=1, 2, \ldots, n).\end{aligned}$$

This proves the lemma.

We denote by $J_{\mathfrak{D}}(m)$, $\bar{A}(m)$, $\Lambda(m)$ and $\bar{J}_{\bar{\mathfrak{D}}}(m)$ the subgroups consisting of all the points of order dividing m on $J_{\mathfrak{D}}$, $\bar{A}$, Λ and $\bar{J}_{\bar{\mathfrak{D}}}$ respectively. We denote by $J_{\mathfrak{D}}(\ ,k)$, $\bar{A}(\ ,k)$, $\Lambda(\ ,k)$ and $\bar{J}_{\bar{\mathfrak{D}}}(\ ,k)$ the (abstract) subgroups consisting of all the k-rational points of $J_{\mathfrak{D}}$, $\bar{A}$, Λ and $\bar{J}_{\bar{\mathfrak{D}}}$ respectively.

LEMMA 4. $\alpha_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(J_{\mathfrak{D}_0}(n))=A_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(n)$.

Proof. Let $\mathfrak{g}$ be the subgroup of points a in $\rho_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(\bar{J}_{\bar{\mathfrak{D}}_0})$ such that $n\dot{a}\in B_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}$. Then we have $\alpha_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(\mathfrak{g})=\bar{A}_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(n)$. On the other hand $J_{\mathfrak{D}_0}(n)=\{a-b\,|\,a\in\rho_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(\bar{J}_{\bar{\mathfrak{D}}_0}),\ b\in B_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0},\ na=nb\}$ and any points of $J_{\mathfrak{D}_0}$ is divisible; hence we have $\mathfrak{g}\subset J_{\mathfrak{D}_0}(n)$ and $\alpha_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(J_{\mathfrak{D}_0}(n))=\bar{A}_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(n)$. Namely we have $\alpha_{\mathfrak{D}_0,\bar{\mathfrak{D}}_0}(\mathfrak{g})$

$= \alpha_{\mathfrak{o}_0, \bar{\mathfrak{o}}_0}(J_{\mathfrak{o}_0}(\mathfrak{n}))$. This proves the lemma.

LEMMA 5. *If k is a finite field, then*

$$\alpha_{\mathfrak{o},\bar{\mathfrak{o}}}(J(\ ,k)) = \bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}(\ ,k).$$

Proof. Let $\bar{a}$ be a point of $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}(\ ,k)$ and a be an algebraic point of $J_{\mathfrak{o}}$ over k such that $\alpha_{\mathfrak{o},\bar{\mathfrak{o}}}a = \bar{a}$. Let σ be a generator of the galois group of $k(a)/k$. Then $a^{\sigma} - a = b$ is a point of $B_{\mathfrak{o},\bar{\mathfrak{o}}}$. Therefore, if we prove that there exists a point b_1 in $B_{\mathfrak{o},\bar{\mathfrak{o}}}(\ ,k(a))$ satisfying $b = b_1^{\sigma} - b_1$, we get a point $a_1 = a - b_1$ in $J_{\mathfrak{o}}(\ ,k)$ such that $\alpha_{\mathfrak{o},\bar{\mathfrak{o}}}a_1 = \bar{a}$. Let q be the number of elements in k and $\mathfrak{p}$ be the endomorphism of $B_{\mathfrak{o},\bar{\mathfrak{o}}}$ induced by the automorphism $x \to x^q$ of the universal domain. Then we may assume that $b^{\sigma} = \mathfrak{p}b$. Since $b = a^{\sigma} - a$, we have

$$(\delta_{B_{\mathfrak{o},\bar{\mathfrak{o}}}} + \mathfrak{p} + \ldots + \mathfrak{p}^{d-1})b = 0,$$

where $d = [k(a) : k]$. On the other hand $(\delta_{B_{\mathfrak{o},\bar{\mathfrak{o}}}} - \mathfrak{p})^{-1}(0)$ is the group of all k-rational points of $B_{\mathfrak{o},\bar{\mathfrak{o}}}$, hence $\delta_{B_{\mathfrak{o},\bar{\mathfrak{o}}}} - \mathfrak{p}$ is an onto endomorphism. Therefore we have a point b_1 in $B_{\mathfrak{o},\bar{\mathfrak{o}}}$ such that $(\delta_{B_{\mathfrak{o},\bar{\mathfrak{o}}}} - \mathfrak{p})b_1 = b$. Hence we have

$$(\delta_{B_{\mathfrak{o},\bar{\mathfrak{o}}}} - \mathfrak{p}^d)b_1 = (\delta_{B_{\mathfrak{o},\bar{\mathfrak{o}}}} + \mathfrak{p} + \ldots + \mathfrak{p}^{d-1})(\delta_{B_{\mathfrak{o},\bar{\mathfrak{o}}}} - \mathfrak{p})b_1 = 0.$$

This shows that $b_1 \in B_{\mathfrak{o},\bar{\mathfrak{o}}}(\ ,k(a))$.

