CAPACITIES OF BORELIAN SETS AND THE CONTINUITY OF POTENTIALS

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Introduction

One of the most important problems in the potential theory is the one of capacitability, that is, whether the inner capacity of an arbitrary borelian subset B is equal to the outer capacity of B. As for the capacities induced by the Newtonian potentials and other classical potentials, Choquet [5] has shown that every borelian and, more generally, every analytic set are capacitable. He goes on as follows: first he shows that, for the Newtonian capacity f, the inequality of strong subadditivity holds, that is,

$$f(A \cup B) + f(A \cap B) \leqq f(A) + f(B),$$

and then, using this inequality, he shows that the outer capacity f^* has the analogous property to one of the outer measure, more precisely, if an increasing sequence $\{A_n\}$ of arbitrary subsets converges to A, then $f^*(A) = \lim f^*(A_n)$. This property plays an important role in his proof.

Recently it tends to investigate the general potentials in a locally compact Hausdorff space. As for the problem of capacitability, anything more than the results of Choquet has not yet been stated. In this paper we deal with this problem and we shall prove that every K-borelian subset and, more generally, every K-analytic subset, contained in a compact set, are capacitable under the two assumptions that every compact subset is metrisable and there exists en equilibrium measure of every compact subset. A K-borelian subset is a subset belonging to the K-borelian field, which is the smallest borelian field which contains each compact subset. As every compact set is metrisable in our case, every classical borelian subset contained in a compact set is K-borelian. A K-analytic subset is the continuous image of a $K_{\sigma\delta}$ set contained in a compact space. It is known that every K-borelian subset is K-analytic.

First, in § 1, we consider the quasi continuity principle, which is a gener-

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alisation of the continuity principle and we shall prove that the quasi continuity principle follows from the assumption that there exists an equilibrium measure of every compact subset. In the following section we shall prove that, if a sequence $\{\mu_n\}$ of positive measures on a compact subset converges vaguely to a measure μ and if the potentials U^{μ_n} are uniformly bounded, then we have $U^{\mu} = \underline{\lim}_n U^{\mu_n}$ quasi everywhere. This fact is very important to assert that the outer capacity, induced by our potentials, has the following property: the sequence of the outer capacities of arbitrary subsets A_n in a compact subset converges increasingly to the outer capacity of A, whenever $\{A_n\}$ increases to A. From this property follows the capacitability of all analytic subsets in a compact set.

In the last section, we associate a function $m^*(P, U^{\mu})$ with each potential U^{μ}. Using this function, first we shall investigate the continuity of potentials, and then we shall discuss an equilibrium potential of an open set G which is constantly equal to 1 in G.

§ 1. Quasi continuity principle

1. Let Ω be a locally compact Hausdorff space. In the sections 1, 2 and 3 we shall assume that every compact subset of Ω is metrisable. In this paper we always consider positive measures[1] μ in Ω with compact carriers denoted by S_μ. We denote by $\mu(1)$ the total mass of a positive measure μ. A sequence $\{\mu_n\}$ of positive measures is called to *converge vaguely* to μ, when we have

$$\int f d\mu = \lim_n \int f d\mu_n$$

for every continuous function f with compact carrier.

The following theorem is important in the potential theory.

THEOREM 1.1.[2] *Suppose that positive measures μ_n $(n = 1, 2, \ldots)$ satisfy the following conditions:*

1° S_{μ_n} $(n = 1, 2, \ldots)$ *are contained in a fixed compact subset K,*

2° $\mu_n(1)$ $(n = 1, 2, \ldots)$ *are uniformly bounded from above.*

[1] For the theory of measures in a locally compact Hausdorff space, see, for example, Bourbaki [2].

[2] Cf. Frostman [8].

Then, from a given $\{\mu_n\}$, *we can take out a subsequence* $\{\mu_{n'}\}$ *which converges vaguely to a positive measure* μ.

Since K is metrisable, this theorem can be proved using the following theorem. We shall omit the proof of Theorem 1.1.

