UNIT GROUPS OF CYCLIC EXTENSIONS

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Let Ω be an algebraic number field of finite degree, which we fix once for all, and let K be a cyclic extension over Ω such that the degree of K/Ω is a power l^{ν} of a prime number l. It is obvious that the norm group $N_{K/\Omega}\mathfrak{e}_K$ of the unit group $\mathfrak{e}_K$ of K, being a subgroup of the unit group $\mathfrak{e}$ of Ω, contains the group $\mathfrak{e}^{l^{\nu}}$ consisting of all l^{ν}-th powers $\varepsilon^{l^{\nu}}$ of $\varepsilon\in\mathfrak{e}$. The main aim of the present work is to prove the converse assertion of this fact in certain special case. Namely, it is verified that, if l is an odd prime number prime to the absolute discriminant $D(\Omega)$ of Ω, then, for any subgroup H of $\mathfrak{e}$ containing $\mathfrak{e}^{l^{\nu}}$, there is an infinite set $\mathfrak{K}$ of cyclic extensions of degree l^{ν} over Ω such that we have $N_{K/\Omega}\mathfrak{e}_K = H$ for every $K\in\mathfrak{K}$. More precisely, the infinite set $\mathfrak{K}$ is so chosen that, for every $K\in\mathfrak{K}$, the first cohomology group of $\mathfrak{e}_K$ is isomorphic to the direct product of the 0-th cohomology group of $\mathfrak{e}_K$ by a cyclic group $\mathfrak{Z}$ of degree l^{ν}, where the cohomology groups are defined by considering $\mathfrak{e}_K$ as an operator module of the Galois group of K/Ω. Thus we can also conclude that, if r_Ω is the dimension of $\mathfrak{e}$ and if A_0 is a subgroup of the direct product of r_Ω groups all isomorphic to $\mathfrak{Z}$, then there is an infinite set $\mathfrak{K}$ of cyclic extensions of degree l^{ν} over Ω such that the 0-th cohomology group of $\mathfrak{e}_K$ is isomorphic to A_0 and the first cohomology group of $\mathfrak{e}_K$ is isomorphic to $A_1 = A_0\times\mathfrak{Z}$, where $K\in\mathfrak{K}$ and l is, still as before, an odd prime number prime to $D(\Omega)$.

In § 1, we introduce the convenient notion of *fixed extensions*,[1] and, after preparations in § 2, we deduce all the results in § 3. As for the case of extensions with prime degree l, the results of this paper are already obtained in the previous paper of the author [4].

§ 1. Preliminaries

1. For a normal field K/Ω, we denote its Galois group by $\mathfrak{G}(K/\Omega)$. In

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[1] This was first introduced and studied in works of Hasse. See, e.g., Hasse [2].

particular, if Ω and Ω_A are respectively the algebraic closure and the maximal abelian extension over Ω in the complex number field, then we put $\mathfrak{g}(\bar{\Omega}/\Omega) = G$ and $\mathfrak{g}(\Omega_A/\Omega) = G'$. Groups G, G' are always considered as compact topological groups in usual manner.

Let $\mathfrak{G}$ be a (discrete) finite group. We call a continuous homomorphism κ of G into $\mathfrak{G}$ a *fixed $\mathfrak{G}$-extension* over Ω. A fixed $\mathfrak{G}$-extension κ uniquely determines an overfield K_κ of Ω, i.e., the invariant field of the kernel of κ. It also determines a natural isomorphism between the Galois group of $\mathfrak{g}(K_\kappa/\Omega)$ and the subgroup $\kappa(G)$ of $\mathfrak{G}$. We call K_κ the *corresponding field* of κ. Some of the properties or invariants of the corresponding field K_κ of a fixed extension κ are expressed in the following as those of κ itself, e.g., we say κ is ramified at a place $\mathfrak{p}$ of Ω if K_κ/Ω is so, and the degree of κ means the degree $(K_\kappa : \Omega)$. If $\mathfrak{G}$ is of order n, then a fixed $\mathfrak{G}$-extension over Ω of degree n is said to be *proper*.

A fixed $\mathfrak{G}$-extension κ is naturally considered as a homomorphism of $\mathfrak{g}(K_\kappa/\Omega)$, and, if $\mathfrak{G}$ is an abelian group $\mathfrak{A}$, then κ is also considered as a homomorphism of G'. Furthermore, by the reciprocity law of class field theory, a fixed $\mathfrak{A}$-extension κ is considered as a homomorphism of the idèle group $\mathbf{I}$ of Ω or of the idèle class group C_Ω of Ω. These various interpretations of fixed extensions are occasionally applied as far as no confusion is possible.