THEOREM 2. *If all the indices of ramification of L/K are coprime to p, then we have*

$$\bar{J}_{\bar{\mathfrak{o}}_0}(\mathfrak{n})/\pi_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(J_{\mathfrak{o}_0}(\mathfrak{n})) \cong G(L/K).$$

Proof. By virtue of proposition 7, we have

$$a_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\varepsilon_\nu) = \beta_{\mathfrak{o}_0}b_{\mathfrak{o}_0}(\varepsilon_\nu) \qquad (\nu = 1, 2, \ldots, n).$$

Let $\mathfrak{o} = k + \bigcap_{i=1}^{r}\bigcap_{j=1}^{h_i}\mathfrak{m}_{ij}^{\nu}$ be a local ring of L satisfying $\mathfrak{o}_0 \supset \mathfrak{o}$ and $\pi_{\mathfrak{o},\mathfrak{o}}^{-1}(0) \cong G(L/K)$, where $\bar{\mathfrak{o}} = \mathfrak{o} \cap K$. By virtue of Proposition 5, such a local ring always exists. We denote by $\tau_{\mathfrak{o},\mathfrak{o}_0}$ the natural homomorphism of $J_{\mathfrak{o}}$ onto $J_{\mathfrak{o}_0}$. Then there exists a homomorphism γ of $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ onto $\bar{A}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}$ such that $\gamma\alpha_{\mathfrak{o},\bar{\mathfrak{o}}} = \alpha_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}\tau_{\mathfrak{o},\mathfrak{o}_0}$ and $\gamma a_{\mathfrak{o},\bar{\mathfrak{o}}}(\varepsilon_\nu) = a_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\varepsilon_\nu)$ $(\varepsilon_\nu \in G(L/K))$. Since the kernel of $\tau_{\mathfrak{o},\mathfrak{o}_0}$ is an affine group, the kernel of γ is also an affine group. Therefore, for any integer r coprime to p, there exists no element in $\gamma^{-1}(0)$ whose order is r. This shows that the kernel of the homomorphism $a_{\mathfrak{o},\bar{\mathfrak{o}}}(\varepsilon_\nu) \to a_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\varepsilon_\nu)$ $(\varepsilon_\nu \in G(L/K))$ is contained in the p-sylow group of $\{a_{\mathfrak{o},\bar{\mathfrak{o}}}(\varepsilon_\nu)\}$. Let L^* be the subfield of L such that $[L : L^*]$

is coprime to p and $[L^* : K]$ is a p's power. Then L^*/K is unramified. Therefore, by virtue of Proposition 5, $\bar{\pi}_{L^*,K}^{-1}(0) = \{a_{L^*,K}(\varepsilon) \mid \varepsilon \in G(L^*/K)\}$ is isomorphic to $G(L^*/K)$. On the other hand there exists a homomorphism γ^* of $A_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}$ onto $\bar{A}_{L^*,K}$ such that $\gamma^* a_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\varepsilon_\nu) = a_{L^*,K}([\varepsilon_\nu])$, where $[\varepsilon_\nu]$ is the class of ε_ν in $G(L^*/K)$. This shows that the p-sylow group of $\bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}^{-1}(0)$ is isomorphic to that of $G(L/K)$. Therefore $\bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}^{-1}(0)$ is isomorphic to $G(L/K)$.

Let μ be the homomorphism of $\bar{J}_{\bar{\mathfrak{o}}_0}$ onto $\bar{A}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}$ such that $\bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}\mu = n\delta_{\bar{J}_{\bar{\mathfrak{o}}_0}}$. Then we have $\mu\bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0} = n\delta_{\bar{A}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}}$, $\bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}^{-1}(0) = \mu(\bar{J}_{\bar{\mathfrak{o}}_0}(n))$ and $\mu^{-1}(0) = \bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\bar{A}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(n))$. This shows that $\bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}^{-1}(0) \cong \bar{J}_{\bar{\mathfrak{o}}_0}(n)/\mu^{-1}(0) = \bar{J}_{\bar{\mathfrak{o}}_0}(n)/\bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(\bar{A}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(n))$. Hence, by virtue of Lemma 4, we have

$$\bar{J}_{\bar{\mathfrak{o}}_0}(n)/\pi_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}(J_{\mathfrak{o}_0}(n)) \cong G(L/K).$$

THEOREM 3. *If k is a finite field, then there exists a local ring $\mathfrak{o}$ of L such that*

$$J_{\bar{\mathfrak{o}}}(\ ,\ k)/\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}(J_{\mathfrak{o}}(\ ,\ k)) \cong G(L/K),$$

where $\bar{\mathfrak{o}} = \mathfrak{o} \cap K$.

Proof. Let $\mathfrak{o}$ be the local ring in Theorem 1. Then we have $\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}^{-1}(0) \cong G(L/K)$. Let q be the number of elements in k. We denote by $\mathfrak{p}_{\bar{J}_{\bar{\mathfrak{o}}}}$ and $\mathfrak{p}_{\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}}$ respectively the endomorphisms of $\bar{J}_{\bar{\mathfrak{o}}}$ and $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ induced by the automorphism $x \to x^q$ of the universal domain. Then we have $\mathfrak{p}_{\bar{J}_{\bar{\mathfrak{o}}}}\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}} = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}\mathfrak{p}_{\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}}$. Therefore $(\delta_{\bar{J}_{\bar{\mathfrak{o}}}} - \mathfrak{p}_{\bar{J}_{\bar{\mathfrak{o}}}})\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}} = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}(\delta_{\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}} - \mathfrak{p}_{\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}})$. Let μ be the homomorphism of $\bar{J}_{\bar{\mathfrak{o}}}$ onto $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ such that $\mu\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}} = (\delta_{\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}} - \mathfrak{p}_{\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}})$. Then, since $\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}\mu\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}} = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}(\delta_{\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}} - \mathfrak{p}_{\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}}) = (\delta_{\bar{J}_{\bar{\mathfrak{o}}}} - \mathfrak{p}_{\bar{J}_{\bar{\mathfrak{o}}}})\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}$, we have $\delta_{\bar{J}_{\bar{\mathfrak{o}}}} - \mathfrak{p}_{\bar{J}_{\bar{\mathfrak{o}}}} = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}\mu$. Hence we have $\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}^{-1}(0) = \mu(J(\ ,\ k))$ and $\mu^{-1}(0) = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}(\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}(\ ,\ k))$. Therefore $\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}^{-1}(0)$ must be isomorphic to $\bar{J}_{\bar{\mathfrak{o}}}(\ ,\ k)/\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}(\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}(\ ,\ k))$. From Lemma 5, we get $G(L/K) \cong \bar{J}_{\bar{\mathfrak{o}}}(\ ,\ k)/\pi_{\mathfrak{o},\bar{\mathfrak{o}}}(J_{\mathfrak{o}}(\ ,\ k))$.