THEOREM.[3] *Suppose that a compact set* K *is metrisable. Then the space* $\mathfrak{C}(K)$ *of all continuous functions in* K *with the uniform convergence topology is separable.*

Conversely we can prove

THEOREM 1.2. *If* $\mathfrak{C}(K)$ *is separable, then the compact set* K *is metrisable.*

Proof. Let $\mathfrak{F}$ be a countable subset of $\mathfrak{C}(K)$ which is dense in $\mathfrak{C}(K)$ with respect to the uniform convergence topology. For each point P_0 of K we define a new base of neighborhoods $N(P_0)$ as follows:

$$N(P_0)=\{P\in K;\ |f_j(P)-f_j(P_0)|<\varepsilon_j,\ \varepsilon_j>0,\ f_j\in\mathfrak{F};\ j=1,2,\ldots,m\}.$$

Let us denote by $\tilde{K}$ the set K with this topology. It is easily seen that $\tilde{K}$ is a Hausdorff space. In fact, for any two points P_1 and P_2 of K, there exists a continuous function $f(P)$ of $\mathfrak{C}(K)$ such that $f(P_1)=0$ and $f(P_2)=1$. Then we can choose a function f_{j_0} of $\mathfrak{F}$ such that $|f(P)-f_{j_0}(P)|<\frac{1}{4}$ at every point P of K. Put

$$N(P_1)=\left\{P;\ |f_{j_0}(P)-f_{j_0}(P_1)|<\frac{1}{4}\right\}$$

and

$$N(P_2)=\left\{P;\ |f_{j_0}(P)-f_{j_0}(P_2)|<\frac{1}{4}\right\}.$$

Then we see that $N(P_1)\cap N(P_2)=\emptyset$. It is easy to verify that K and $\tilde{K}$ are homeomorphic, and that by our topology $\tilde{K}$ satisfies the first axiom of countability, and so does K.

Now we shall show that K is separable. For the purpose, first we show that, for any neighborhood U of an arbitrary point P_0 of K, there exists a continuous function $f(P)$ of $\mathfrak{C}(K)$ such that $0\leqq f(P)\leqq 1$ in K, $f(P_0)=1$, $f(P)=0$ at every point P of $K-U$ and $f(P)<1$ at each point $P\neq P_0$. In fact, since K satisfies the first axiom of countability, there exists a sequence $\{U_n\}$ $(n=1,$

[3] Cf. Kryloff and Bogoliouboff [12].

2, . . .) of neighborhoods of P_0 such that

$$U \supset U_n \supset \bar{U}_{n+1} \supset U_{n+1} \quad \text{and} \quad \bigcap_n U_n = \{P_0\}.$$

Since K is normal by the compactness of K, we can find a continuous function $f_n(P)$ of $\mathfrak{C}(K)$ for each n such that $f_n(P_0) = 1$, $f_n(P) = 0$ at every point P of $K - U_n$ and $0 \leqq f_n(P) \leqq 1$ in K. Then we see that the continuous function $f(P) = \sum \frac{1}{2^n} f_n(P)$ satisfies our requirements. Now, for every function $f_j \in \mathfrak{F}$ and for every integer k, we choose a point $P_{j,k}$ such that $1 - \frac{1}{k} < f_j(P_{j,k}) < 1 + \frac{1}{k}$, when such a point exists. To verify that K is separable, it is sufficient to show that $P_{j,k}$ (j, $k = 1, 2, \ldots$) is dense in K. For any point P_0 of K, by our above observation, there exists a continuous function $f(P)$ such that $0 \leqq f(P) \leqq 1$ in K, $f(P_0) = 1$, $f(P) = 0$ in the complement of a neighborhood U of P_0 and at each point $P \neq P_0$, $f(P) < 1$. By our assumption there exists a sequence $\{f_{j'}(P)\}$ of $\mathfrak{F}$ such that $f(P)$ is a uniform convergence limit of $f_{j'}(P)$. Let $\{P_{j'}\}$ be a subsequence of $\{P_{j,k}\}$ such that

$$1 - \frac{1}{j'} < f_{j'}(P_{j'}) < 1 + \frac{1}{j'}.$$

Then we see that $\lim_{j'} f_{j'}(P_{j'}) = 1$. Let $\tilde{P}$ be an accumulation point of $\{P_{j'}\}$. Since K satisfies the first axiom of countability, there exists a subsequence $\{P_{j''}\}$ of $\{P_{j'}\}$ which tends to $\tilde{P}$. It is easy to verify that $f(\tilde{P}) = \lim_{j''} f_{j''}(P_{j''}) = 1$, and that $\tilde{P} = P_0$, that is, $\{P_{j,k}\}$ is dense in K.