The set of all fixed $\mathfrak{A}$-extensions over Ω forms an abelian group if we define the product $\kappa\kappa'$ of two fixed $\mathfrak{A}$-extensions κ, κ' by setting $\kappa\kappa'(\sigma) = \kappa(\sigma)\kappa'(\sigma)$ for any $\sigma \in G$.

Let κ be a fixed $\mathfrak{A}$-extension over Ω and $\mathfrak{p}$ be a finite or infinite place of Ω, then, using as usual the $\mathfrak{p}$-component of an idèle of Ω, we can attach to κ a continuous homomorphism $\kappa_\mathfrak{p}$ into $\mathfrak{A}$ of the multiplicative group $\Omega_\mathfrak{p}^\times$[2] of the $\mathfrak{p}$-completion $\Omega_\mathfrak{p}$ of Ω. By local class field theory, $\kappa_\mathfrak{p}$ is regarded as a homomorphism of the Galois group of a maximal abelian extension over $\Omega_\mathfrak{p}$ and therefore as a fixed $\mathfrak{A}$-extension over $\Omega_\mathfrak{p}$. We call $\kappa_\mathfrak{p}$ the $\mathfrak{p}$-component of κ.

2. Let $\mathbf{I}$, $\mathbf{U}$ be the idèle group and the unit idèle group[3] of Ω, respectively, and denote by $\Omega^\times$ the principal idèle group of Ω. Let $\mathfrak{S}$ be a finite set of places

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[2] We always use the mark $\times$ to stand for the multiplicative group of non-zero elements of a field.

[3] In this paper, we settle no sign condition for the infinite components of a unit idèle, somewhat differently from the definition of Weil [6].

of Ω and κ_{U} be a homomorphism of $\mathbf{U}$ into $\mathfrak{Z}$ such that the $\mathfrak{q}$-component[4] of κ_{U} is trivial for every place $\mathfrak{q}$ of Ω outside $\mathfrak{S}$, where $\mathfrak{Z}$ is a cyclic group whose order l^{ν} is a power of a prime number l. Then κ_{U} is, in a natural way, regarded as a homomorphism of the group $\mathrm{U}_{\mathfrak{S},\nu}=\prod_{\mathfrak{p}\in\mathfrak{S}} U_{\mathfrak{p}}/U_{\mathfrak{p}}^{l^{\nu}}$, where $\mathrm{U}_{\mathfrak{p}}$ is the unit group of the $\mathfrak{p}$-completion $\Omega_{\mathfrak{p}}$ of Ω. On the other hand, set $B^{(\nu)}=\Omega^{\times}\cap \mathbf{I}^{l^{\nu}}\mathbf{U}$; then $B^{(\nu)}$ consists of numbers β of $\Omega^{\times}$ such that the principal ideal (β) is the l^{ν}-th power of an ideal of Ω, and, writing $\beta=\mathbf{a}^{l^{\nu}}\mathbf{u}$ $(\mathbf{a}\in\mathbf{I},\ \mathbf{u}\in\mathbf{U})$, the mapping $\beta\to\mathbf{u}$ followed by the natural mapping of $\mathbf{u}$ into $U_{\mathfrak{S},\nu}$ gives rise to a homomorphism $\iota_{\mathfrak{S},\nu}$ of $B^{(\nu)}$ into $U_{\mathfrak{S},\nu}$.

Now we state the following three Lemmas.[5]

LEMMA 1. *Let l^{ν} be a power of a prime number l, $\mathfrak{Z}$ be a cyclic group of order l^{ν} and let $\mathfrak{S}$ be a finite set of places of Ω. Then the restriction to* U *of a fixed $\mathfrak{Z}$-extension κ over Ω which is unramified at every place of Ω outside $\mathfrak{S}$ is characterized as a homomorphism κ_{U} of* U *into $\mathfrak{Z}$ which has trivial $\mathfrak{q}$-component for every place $\mathfrak{q}$ of Ω outside $\mathfrak{S}$ and which satisfies* $\kappa_{\mathrm{U}}(\iota_{\mathfrak{S},\nu}(B^{(\nu)}))=1$.

LEMMA 2. *Let h_{ν} be the index* $(\mathbf{I}:\Omega^{\times}\mathbf{I}^{l^{\nu}}\mathbf{U})$. *Then the number of all fixed $\mathfrak{Z}$-extensions κ over Ω unramified at every place $\mathfrak{q}$ of Ω outside $\mathfrak{S}$ is equal to* $h_{\nu}\cdot(U_{\mathfrak{S},\nu}:\iota_{\mathfrak{S},\nu}(B^{(\nu)}))$.