3.4. In the following, if $\mathfrak{o}'$ and $\bar{\mathfrak{o}}'$ are respectively local ring in the function field L and K such that $N_{L/K}\mathfrak{o}' \subset \bar{\mathfrak{o}}'$, we shall mean by $J_{\mathfrak{o}'}$, $B_{\mathfrak{o}',\bar{\mathfrak{o}}'}$ and $\bar{A}_{\mathfrak{o}',\bar{\mathfrak{o}}'}$ respectively the objects associating with the system $(L, K, \mathfrak{o}', \bar{\mathfrak{o}}')$ corresponding to $J_{\mathfrak{o}}$, $B_{\mathfrak{o},\bar{\mathfrak{o}}}$ and $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ associating with the system $(L, K, \mathfrak{o}, \bar{\mathfrak{o}})$

LEMMA 8. *Let A be a commutative group variety defined over k and λ be a homomorphism of A onto $\bar{J}_{\bar{\mathfrak{o}}}$ whose kernel $\lambda^{-1}(0)$ is a finite group consisting of k-rational points of A. Let y be a point of A such that λy is a generic point of $\bar{\varphi}_{\bar{\mathfrak{o}}}(\bar{C}_{\bar{\mathfrak{o}}}^*)$. Then, if $\bar{\varphi}_{\bar{\mathfrak{o}}}$ is defined over k' and $k' \supset k$, $k(y)$ is normal over $k(\lambda y)$*

and the galois group $G(k'(y)/k'(\lambda y))$ *is isomorphic to* $\lambda^{-1}(0)$.

Proof. Let y' be a conjugate of y over $k(\lambda y)$. Then we have $\lambda y' = \lambda y$. Let $\mathfrak{g}$ be the group of all the points t of A such that $y+t$ is a conjugate of y over $k'(\lambda y)$. Then $\mathfrak{g} \subset \lambda^{-1}(0)$. Let η be the natural homomorphism of A onto $A/\mathfrak{g}$ and ξ be the homomorphism of $A/\mathfrak{g}$ onto $J_{\bar{\mathfrak{o}}}$ such that $\lambda = \xi\eta$. Then, since $k'(\eta y)/k'(\lambda y)$ is purely inseparable, by virtue of Lemma 1 in §2, ξ must be purely inseparable. This shows that

$$\lambda^{-1}(0) = \mathfrak{g} \quad \text{and} \quad G(k'(y)/k'(\lambda y)) \cong \lambda^{-1}(0).$$

LEMMA 9. *Let* $\tilde{C}$ *be the locus of* y *in Lemma 8 over* k' *and* $\bar{\mathfrak{o}}$ *be a local ring in* $k'(y)$ *such that* $N_{k'(y)/k'(\lambda y)}\bar{\mathfrak{o}} \subseteqq \bar{\mathfrak{o}}$. *Then there exists a homomorphism* μ *of* $J_{\bar{\mathfrak{o}}}$ *onto* A *such that* $\lambda\mu = \pi_{\bar{\mathfrak{o}},\bar{\mathfrak{o}}}$.

Proof. Let C^* be a copy of $\tilde{C}$ and f be the biregular mapping of C^* onto $\tilde{C}$. Let r be a positive integer greater than $2(\dim J_{\bar{\mathfrak{o}}})$ and $Q_1, Q_2, \ldots, Q_r$ be independent generic points of $C^*_{\bar{\mathfrak{o}}}$ over k. Then $f(Q_1 + \ldots + Q_r)$ is a generic point of A over k. Let l be an integer such that (i) $r \leqq l$ and (ii) there exists a k-rational positive divisor of degree l $S_1 + S_2 + \ldots + S_l$ on C^*. Put $H^*_{\bar{\mathfrak{o}}} = \{Q_1 \times Q_2 \times \ldots \times Q_l \mid Q_i \in C^*,\ \varphi_{\bar{\mathfrak{o}}}((Q_1 + \ldots + Q_l) - (S_1 + \ldots + S_l)) = 0\}$ and $H_{\bar{\mathfrak{o}}} = \{f((Q_1 + \ldots + Q_l) - (S_1 + \ldots + S_l)) \mid Q_1 \times Q_2 \times \ldots \times Q_l \in H^*_{\bar{\mathfrak{o}}}\}$. Then $H_{\bar{\mathfrak{o}}}$ is a subvariety of A. From $N_{k'(y)/k'(\lambda y)}\bar{\mathfrak{o}} \subset \bar{\mathfrak{o}}$, we have $\lambda(H_{\bar{\mathfrak{o}}}) = 0$. On the other hand $H_{\bar{\mathfrak{o}}}$ is irreducible, hence we have $H_{\bar{\mathfrak{o}}} = 0$.