Finally, we shall prove that K satisfies the second axiom of countability. For the purpose we shall show that, for any neighborhood U of an arbitrary point P_0 of K, there exist a finite family $\{f_l\}$ ($l = 1, 2, \ldots, m_0$) of continuous functions of $\mathfrak{F}$, and a finite family $\{\varepsilon_l\}$ ($l = 1, 2, \ldots, m_0$) of positive numbers and a point P_{n_0} belonging to the family $\{P_{j,k}\}$ chosen above, such that an open set

$$N(P_{n_0};\ f_l,\ \varepsilon_l) = \{P;\ |f_l(P) - f_l(P_{n_0})| < \varepsilon_l,\ l = 1, 2, \ldots, m_0\}$$

is a neighborhood of P_0 which is contained in U. In fact, there exist a finite family $\{f_l\}$ ($l = 1, 2, \ldots, m_0$) of continuous functions of $\mathfrak{F}$ and a finite family $\{\varepsilon_l\}$ ($l = 1, 2, \ldots, m_0$) of positive numbers such that an open neighborhood

$$N(P_0;\ f_l,\ 2\varepsilon_l) = \{P;\ |f_l(P) - f_l(P_0)| < 2\varepsilon_l,\ l = 1, 2, \ldots, m_0\}$$

is contained in U. Since $\{P_{j,k}\}$ $(j, k=1, 2, \dots)$ is dense in K, we can choose a point P_{n_0} of $\{P_{j,k}\}$ such that

$$|f_l(P_{n_0})-f_l(P_0)|<\varepsilon_l \quad \text{for} \quad l=1, 2, \dots, m_0.$$

We put

$$N(P_{n_0};\ f_l,\ \varepsilon_l)=\{P;\ |f_l(P)-f_l(P_{n_0})|<\varepsilon_l,\ l=1, 2, \dots, m_0\},$$

then we obtain that $P_0 \in N(P_{n_0};\ f_l,\ \varepsilon_l)$ and $N(P_{n_0};\ f_l,\ \varepsilon_l) \subset N(P_0;\ f_l,\ 2\varepsilon_l) \subset U$. This shows that the family $\{N(P_{j,k};\ f_l,\ \varepsilon_l)\}$ $(f_l \in \mathfrak{F};\ l=1, 2, \dots, m;\ j, k=1, 2, \dots)$ of neighborhoods is a base of neighborhoods of an arbitrary point of K, and hence K satisfies the second axiom of countability. Thus we conclude that K is metrisable.

2. Now let $\Phi(P, Q)$ be a continuous real-valued function defined on the product space $\Omega \times \Omega$, which satisfies the following conditions:

1° $0<\Phi(P, Q) \leqq +\infty$,

2° $\Phi(P, Q)$ is finite except at most at the points of the diagonal set of $\Omega \times \Omega$,[4]

3° $\Phi(P, Q)$ is symmetric, that is, $\Phi(P, Q)=\Phi(Q, P)$. The potential $U^{\mu}(P)$ of a positive measure μ is defined by

$$U^{\mu}(P)=\int \Phi(P, Q)\, d\mu(Q).$$

Then $U^{\mu}(P)$ is lower semi-continuous in Ω and continuous in $\Omega - S_{\mu}$. By the condition 3°, we have always the reciprocal law, $\int U^{\mu} d\nu = \int U^{\nu} d\mu$, for any two positive measures μ and ν. In this paper, μ will be called *admissible* on a compact set K, if $S_{\mu} \subset K$ and $U^{\mu}(P) \leqq 1$ everywhere in Ω. The family of all admissible measures on K is denoted by $\mathfrak{A}(K)$. We associate every compact subset K with the number $c(K)$ defined by sup $\mu(1)$ for all $\mu \in \mathfrak{A}(K)$. By this set-function $c(K)$ we define the inner and the outer capacities of an arbitrary subset A of Ω as follows: the *inner capacity* $\mathrm{cap}_i(A)$ is equal to sup $c(K)$ for all compact sets $K \subset A$, and the *outer capacity* $\mathrm{cap}_e(A)$ is equal to inf $\mathrm{cap}_i(G)$ for all open sets $G \supset A$. It follows immediately that we have $\mathrm{cap}_i(K) = c(K)$ for every compact set K, $\mathrm{cap}_i(A) \leqq \mathrm{cap}_e(A)$ for an arbitrary subset

[4] If $\Phi(P, P)=+\infty$ at every point of the diagonal set, every compact subset of Ω is necessarily metrisable.

A and $\mathrm{cap}_i(G) = \mathrm{cap}_e(G)$ for every open set G. When the inner capacity $\mathrm{cap}_i(A)$ of A is equal to the outer capacity $\mathrm{cap}_e(A)$, we shall say that A is *capacitable* and we shall denote the common value of these two capacities by $\mathrm{cap}(A)$, which we shall call the capacity of A. Every open set is capacitable and as we shall show later, every compact G_δ set is capacitable. It may happen that an open set G is of capacity zero; only in § 4 we assume that every open set is of positive capacity.