LEMMA 3. *The kernel of $\iota_{\mathfrak{S},\nu}$ consists of the numbers $\beta\in B^{(\nu)}$ such that β is, for every $\mathfrak{p}\in\mathfrak{S}$, an l^{ν}-th power in the $\mathfrak{p}$-completion $\Omega_{\mathfrak{p}}$ of Ω.*

§ 2. Covering of an unramified field

3. Denote by $\mathbf{I}$, $\mathbf{U}$ the idèle group and the unit idèle group of Ω, respectively, and let $\mathfrak{Z}$ be a cyclic group whose order is a power l^{ν} of a prime number l. Set, as in § 1, **2**, $B^{(\nu)}=\Omega^{\times}\cap\mathbf{I}^{l^{\nu}}\mathbf{U}$ and consider the mapping $\beta\to\mathbf{a}$ defined for an element $\beta\in B^{(\nu)}$ with $\beta=\mathbf{a}^{l^{\nu}}\mathbf{u}$ $(\mathbf{a}\in\mathbf{I},\ \mathbf{u}\in\mathbf{U})$. If $\mathbf{e}$ is the unit group of Ω, then the above mapping gives rise to an isomorphism of $B^{(\nu)}/\mathbf{e}\Omega^{\times l^{\nu}}$ onto the group $C^{(\nu)}$ consisting of all elements of $\mathbf{I}/\Omega^{\times}\mathbf{U}$ whose orders divide l^{ν}. Any homomorphism χ of $C^{(\nu)}$ into $\mathfrak{Z}$ is therefore regarded as a homomorphism of $B^{(\nu)}/\mathbf{e}\Omega^{\times l^{\nu}}$ into $\mathfrak{Z}$, and *vice versa*. Whenever no confusion is possible, χ may also be considered as a homomorphism of $B^{(\nu)}$ or of a subgroup of $\mathbf{I}$. Take

[4] This can be defined quite similarly to that of a fixed abelian extension.
[5] As for the proofs of these lemmas, see Kubota [5], §1.

such a homomorphism χ and denote by $B_\chi^{(\nu)}$ the subgroup of $B^{(\nu)}$ which is the kernel of χ. Suppose furthermore that $l \neq 2$. Then we have $(B^{(\nu)} : B_\chi^{(\nu)}) = (\Omega(\zeta_{l^\nu}, \sqrt[l^\nu]{B^{(\nu)}}) : \Omega(\zeta_{l^\nu}, \sqrt[l^\nu]{B_\chi^{(\nu)}}))$, where ζ_{l^ν} is a primitive l^ν-th root of unity.[6] Therefore, by Lemma 3 and by the theory of Kummer extensions, there are infinitely many prime ideals $\mathfrak{p}$ of Ω prime to l such that $N\mathfrak{p} - 1 \equiv 0 \pmod{l^\nu}$ and that, if we denote by $\iota_{\mathfrak{p},\nu}$ the homomorphism of §1, **2** with the set $\mathfrak{S} = \{\mathfrak{p}\}$ of a single place $\mathfrak{p}$, then the kernel of $\iota_{\mathfrak{p},\nu}$ coincides with $B_\chi^{(\nu)}$. We call such a $\mathfrak{p}$ a prime ideal of Ω which *belongs* to the homomorphism χ.

4. Let K' be an unramified cyclic extension over Ω such that the degree $(K' : \Omega)$ divides a power l^ν of a prime number l, and let K be an overfield of K' such that K/Ω is cyclic of degree l^ν and that there is at most one prime ideal of Ω which is ramified in K/Ω. Then we say that K is a *covering* of degree l^ν of K'. We propose to show that, for any K' and l^ν, we can always find a covering of degree l^ν of K', provided that $l \neq 2$. It suffices to prove that, if $\mathfrak{Z}$ is a cyclic group of order l^ν, then, for any unramified fixed $\mathfrak{Z}$-extension κ' over Ω, there is a covering κ of degree l^ν of κ', i.e., a proper fixed $\mathfrak{Z}$-extension κ over Ω such that κ is ramified at most at one prime ideal of Ω and that κ' is a power of κ.