LEMMA 10. *Let* L/K *be purely inseparable. If* $\mathfrak{o} = k + \bigcap_{i=1}^{r} \mathfrak{m}^{\nu_i}$ *and* $\bar{\mathfrak{o}} = k + \bigcap_{i=1}^{r} \bar{\mathfrak{m}}_i^{\nu_i}$, *then* $\pi_{\mathfrak{o},\bar{\mathfrak{o}}}$ *is purely inseparable.*

Proof. From the inseparability of L/K, we have $N_{L/K}\mathfrak{o} \subset \bar{\mathfrak{o}}$ and $\deg \mathfrak{m}_i = \deg \bar{\mathfrak{m}}_i$ $(i = 1, 2, \ldots, r)$. This shows that $\pi_{\mathfrak{o},\bar{\mathfrak{o}}}$ exists and the dimension of $J_{\mathfrak{o}}$ equal to that of $\bar{J}_{\bar{\mathfrak{o}}}$. Let $Q_1, Q_2, \ldots, Q_g$ be independent generic points of $C^*_{\mathfrak{o}}$ over k, where g is the dimension of $J_{\mathfrak{o}}$. Then $\varphi_{\mathfrak{o}}(Q_1 + \ldots + Q_g)$ and $\pi_{\mathfrak{o},\bar{\mathfrak{o}}}\varphi_{\mathfrak{o}}(Q_1 + \ldots Q_g)$ are generic points of $J_{\mathfrak{o}}$ and $\bar{J}_{\bar{\mathfrak{o}}}$, respectively. Since $\bar{k}(Q_i)/k(\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}(Q_i))$, we observe that $\bar{k}(\varphi_{\mathfrak{o}}(Q_1 + \ldots + Q_g)) = \bar{k}(Q_1, \ldots, Q_g)_s$ is are purely inseparable over $\bar{k}(\pi_{\mathfrak{o},\bar{\mathfrak{o}}}\varphi_{\bar{\mathfrak{o}}}(Q_1 + \ldots + Q_g)) = \bar{k}(\pi_{\mathfrak{o},\bar{\mathfrak{o}}}\varphi_{\mathfrak{o}}(Q_1), \ldots, \pi_{\mathfrak{o},\bar{\mathfrak{o}}}\varphi_{\mathfrak{o}}(Q_g))_s$, where $\bar{k}(\ )_s$ means the subfield of $k(\ )$ consisting of all the elements fixed by any permutation of sufficis. This prove that $\pi_{\mathfrak{o},\bar{\mathfrak{o}}}$ is purely inseparable.

LEMMA 11. *Let L be the maximal separable subfield of $k'(y)$ in Lemma 8 over $k'(\lambda y)$ writting $\mathfrak{o} = k + \bigcap_i \bigcap_j \mathfrak{m}_{ij}^{\nu_i}$, we put $\tilde{\mathfrak{o}} = k + \bigcap_i \bigcap_j (\mathfrak{m}_{ij} \cap L)^{\nu_i}$. Then there exist a purely inseparable homomorphisms ζ and ξ of $\bar{A}_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}}$ respectively onto Λ and $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ such that $\pi_{\mathfrak{o},\bar{\mathfrak{o}}} = \lambda\zeta$ and $\pi_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}} = \pi_{\mathfrak{o},\bar{\mathfrak{o}}}\xi$. Moreover there exists a purely inseparable homomorphism β of Λ onto $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ such that $\lambda = \pi_{\mathfrak{o},\bar{\mathfrak{o}}}\beta$.*

Proof. Since $\alpha_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}}^{-1}(0) \supset (\alpha_{\mathfrak{o},\bar{\mathfrak{o}}}\pi_{\tilde{\mathfrak{o}},\mathfrak{o}})^{-1}(0)$, there exists a homomorphism ξ of $\bar{A}_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}}$ onto $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ such that $\pi_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}} = \pi_{\mathfrak{o},\bar{\mathfrak{o}}}\xi$. By virtue of Lemma 10, ξ must be purely inseparable. Let μ be the homomorphism of $J_{\tilde{\mathfrak{o}}}$ onto Λ such that $\pi_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}} = \lambda\mu$. Then $\mu^{-1}(0)$ contains $B_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}}$. This shows that there exists a homomorphism ζ of $\bar{A}_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}}$ onto Λ such that $\pi_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}} = \lambda\zeta$. On the other hand the order of $\pi_{\mathfrak{o},\bar{\mathfrak{o}}}^{-1}(0)$ is at most that of $G(k'(y)/k'(\lambda y))$ and $G(k'(y)/k'(\lambda y))$ is isomorphic to $\lambda^{-1}(0)$, hence ζ is purely inseparable. Next we shall prove the existence of a purely inseparable homomorphism β of Λ onto $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$. Let x be a generic point of $\bar{A}_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}}$ over k. Then $k'(\xi x)$ is the maximal separable subfield of $k'(x)$ over $k'(\pi_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}}x)$. On the other hand the degree of separability of $k'(\zeta x)/k'(\pi_{\tilde{\mathfrak{o}},\bar{\mathfrak{o}}}x)$ equals to the order of $\lambda^{-1}(0)$, hence $k'(\zeta x) \supset k'(\xi x)$. This shows that there exists a purely inseparable homomorphism β of Λ onto $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ such that $\lambda = \pi_{\mathfrak{o},\bar{\mathfrak{o}}}\beta$.

LEMMA 12. *In Lemma 8, if λ is separable, $k'(y)/k'(\lambda y)$ is separable.*

Proof. Using the notation of Lemma 11, we observe that β must be an isomorphism. Let z be the point of $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ such that $\beta y = z$. Then, since $k'(z)/k'(\lambda y)$ is separable, $k'(y)/k'(\lambda y)$ must be separable.