Using the notations in **3**, let χ be the homomorphism of $C^{(\nu)}$ into $\mathfrak{Z}$ which is naturally induced by κ' and let $\mathfrak{p}$ be a prime ideal belonging to χ. If $U_\mathfrak{p}$ is the unit group of the $\mathfrak{p}$-completion $\Omega_\mathfrak{p}$ of Ω, then we can find an isomorphic mapping $\chi_\mathfrak{p}$ of $U_\mathfrak{p}/U_\mathfrak{p}^{l^\nu}$ onto $\mathfrak{Z}$ such that we have $\chi_\mathfrak{p}(\iota_{\mathfrak{p},\nu}(\beta)) = \chi(\beta)$ for every $\beta \in B^{(\nu)}$, where χ is considered as a homomorphism of $B^{(\nu)}$ as in **3**. Let $l^{\nu-r}$ be the degree of κ' and denote by κ_U the homomorphism of $\mathbf{U}$ into $\mathfrak{Z}$ whose $\mathfrak{p}$-component coincides with $\chi_\mathfrak{p}^{l^{\nu-r}}$ and whose $\mathfrak{q}$-component is trivial for every place $\mathfrak{q} \neq \mathfrak{p}$ of Ω. Then, since we have $\chi_\mathfrak{p}^{l^{\nu-r}}(\iota_{\mathfrak{p},\nu}(\beta)) = \chi^{l^{\nu-r}}(\beta) = 1$, there is, by Lemma 1, a fixed $\mathfrak{Z}$-extension κ_1 over Ω such that the restriction to $\mathbf{U}$ of κ_1 coincides with κ_U. If now $\mathbf{a}$ is an idèle of Ω which represents an element of $C^{(r)}$, then we have $\mathbf{a}^{l^r}\mathbf{u} = \alpha$ ($\mathbf{u} \in \mathbf{U}$, $\alpha \in \Omega^\times$) and consequently $\mathbf{a}^{l^\nu}\mathbf{u}^{l^{\nu-r}} = \alpha^{l^{\nu-r}} \in B^{(\nu)}$. Therefore we have $\kappa(\mathbf{a}) = \chi(\alpha^{l^{\nu-r}}) = \chi_\mathfrak{p}(\iota_{\mathfrak{p},\nu}(\alpha^{l^{\nu-r}})) = \chi_\mathfrak{p}(\mathbf{u}^{l^{\nu-r}}) = \kappa_1(\mathbf{u}) = \kappa_1^{-l^r}(\mathbf{a})$, where χ is considered as a homomorphism of $B^{(\nu)}$. This shows that $\kappa'\kappa_1^{l^r}$ induces a trivial mapping on $C^{(r)}$ and therefore we have $\kappa'\kappa_1^{l^r} = \kappa_2^{l^r}$ with an unramified fixed $\mathfrak{Z}$-extension κ_2 over Ω. Setting

6) See Hasse [3], §1, Satz 1, 2.

$\kappa=\kappa_1^{-1}\kappa_2$, we have $\kappa'=\kappa^{l^r}$. Thus we see that, for every prime ideal $\mathfrak{p}$ of Ω belonging to χ, there is a fixed $\mathfrak{Z}$-extension κ over Ω with $\kappa'=\kappa^{l^r}$ and with at most one ramification place $\mathfrak{p}$, which proves our assertion.

5. Still using the same notations, we make another observation. We denote by $\mathfrak{S}=\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ a set of prime ideals, prime to l, of Ω and by $\mathbf{U}_{\mathfrak{S}}$ the group of unit idèles $\mathbf{u}$ of Ω such that, for every i, the $\mathfrak{p}_i$-component u_i of $\mathbf{u}$ satisfies the condition $u_i \equiv 1 \pmod{\mathfrak{p}_i}$. If $\mathfrak{p}_i$ completely decomposes in the field $\Omega(\zeta_{l^\nu}, \sqrt[l^\nu]{\mathrm{e}})$ and if the factor group $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu}\mathbf{U}_{\mathfrak{S}}$ is isomorphic to the direct product of t cyclic groups of order l^ν, where t is the rank of the group $\mathbf{I}/\Omega^\times \mathbf{I}^l \mathbf{U}$, then we call $\mathfrak{S}$ a *parametric set* of degree l^ν of Ω and the class field $\tilde{Z}_\nu$ over $\Omega^\times \mathbf{I}^{l^\nu}\mathbf{U}_{\mathfrak{S}}$ the *complete covering* attached to $\mathfrak{S}$. It follows from Lemma 2 that, for any parametric set of degree l^ν of Ω, the order of $\iota_{\mathfrak{S},\nu}(B^{(\nu)})$ is equal to that of $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu}\mathbf{U}$ and therefore the kernel of $\iota_{\mathfrak{S},\nu}$ is $\mathrm{e}\Omega^{\times l^\nu}$.

Now, we propose to prove the existence of parametric sets of arbitrary degree l^ν, provided that $l \neq 2$. Let $\tilde{c}_1, \ldots, \tilde{c}_t$ be a base of the group $\tilde{C}$ consisting of all elements of $\mathbf{I}/\Omega^\times \mathbf{U}$ whose orders are powers of l, and let $\chi_1, \ldots, \chi_t$ be a set of homomorphisms of $C^{(\nu)}$ into $\mathfrak{Z}$ such that the restriction of χ_i to the group $\{\tilde{c}_i\} \cap C^{(\nu)}$ is an isomorphism into $\mathfrak{Z}$ and that χ_i is trivial on $\{\tilde{c}_j\} \cap C^{(\nu)}$ $(i \neq j)$, where $\{\tilde{c}_i\}$ is the group generated by $\tilde{c}_i$. Then χ_i's form a base of the group consisting of all homomorphisms of $C^{(\nu)}$ into $\mathfrak{Z}$. Choose for every i a prime ideal $\mathfrak{p}_i$ of Ω belonging to χ_i and set $\mathfrak{S}=\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Then it follows from the results of 4 that, for every unramified fixed $\mathfrak{Z}$-extension κ'_i over Ω which is trivial on every $\{\tilde{c}_j\}$ with $i \neq j$, there is a covering κ_i of κ'_i which is unramified at every place of Ω except $\mathfrak{p}_i$. This means that the factor group $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu}\mathbf{U}_{\mathfrak{S}}$ contains the direct product of t cyclic groups of order l^ν. On the other hand, $\chi_1, \ldots, \chi_t$ regarded as homomorphisms of $B^{(\nu)}$ form a base of the group consisting of all homorphisms of $B^{(\nu)}/\mathrm{e}\Omega^{\times l^\nu}$ into $\mathfrak{Z}$. Therefore it follows from the definition of $\mathfrak{S}$ that the kernel of $\iota_{\mathfrak{S},\nu}$ is $\mathrm{e}\Omega^{\times l^\nu}$ and consequently the order of $\iota_{\mathfrak{S},\nu}(B^{(\nu)})$ is equal to the order of the group $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu}\mathbf{U}$. Hence, by Lemma 2, the number of all $\mathfrak{Z}$-extensions κ over Ω unramified at every place of Ω outside $\mathfrak{S}$ is equal to $l^{\nu t}$. Thus we see that the group $\mathbf{I}/\Omega^\times \mathbf{I}^{l^\nu}\mathbf{U}_{\mathfrak{S}}$ is just the direct product of t cyclic groups of order l^ν, whence $\mathfrak{S}$ is a parametric set of degree l^ν of Ω.

§ 3. Unit groups and their norms

6. The main purpose of this section is to prove the following

THEOREM 1. *Let $\mathfrak{Z}$ be a cyclic group whose order l^ν is a power of an odd prime number l prime to the absolute discriminant $D(\Omega)$ of Ω. Denote by* e *the unit group of Ω and let H be a subgroup of* e *containing* e^{l^ν}. *Then there are infinitely many proper fixed $\mathfrak{Z}$-extensions κ over Ω such that we have $N_{K_\kappa/\Omega}\mathrm{e}_\kappa = H$, where e_κ is the unit group of the corresponding field K_κ of κ.*

7. Let $\mathbf{I}$, $\mathbf{U}$ be the idèle group and the unit idèle group of Ω, respectively, and let Z_1 be the class field over $\Omega^\times \mathbf{I}^l \mathbf{U}$. Then, under the assumptions in Theorem 1, we have $Z_1 \cap \tilde{\Omega}_\nu = \Omega$, where $\tilde{\Omega}_\nu = \Omega(\zeta_{l^\nu}, \sqrt[l^\nu]{B^{(\nu)}})$, $B^{(\nu)} = \Omega^\times \cap \mathbf{I}^{l^\nu}\mathbf{U}$ and ζ_{l^ν} is a primitive l^ν-th root of unity. For, since the assumptions imply that $\Omega(\zeta_{l^\nu})/\Omega$ is an extension of degree $l^{\nu-1}(l-1)$ containing no unramified subfield except Ω itself, $\tilde{\Omega}_\nu/\Omega$ has $\Omega(\zeta_{l^\nu})$ as the largest abelian subfield and has Ω itself as the largest unramified abelian subfield. From this follows that there is a parametric set $\mathfrak{S} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ of degree l^ν of Ω such that the substitutions $\left(\frac{Z_1/\Omega}{\mathfrak{p}_i}\right)$ form a base of the Galois group $\mathfrak{g}(Z_1/\Omega)$, because the signifying condition in §2 of $\mathfrak{p}_i$ concerns only the decomposition of $\mathfrak{p}_i$ in $\tilde{\Omega}_\nu$. We take such a parametric set $\mathfrak{S}$ and fix homomorphisms $\chi_{\mathfrak{p}_i}$ of $U_{\mathfrak{p}_i}$ onto $\mathfrak{Z}$, where $U_{\mathfrak{p}_i}$ is the unit group of $\mathfrak{p}_i$-completion $\Omega_{\mathfrak{p}_i}$ of Ω.