PROPOSITION 8. *If k is a finite field, then the canonical mapping $\varphi_{\mathfrak{o}}(\bar{\varphi}_{\bar{\mathfrak{o}}})$ can be defined over k.*

Proof. Let k' be a finite extension of k over which $\varphi_{\mathfrak{o}}$ is defined. We denote by σ the generator of the galois group of k'/k such that $a^{\sigma} = \mathfrak{p}\ a$ for any point of $J_{\mathfrak{o}}(\ ,k')$, where $\mathfrak{p}$ is the endomorphism of $J_{\mathfrak{o}}$ corresponding to the automorphism $x \to x^q$ of the universal domain. Then $\varphi_{\mathfrak{o}}^{\sigma^\nu} - \varphi_{\mathfrak{o}}$ is a constant mapping of $C_{\mathfrak{o}}^*$ onto a k'-rational point c_{σ^ν} and $\{c_{\sigma^\nu}\}$ satisfies the relation

$$c_{\sigma^\nu} = c_{\sigma^{\nu-1}}^{\sigma} + c_\sigma = \mathfrak{p}c_{\sigma^{\nu-1}} + c_\sigma = (\delta_{J_{\mathfrak{o}}} + \mathfrak{p} + \ . \ . \ . + \mathfrak{p}^{\nu-1})c_\sigma.$$

On the other hand $(\delta_{J_{\mathfrak{o}}} - \mathfrak{p})$ is an onto separable endomorphism of $J_{\mathfrak{o}}$, there exists a point b in $J_{\mathfrak{o}}$ such that $(\delta_{J_{\mathfrak{o}}} - \mathfrak{p})b = c_\sigma$. If k'/k is of degree a, we have

$$\begin{aligned}(\delta_{J_{\mathfrak{d}}} - \mathfrak{p}^{d})b &= (\delta_{J_{\mathfrak{d}}} + \mathfrak{p} + \dots + \mathfrak{p}^{d-1})(\delta_{J_{\mathfrak{d}}} - \mathfrak{p})b \\ &= (\delta_{J_{\mathfrak{d}}} + \mathfrak{p} + \dots + \mathfrak{p}^{d-1})c_{\sigma} \\ &= c_{\sigma^d} = 0.\end{aligned}$$

This shows that $b \in J_{\mathfrak{d}}(\ ,k)$. We put $\varphi'_{\mathfrak{d}} = \varphi_{\mathfrak{d}} + b$. Then

$$\begin{aligned}\varphi_{\mathfrak{d}}^{\sigma} + b^{\sigma} = \varphi_{\mathfrak{d}} + c_{\sigma} + b^{\sigma} = \varphi'_{\mathfrak{d}} - b + c_{\sigma} + b^{\sigma} = \varphi'_{\mathfrak{d}} + c_{\sigma} - (b - b^{\sigma}) \\ = \varphi'_{\mathfrak{d}} + c_{\sigma} - (\delta_{J_{\mathfrak{d}}} - \mathfrak{p})b = \varphi'_{\mathfrak{d}}.\end{aligned}$$

This shows that $\varphi'_{\mathfrak{d}}$ is defined over k.

PROPOSITION 9. *If L/K is ramified, then the mapping $\varepsilon_{\nu} \to a_{L,K}(\varepsilon_{\nu})$ is not isomorphic, where $a_{L,K}(\varepsilon_{\nu})$ means the point on the ordinary jacobian variety corresponding to $a_{\mathfrak{d},\bar{\mathfrak{d}}}(\varepsilon_{\nu})$.*

Proof. From the proof of Lemma 7 in §2 it is sufficient to prove the proposition for any extension of prime degree. Let P be the place of L. Then, denoting by the same P the point of c_L corresponding to P, we have $P^{\varepsilon} = P$. This shows that

$$b_L(\varepsilon) = \varphi_L(P^{\varepsilon}) - \eta(\varepsilon)\varphi_L(P) = (\delta_{J_L} - \eta(\varepsilon))\varphi_L(P)$$

Hence

$$a_{L,K}(\varepsilon) = \alpha_{L,K} b_L(\varepsilon) = 0.$$

PROPOSITION 10. *If L/K has an index of ramification which is divisible by p, then the mapping $\varepsilon_{\nu} \to a_{\mathfrak{d}_0,\bar{\mathfrak{d}}_0}(\varepsilon_{\nu})$ is not isomorphic.*

Proof. From the proof of Lemma 7 in §2, it is sufficient to prove the proposition for any extension of degree p. We assume that $a_{\mathfrak{d}_0,\bar{\mathfrak{d}}_0}(\varepsilon) \neq 0$. Since $J_{\bar{\mathfrak{d}}_0}$ has no affine subgroup, $\bar{A}_{\mathfrak{d}_0,\bar{\mathfrak{d}}_0}$ has no affine subgroup. Therefore the maximal linear subgroup of $\bar{A}_{\mathfrak{d}_0,\bar{\mathfrak{d}}}$ has no point of order p. This shows that $a_{\mathfrak{d}_0,\bar{\mathfrak{d}}_0}(\varepsilon) \neq 0$. This contradicts to Proposition 9.

THEOREM 4. *Let $\bar{\varphi}_{\bar{\mathfrak{d}}}$ be any canonical mapping defined over k and $\mathfrak{g}$ be a subgroup of $\bar{J}_{\bar{\mathfrak{d}}}(\ ,k)$. If k is a finite field, then for the pair $(\bar{\varphi}_{\bar{\mathfrak{d}}}, g)$ there exists a separable extension of K such that*

(I) *L/k is regular,*

(II) *all the place ramifying in L/K belong to $\bar{\mathfrak{d}}$*

and

(III) *$\pi_{\mathfrak{d},\bar{\mathfrak{d}}} J(\ ,k) = \mathfrak{g}$ for any local ring $\mathfrak{d}$ in L satisfying $N_{L/K}\mathfrak{d} \subset \bar{\mathfrak{d}}$. Moreover* *for any separable abelian extension L/K satisfying* (I) *and* (II) *there exist a local ring $\bar{\mathfrak{o}}'$ in K which has the same places as $\bar{\mathfrak{o}}$, a canonical mapping $\bar{\varphi}_{\bar{\mathfrak{o}}'}$ which is defined over k and a subgroup $\mathfrak{g}$ of $J_{\bar{\mathfrak{o}}'}(\ ,k)$ such that L is the extension corresponding to* $(\bar{\varphi}_{\bar{\mathfrak{o}}'}, \mathfrak{g})$.

Proof. Let Λ be the quotient group variety of $\bar{J}_{\bar{\mathfrak{o}}}$ by $\mathfrak{g}$ and μ be the natural homomorphism of $\bar{J}_{\bar{\mathfrak{o}}}$ onto Λ and λ be the homomorphism of Λ onto $J_{\mathfrak{o}}$ such that $\mu\lambda = \delta_{J_{\mathfrak{o}}} - \mathfrak{p}$. Then Λ and λ are defined over k and any point of $\lambda^{-1}(0)$ is k-rational. Let y be the point of Λ such that $k(\lambda y) = K$ and $\lambda y = \bar{\varphi}_{\bar{\mathfrak{o}}}(\bar{P})$ with a point $\bar{P}$ of $C^*_{\bar{\mathfrak{o}}}$. Then, by virtue of Lemma 8 and 12 $k(y)/k(\lambda y)$ is separable and $G(k(y)/K) \cong \lambda^{-1}(0)$. Let $\mathfrak{o}$ be any local ring in $k(y)$ satisfying $N_{k(y)/K}\mathfrak{o} \subset \bar{\mathfrak{o}}$. Then, by virtue of Lemma 11, there exists an isomorphism ζ of $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}$ onto Λ such that $\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}} = \lambda\zeta$, where we notice that η is defined over k. Therefore we have

$$\begin{aligned}\pi_{\mathfrak{o},\bar{\mathfrak{o}}}\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}(\ ,k) &= \lambda\zeta\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}}(\ ,k)\\ &= \lambda\Lambda(\ ,k)\\ &= \mu^{-1}(0) = \mathfrak{g}.\end{aligned}$$

This $k(y)$ is the extension of K in the theorem.

Conversely we assume that L is an separable abelian extension of K satisfying (I) and (II). Let $\mathfrak{o}$ be the local ring in L such that (i) $\mathfrak{o} \cap K$ has the same places as $\mathfrak{o}$ and (ii) $\bar{\pi}^{-1}_{\mathfrak{o},\bar{\mathfrak{o}}}(0) \cong G(L/K)$ and $\varphi_{\mathfrak{o}}$ be the canonical mapping defined over k. By virtue of proposition 5 and 8, such $\mathfrak{o}$ and $\varphi_{\mathfrak{o}}$ always exist. We choose the canonical mapping $\bar{\varphi}_{\bar{\mathfrak{o}}}$ of $\bar{C}^*_{\bar{\mathfrak{o}}}$ into $\bar{J}_{\bar{\mathfrak{o}}}$ such that $\pi_{\mathfrak{o},\bar{\mathfrak{o}}}\varphi_{\mathfrak{o}} = \bar{\varphi}_{\bar{\mathfrak{o}}}\hat{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}$. This $\bar{\varphi}_{\bar{\mathfrak{o}}}$ is defined over k. Putting $\mathfrak{g} = \mu_{\mathfrak{o},\bar{\mathfrak{o}}}J_{\mathfrak{o}}(\ ,k)$, we get a system $(\bar{\varphi}_{\bar{\mathfrak{o}}}, \mathfrak{g})$ which corresponds to L/K.

3.5. In this section, we shall treat the case that $\bar{\varphi}_{\bar{\mathfrak{o}}}$ is not defined over k.

We need the following A. Weil's theorem on the field of definion of a variety:

Theorem (A. Weil) Let k'/k be a separable algebraic extension and $\theta = \{\sigma_1, \ldots, \sigma_r\}$ be the set of all isomorphism of k' into $\bar{k}$. Let V be a projective vareity defined over k and $V^{\sigma}(\sigma \in \theta)$ be the σ-conjugate of V. Let $f_{\sigma_i,\sigma_j}(\sigma_i, \sigma_j \in \theta)$ be a biregular correspondence between V^{σ_j} and V^{σ_i}. Then, if $\{f_{\sigma_i,\sigma_j}\}$ satisfies the conditions:

(i) $f_{\sigma_i,\sigma_h}=f_{\sigma_i,\sigma_j}\circ f_{\sigma_j,\sigma_h}$ for all $\sigma_i,\ \sigma_j,\ \sigma_h\in\theta$,

(ii) $f_{\sigma_i\omega,\sigma_j\omega}=(f_{\sigma_i,\sigma_j})^{\omega}$ for any automorphism ω of k'/k,

we have a variety V_0 defined over k and a biregular correspondece defined over k' between V_0 and V such that

$$f_{\sigma_i,\sigma_j}=f^{\sigma_i}\circ(f^{\sigma_j})^{-1}.$$

Moreover V_0 and f is uniquely determined up to a biregular transformation over k.

LEMMA 13. *Let k'/k be a finitely normal extension of k over which φ_D is defined. Then there exists a $J_D(\ ,k)$-valued cocycle $(c_{\sigma_i})_{\sigma_i\in G(k'/k)}$ of $G(k'/k)$ such that*

$$\varphi_D^{\sigma_i}-\varphi_D=c_{\sigma_i}\qquad(\sigma_i\in G(k'/k).$$

Proof. We observe that

$$\begin{aligned}b_{\sigma_i\sigma_j}=\varphi_\sigma^{\sigma_i\sigma_j}-\varphi_D&=(\varphi_D^{\sigma_i}-\varphi_D)^{\sigma_j}+(\varphi_D^{\sigma_j}-\varphi_D)\\&=c_{\sigma_i}^{\sigma_j}+c_{\sigma_j}.\end{aligned}$$

This shows that $(c_\sigma)_{\sigma\in G(k'/k)}$ is a cocycle.

We call this cocycle $(c_\sigma)_{\sigma\in G(k'/k)}$ in the abave lemma the cocycle associatting with $(C_D^*,\ \varphi_D)$.