Now, we can find subgroups $H_1, \ldots, H_s$ of e containing H such that e/H_i is cyclic and that we have $\bigcap_i H_i = H$. Let c_i be the index $(\mathrm{e} : H_i)$ and ζ_{c_i} be a primitive c_i-th root of unity. Then, since we have $(\Omega(\zeta_{c_i}, \sqrt[c_i]{\mathrm{e}}) : \Omega(\zeta_{c_i}, \sqrt[c_i]{H_i})) = c_i$,[7] there is a prime ideal $\mathfrak{q}_i$ of Ω prime to l such that $N\mathfrak{q}_i - 1 \equiv 0 \pmod{c_i}$ and that we have $H_i = \mathrm{e} \cap U_{\mathfrak{q}_i}^{c_i}$, where $U_{\mathfrak{q}_i}$ is the unit group of the $\mathfrak{q}_i$-completion $\Omega_{\mathfrak{q}_i}$ of Ω and e is regarded as a subgroup of $U_{\mathfrak{q}_i}$. We take such a prime ideal $\mathfrak{q}_i$ for every i, and fix homomorphisms $\chi_{\mathfrak{q}_i}$ of $U_{\mathfrak{q}_i}$ into $\mathfrak{Z}$ with the kernel $U_{\mathfrak{q}_i}^{c_i}$. Let π_i be a generator of the prime ideal of $\Omega_{\mathfrak{p}_i}$ and σ be a generator of $\mathfrak{Z}$. Then, setting $\chi_{\mathfrak{p}_i}(\pi_i) = 1$, we can extend $\chi_{\mathfrak{p}_i}$ to a homomorphism of the whole multiplicative group $\Omega_{\mathfrak{p}_i}^\times$. We also extend $\chi_{\mathfrak{q}_i}$ to a homomorphism into $\mathfrak{Z}$ of $\Omega_{\mathfrak{q}_i}^\times$ in an arbitrary way.

By the existence theorem of Grunwald,[8] there are infinitely many proper

[7] See footnote 6.
[8] See Hasse [3].

fixed $\mathfrak{Z}$-extensions κ over Ω such that we have $\chi_{\mathfrak{p}_i}=\bar{\chi}_{\mathfrak{p}_i}$, $\kappa_{\mathfrak{q}_i}=\bar{\chi}_{\mathfrak{q}_i}$ for the local components $\kappa_{\mathfrak{p}_i}$, $\kappa_{\mathfrak{q}_i}$ of κ and that there is only one ramification prime ideal $\mathfrak{x}$ outside the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t, \mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$.

8. We propose to show that the proper fixed $\mathfrak{Z}$-extensions κ in 7 have the required properties of Theorem 1. Since we have $\kappa\left(\frac{\alpha, K_\kappa/\Omega}{\mathfrak{p}_i}\right)^{9)}=\bar{\chi}_{\mathfrak{p}_i}(\alpha)$, $\kappa\left(\frac{\alpha, K_\kappa/\Omega}{\mathfrak{q}_i}\right)=\bar{\chi}_{\mathfrak{q}_i}(\alpha)$ for $\alpha\in\Omega^\times$, it follows from the definition of $\bar{\chi}_{\mathfrak{q}_i}$ that we have $N_{K_\kappa/\Omega}\mathfrak{e}_\kappa\subset H$ and, on the other hand, it follows from the definition of $\chi_{\mathfrak{p}_i}$ and from a property in 5 of parametric sets that no element of $B^{(\nu)}$ outside $\mathfrak{e}\Omega^{\times l^\nu}$ is a norm of K_κ/Ω. The latter result implies that, if an ideal $\mathfrak{a}$ of Ω is principal in K_κ, then it is principal in Ω, because from $\mathfrak{a}=(\alpha^\kappa)(\alpha^\kappa\in K_\kappa)$ necessarily follows $\mathfrak{a}^{l^\nu}=(N_{K_\kappa/\Omega}\alpha^\kappa)$ and $N_{K_\kappa/\Omega}\alpha^\kappa\in B^{(\nu)}$. Hence, denoting by (α) a principal ideal of Ω, by (α_0^κ) an "ambig" principal ideal of K_κ/Ω and by $\mathfrak{a}$ an ideal of Ω, we heve $((\alpha_0^\kappa)\cap\mathfrak{a} : (\alpha))=1$, where a general element of a group stands for the group itself. Therefore, if $\mathfrak{a}_0^\kappa$ is an "ambig" ideal of K_κ/Ω, we have $((\alpha_0^\kappa) : (\alpha))=((\alpha_0^\kappa)\mathfrak{a} : \mathfrak{a})=(\mathfrak{a}_0^\kappa : \mathfrak{a})/(\mathfrak{a}_0^\kappa : (\alpha_0^\kappa)\mathfrak{a})$. Since the group $(\alpha_0^\kappa)/(\alpha)$ is isomorphic to the first cohomology group of $\mathfrak{e}_\kappa$ as a $\mathfrak{g}(K_\kappa/\Omega)$-group, we have, by Herbrand's relation,[10] $((\alpha_0^\kappa) : (\alpha))=l^\nu\cdot(\mathfrak{e} : N_{K_\kappa/\Omega}\mathfrak{e}_\kappa)$. Thus we obtain $(\mathfrak{e} : N_{K_\kappa/\Omega}\mathfrak{e}_\kappa)=l^{-\nu}\cdot(\mathfrak{a}_0^\kappa : \mathfrak{a})/(\mathfrak{a}_0^\kappa : (\alpha_0^\kappa)\mathfrak{a})$. The factor $l^{-\nu}\cdot(\mathfrak{a}_0^\kappa : \mathfrak{a})$ of this formula is estimated as follows: $l^{-\nu}\cdot(\mathfrak{a}_0^\kappa : \mathfrak{a})=l^{-\nu}\cdot\prod_i e(\mathfrak{p}_i)\cdot\prod_i e(\mathfrak{q}_i)\cdot e(\mathfrak{x}) \leqq l^{\nu t}\cdot(\mathfrak{e} : H)$, where we denote by $e(\ )$ the ramification order with respect to K_κ/Ω. As for $(\mathfrak{a}_0^\kappa : (\alpha_0^\kappa)\mathfrak{a})$, we make the following investigation. Suppose that $\mathfrak{p}_i=\mathfrak{P}_i^{l^\nu}$ in K_κ and let $K_{\kappa,\mathfrak{P}_i}$ be the $\mathfrak{P}_i$-completion of K_κ. Then, since we have $\bar{\chi}_{\mathfrak{p}_i}(\pi_i)=\kappa\left(\frac{\pi_i, K_\kappa/\Omega}{\mathfrak{p}_i}\right)=1$, there is a generator Π_i of the prime ideal of $K_{\kappa,\mathfrak{P}_i}$ with the norm π_i to $\Omega_{\mathfrak{p}_i}$. If there is a relation $\mathfrak{P}_1^{m_1}\ldots\mathfrak{P}_t^{m_t}=(\alpha_0^\kappa)\mathfrak{a}$, then we have $\Pi_1^{m_1}\ldots\Pi_t^{m_t}\in K_\kappa^\times\mathbf{I}\mathbf{U}_\kappa$, where $K_\kappa^\times$ is the principal idèle group of K_κ, $\mathbf{U}_\kappa$ is the unit idèle group of K_κ and Π_i is regarded as an idèle of K_κ with $\mathfrak{P}_i$-component Π_i and with other components 1. Denoting by $\mathfrak{S}$ the parametric set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ and by $\mathbf{U}_\mathfrak{S}$ the group defined in 5, we have $K_{K_\kappa/\Omega}\mathbf{U}_\kappa\subset\mathbf{U}^{l^\nu}\mathbf{U}_\mathfrak{S}$, whence $\pi_1^{m_1}\ldots\pi_t^{m_t}\in\Omega^\times\mathbf{I}^{l^\nu}\mathbf{U}_\mathfrak{S}$. Since, however, the set of $\left(\frac{Z_1/\Omega}{\mathfrak{p}_i}\right)$ is a base of $\mathfrak{g}(Z_1/\Omega)$, an elementary property of finite abelian groups

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9) This notation expresses the image by κ of the automorphism determined by the norm residue symbol.