LEMMA 14. *Let A be a commtative group variety defined over k and X be an irreducible subvariety in A which is defined over a finitely normal extension k' of k. Let $(d_\sigma)_{\sigma\in G(k'/k)}$ be a cocycle of $G(k'/k)$ valued in $A(\ ,k')$. If the conjugate X^σ is written $X+d_\sigma$ $(\sigma\in G(k'/k))$, then there exist a variety X_0 defined over k and a biregular correspondence f between X_0 and X such that*

$$(f^{\sigma_i})\circ(f^{\sigma_j})^{-1}(x+d_{\sigma_j})=x+d_{\sigma_i},\qquad \textit{where}\quad x\in X.$$

Proof. Let P be a generic point of X over k and f_{σ_i,σ_j} be the locus of $(P+d_{\sigma_i},\ P+d_{\sigma_j})$. Then, since $(d_\sigma)_{\sigma\in G(k'/k)}$ is a cocycle, f_{σ_i,σ_j} satisfies (i) and (ii) in the Weils theorem. Therefore by virtue of the theorem, we get X_0 and f in Lemma 14.

We call X_0 and f in Lemma 14 respectively the variety and the biregular correcepondence associatting with $(X,\ (d_\sigma)_{\sigma\in G(k'/k)})$.

LEMMA 15. *Let $(c_\sigma)_{\sigma\in G(k'/k)}$ be the cocycle associatting with $(C_D^*,\ \varphi_D)$. If $\alpha_{D,\bar{D}}$*

is a biregular, mapping of $\varphi_{\mathfrak{d}}(C_{\mathfrak{d}}^*)$ *onto* $\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}\varphi_{\mathfrak{d}}(C_{\mathfrak{d}}^*)$ *then* $C_{\mathfrak{d}}^*$ *and* $\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}\varphi_{\mathfrak{d}}$ *are respectively the curve and biregular correspondence associatting with* $(\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}\varphi_{\mathfrak{d}}(C_{\mathfrak{d}}^*),$ $(\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}c_\sigma)_{\sigma\in G(k'/k)})$.

Proof. Put $f=\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}\varphi_{\mathfrak{d}}$. Then we have $f^{\sigma_i}=\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}\varphi_{\mathfrak{d}}^{\sigma_i}$ and $(f^{\sigma_i}\circ(f^{\sigma_j})^{-1})$ $(x+\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}c_{\sigma_j})=x+\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}c_{\sigma_i}$, where $x\in\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}\varphi_{\mathfrak{d}}(C_{\mathfrak{d}}^*)$. This shows that $C_{\mathfrak{d}}^*$ and f are respectively the curve and biregular correspondence associatting with $(\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}\varphi_{\mathfrak{d}}(C_{\mathfrak{d}}^*),\ (\alpha_{\mathfrak{d},\bar{\mathfrak{d}}}c_\sigma)_{\sigma\in G(k'/k)})$.

THEOREM 5. *Let k'/k be a finitely normal extension of k over which $\bar{\varphi}_{\bar{\mathfrak{d}}_0}$ is defined and $(\bar{c}_\sigma)_{\sigma\in G(k'/k)}$ be the cocycle associatting with $(\bar{C}_{\bar{\mathfrak{d}}_0}^*,\ \bar{\varphi}_{\bar{\mathfrak{d}}_0})$. Let k'' be the minimal normal extension of k over which all the points in $\{a\,|\,na=c_\sigma;\ \sigma\in G(k'/k)\}$ are rational. Let $\mathfrak{g}$ be a subgroup of $\bar{J}_{\bar{\mathfrak{d}}_0}(n)$ such that $a+\mathfrak{g}$ $(a\in\bar{J}_{\bar{\mathfrak{d}}_0}(n))$ are k-rational cycle as cycles of dimension zero on $\bar{J}_{\bar{\mathfrak{d}}_0}$ and $(z_\omega)_{\omega\in G(k''/k)}$ be a relative cocycle of $G(k''/k)$ valued in $J_{\bar{\mathfrak{d}}_0}(\ ,k')$ modulo $\mathfrak{g}$, such that $nz_\omega=\bar{C}_{[\omega]}$, where $[\omega]$ is the class of ω in $G(k'/k)$. Then for the system of $(\bar{\varphi}_{\bar{\mathfrak{d}}_0},\ \mathfrak{g},\ (z_\omega))$ there exists a separable abelian extension L of K satisfying the following conditions:*

- (I) *L/k is regular,*
- (II) *the indicis of ramification of L/K are all coprime to p,*
- (III) *all the placis of K/k ramifying in L/K belong to $\bar{\mathfrak{d}}_0$,*
- (IV) *if $\mathfrak{o}$ is a locall ring of L such that $N_{L/K}\mathfrak{o}\subset\bar{\mathfrak{o}}_0$, then $\pi_{\mathfrak{d},\bar{\mathfrak{d}}_0}J_{\mathfrak{d}}(n)=\mathfrak{g}$.*
- (V) *if μ is the homomorphism of $\bar{J}_{\bar{\mathfrak{d}}_0}$ onto $\bar{A}_{\mathfrak{d},\bar{\mathfrak{d}}_0}$ such that $\bar{\pi}_{\mathfrak{d},\bar{\mathfrak{d}}_0}\mu=n\delta_{\bar{J}_{\bar{\mathfrak{d}}_0}}$, then (μz_ω) is the cocycle associatting with $(C_{\mathfrak{d}}^*,\ \alpha_{\mathfrak{d},\bar{\mathfrak{d}}_0}\varphi_{\mathfrak{d}})$.*

Moreover for any separable abelian extension satisfying (I), (II), (III) *there exist a subgroup g of $\bar{J}_{\bar{\mathfrak{d}}_0}(n)$ and a relative cocycle $(z_\omega)_{\omega\in G(k''/k)}$ such that L is the extension corresponding to $(\bar{\varphi}_{\bar{\mathfrak{d}}_0},\ \mathfrak{g},\ (z_\omega))$.*