10) See Chevalley [1], §10.

of type $(l^\nu, \ldots, l^\nu)$ shows that, if $\tilde{Z}_\nu$ is the complete covering attached to $\mathfrak{Z}$ of Ω, then the set of reciprocal images $(\pi_i, \tilde{Z}_\nu/\Omega)$ form also a base of $\mathfrak{g}(\tilde{Z}_\nu/\Omega)$. This means that the relation $\pi_1^{m_1} \ldots \pi_t^{m_t} \in \Omega^\times \mathbf{I}^{l^\nu} \mathbf{U}_{\mathfrak{Z}}$ is impossible unless we have $m_1 \equiv \ldots \equiv m_t \equiv 0 \pmod{l^\nu}$. Thus we have $(\mathfrak{a}_0^\kappa : (\alpha_0^\kappa)\mathfrak{a}) \geqq l^{\nu t}$ and therefore $(\mathbf{e} : N_{K_\kappa/\Omega}\mathbf{e}_\kappa) \leqq (\mathbf{e} : H)$. This, together with $N_{K_\kappa/\Omega}\mathbf{e}_\kappa \subset H$ obtained above, proves our assertion.

9. We incidentally observe here the structure of the group $(\alpha_0^\kappa)/(\alpha)$ of **8.** Since $((\alpha_0^\kappa) \cap \mathfrak{a} : (\alpha)) = 1$, we have $(\alpha_0^\kappa)/(\alpha) \cong (\alpha_0^\kappa)\mathfrak{a}/\mathfrak{a}$. It is eventually shown in 8 that we have $((\alpha_0^\kappa) : (\alpha)) = l^\nu \cdot (\mathbf{e} : H)$ and that $\mathfrak{a}_0^\kappa/(\alpha_0^\kappa)\mathfrak{a}$ is the direct product of t cyclic groups of order l^ν. Therefore the character group of $\mathfrak{a}_0^\kappa/(\alpha_0^\kappa)\mathfrak{a}$ is a direct factor of the character group of $\mathfrak{a}_0^\kappa/\mathfrak{a}$. Since $\mathfrak{a}_0^\kappa/\mathfrak{a}$ is isomorphic to the direct product of $t+1$ cyclic groups of order l^ν by the group $\mathbf{e}/H$, $(\alpha_0^\kappa)\mathfrak{a}/\mathfrak{a} \cong (\alpha_0^\kappa)/(\alpha)$ must be isomorphic to the direct product of $\mathbf{e}/H$ by $\mathfrak{Z}$.

10. The unit group $\mathbf{e}_\kappa$ of the corresponding field K_κ of a proper fixed $\mathfrak{Z}$-extension κ over Ω is considered as a $\mathfrak{Z}$-group because the Galois group $\mathfrak{g}(K_\kappa/\Omega)$ is canonically isomorphic to $\mathfrak{Z}$, and the results which we hitherto obtained allow us to know a little about the cohomology groups of the $\mathfrak{Z}$-group $\mathbf{e}_\kappa$. Since $\mathfrak{Z}$ is cyclic, we may consider only the 0-th and the first cohomology groups. Namely, Theorem 1, together with **9**, immediately yields

THEOREM 2. *Let $\mathfrak{Z}$ be a cyclic group whose order l^ν is a power of an odd prime number l prime to the absolute discriminant $D(\Omega)$ of Ω. Denote by $\mathbf{e}_\kappa$ the unit group of the corresponding field K_κ of a proper fixed $\mathfrak{Z}$-extension κ over Ω and by $H^0(\mathfrak{Z}, \mathbf{e}_\kappa)$ resp. $H^1(\mathfrak{Z}, \mathbf{e}_\kappa)$ the 0-th resp. the first cohomology group of the $\mathfrak{Z}$-module $\mathbf{e}_\kappa$. Furthermore, let A_0 be any subgroup of the direct product of r_Ω groups all isomorphic to $\mathfrak{Z}$, where r_Ω is the dimension of the unit group* $\mathbf{e}$ *of Ω, and set $A_1 = A_0 \times \mathfrak{Z}$. Then there are infinitely many fixed $\mathfrak{Z}$-extensions κ over Ω such that we have $H^0(\mathfrak{Z}, \mathbf{e}_\kappa) \cong A_0$, $H^1(\mathfrak{Z}, \mathbf{e}_\kappa) \cong A_1$.*

It is easily seen that Theorem 1 and Theorem 2 hold even in the case where the order of the cyclic group $\mathfrak{Z}$ is not a power of a single prime number l but an odd natural number prime to the absolute discriminant $D(\Omega)$ of Ω.

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