Proof. Let μ be the natural homomorphism of $\bar{J}_{\bar{\mathfrak{d}}_0}$ onto $\Lambda=\bar{J}_{\bar{\mathfrak{d}}_0}/\mathfrak{g}$ and λ be the homomorphism of Λ onto $\bar{J}_{\bar{\mathfrak{d}}_0}$ such that $\mu\lambda=n\delta_\Lambda$. Then A and λ are defined over k and each point of $\lambda^{-1}(0)$ is k-rational. Let y be the point of A such that λy is a point of $\bar{\varphi}_{\bar{\mathfrak{d}}_0}(\bar{C}_{\bar{\mathfrak{d}}_0})$ and $k(\bar{P})=K$, where $\bar{\varphi}_{\bar{\mathfrak{d}}}(\bar{P})=\lambda y$. Then, by virtue of Lemma 8, we have $G(k'(y)/k'(\lambda y))\cong\lambda^{-1}(0)$.

Let $\tilde{C}$ be the locus of y over k'. Then, if σ is an element of $G(k'/k)$, we have $\tilde{C}^\sigma=\tilde{C}+\mu z_\omega$ with $\omega\in G(k''/k)$, where ω is a representative of σ in $G(k''/k)$. Denoting $\tilde{C}^\omega$ instead of $\tilde{C}^\sigma$, we have $\tilde{C}^\omega=\tilde{C}+\mu z_\omega(\omega\in G(k''/k))$.

Since $(\mu z_{\omega})_{\omega \in G(k''/k)}$ is a cocycle, there existe curve C^* defined over k and a biregular correspondence f of C^* and $\tilde{C}$ which are associatting with $(\tilde{C}, (\mu z_{\omega})_{\omega \in G(k''/k)})$. Let P be the point of C^* such that $f(P)=y$. We denote by ε_t the automorphism of $k(P)$ defined as follows $P^{\varepsilon_t}=f^{-1}(f(P)+t)$ $(t \in \lambda^{-1}(0))$.

Then the conjugate ε_t^{ω} of ε_t is defined as follows;

$$P^{\varepsilon_t^{\omega}} = (f^{\omega})^{-1}((f^{\omega})^{-1}(P)+t) \qquad (\omega \in G(k''/k)).$$

Now we observe that

$$\begin{aligned} &(f^{\omega})(P^{\varepsilon_t^{\omega}}) - (f^{\omega}(P)+t) \\ &\quad = (f(P^{\varepsilon_t^{\omega}}) + \mu z_{\omega}) - (f(P) + \mu z_{\omega} + t) \\ &\quad = f(P^{\varepsilon_t^{\omega}}) - (f(P)+t). \end{aligned}$$

This shows that $P^{\varepsilon_t^{\omega}} = P^{\varepsilon_t}$. Namely ε_t $(t \in \lambda^{-1}(0))$ are defined over k.

Next we shall prove that the maximal separable subfield L of $k(P)$ over K is the extension satisfying (I), (II), (III) and (IV). Since λ is an unramified covering mapping and $\{\varepsilon_t\}$ are defined over k, L/K is a separable abelian extension satisfying (I) and (III). Let $\mathfrak{o}$ be a local ring of L such that $N_{L/K}\mathfrak{o} \subset \mathfrak{o}_0$. Then, by virtue of Lemma 12, there exists a purely inseparable homomorphism β of A onto $\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}_0}$ such that $\lambda = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}\beta$. This shows that

$$\mathfrak{g} = \lambda A(n) = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}\beta A_{\bar{\mathfrak{o}}_0}(n) = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}_0}\bar{A}_{\mathfrak{o},\bar{\mathfrak{o}}_0}(n) = \bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}_0}J_{\mathfrak{o}_0}(n).$$

Hence L/K satisfies (IV). By virtue of proposition 10, we observe that L/K satisfies also (III).

Conversely for any separable abelian extension L/K satisfying (I), (II), (III), (IV) we shall constract a relative cocycle $(z_{\omega})_{\omega \in G(k''/k)}$. Let $\varphi_{\mathfrak{o}_0}$ be the canonical mapping of $C^*_{\mathfrak{o}_0}$ into $J_{\mathfrak{o}_0}$ such that $\pi_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}\varphi_{\mathfrak{o}_0} = \bar{\varphi}_{\bar{\mathfrak{o}}_0}\tilde{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}$ and k'''/k be the normal extension of k over which $\varphi_{\mathfrak{o}_0}$ is defined. Let $(c_{\tau})_{\tau \in G(k'''/k)}$ be the cocycle associatting with $(C^*_{\mathfrak{o}_0}, \varphi_{\mathfrak{o}_0})$. Then $(\alpha_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}c_{\tau})_{\tau \in G(k''/k)}$ is a cocycle and

$$\bar{\pi}_{\mathfrak{o},\bar{\mathfrak{o}}}\alpha_{\mathfrak{o},\bar{\mathfrak{o}}}c_{\tau} = \bar{\varphi}^{\tau}_{\bar{\mathfrak{o}}_0}(P) - \bar{\varphi}_{\bar{\mathfrak{o}}_0}(P) = \bar{c}_{\sigma_{\tau}}$$

with $\sigma_{\tau} \in G(k'/k)$. Since there exists a homomorphism μ satisfying $\mu\bar{\pi}_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0} = n\delta_{\bar{A}\mathfrak{o}_0,\bar{\mathfrak{o}}_0}$, there exists a relative cocycle $(z_{\omega})_{\omega \in G(k''/k)}$ valued $J_{\bar{\mathfrak{o}}_0}(\ ,k'')$ modulo $\mathfrak{g}$ such that $\mu z_{\sigma_{\tau}} = \alpha_{\mathfrak{o}_0,\bar{\mathfrak{o}}_0}c_{\tau}$.

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