We say that a property holds *nearly everywhere* (resp. *quasi everywhere*) in a subset A, when the property holds at each point of A except at the points of a set of inner (resp. outer) capacity zero.

3. The following theorems are well-known.

THEOREM 1.3. *If B_n $(n = 1, 2, \ldots)$ are borelian sets, then we have*

$$\mathrm{cap}_i\left(\bigcup_{n=1}^{\infty} B_n\right) \leqq \sum_{n=1}^{\infty} \mathrm{cap}_i(B_n).$$

THEOREM 1.4. *For any sequence $\{A_n\}$ $(n = 1, 2, \ldots)$ of arbitrary subsets of Ω, it holds that*

$$\mathrm{cap}_e\left(\bigcup_{n=1}^{\infty} A_n\right) \leqq \sum_{n=1}^{\infty} \mathrm{cap}_e(A_n).$$

THEOREM 1.5. *If a sequence $\{\mu_n\}$ $(n = 1, 2, \ldots)$ of positive measures converges vaguely to μ, then we have*

$$U^{\mu}(P) \leqq \varliminf_{n} U^{\mu_n}(P)$$

at every point P of Ω.

4. DEFINITION 1.1. *We say that Φ satisfies Frostman's maximum principle, if, for every potential U^{μ} such that $U^{\mu}(P) \leqq 1$ at every point P of S_μ, we have the same inequality everywhere in Ω.*

DEFINITION 1.2. *A positive measure μ_K associated with a compact set K is called an equilibrium measure of K, if it holds the following properties:*

$$S_{\mu_K} \subset K, \quad U^{\mu_K}(P) \leqq 1 \quad \textit{everywhere in } \Omega \textit{ and } \quad U^{\mu_K}(P) = 1$$

nearly everywhere on K.

It is known that the equality, $\mu_K(1) = \mathrm{cap}_i(K)$, holds for an equilibrium measure μ_K of a compact set K. (See, for example, Theorem 3.5.)

We call the potential U^{μ_K} of an equilibrium measure μ_K the equilibrium potential of K. For a compact set K of inner capacity zero, we have $\mu_K \equiv 0$ as an equilibrium measure.

DEFINITION 1.3. *We shall say that a potential U^{μ} is quasi continuous in Ω, if, for any $\varepsilon > 0$, there exists an open set G_{ε} such that* $\mathrm{cap}(G_{\varepsilon}) \leqq \varepsilon$ *and the restriction of U^{μ} to $\Omega - G_{\varepsilon}$ is continuous.*

DEFINITION 1.4.[5] *We say that Φ satisfies the quasi continuity principle, if the continuity of the restriction of any potential U^{μ} to S_{μ} implies the quasi continuity of U^{μ} in Ω.*

DEFINITION 1.5. *We say that Φ satisfies the continuity principle, if the continuity of the restriction of any potential U^{μ} to S_{μ} implies the continuity of U^{μ} in Ω.*

For the continuity principle, see Ohtsuka [15, 16, 17], Kishi [10], Choquet [6] and Ninomiya [14]. The quasi continuity principle follows immediately from the continuity principle, but the latter does not follow from the former.

5. If every open set is of positive capacity, the existence of an equilibrium measure of every compact set K implies Frostman's maximum principle.[6] and then it assures us the continuity principle.[7] In our case, since there may exist an open set of capacity zero, we can only assert the following

THEOREM 1.6.[8] *Suppose that we have an equilibrium measure of every compact set. Then Φ satisfies the quasi continuity principle.*

To prove this theorem we shall use the following

THEOREM 1.7. *There exists the largest open set G_0 of capacity zero, that is,* $\mathrm{cap}(G_0) = 0$ *and, when Ω is of positive capacity, it holds that* $\mathrm{cap}(G) > 0$ *for any open set $G \supsetneqq G_0$.*

Proof. Let $\mathfrak{G}$ be the family of all open sets of capacity zero. Then the open set $G_0 = \bigcup_{G \in \mathfrak{G}} G$ is the largest open set of capacity zero. In fact, for any compact set $K \subset G_0$, we have $\mathrm{cap}_i(K) = 0$, since K is covered by a finite number

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5) Cf. Kishi [11].
6) Cf. Ninomiya [13].
7) See, for example, Ugaheri [19] or Ohtsuka [16].
8) We can construct a kernel function Φ in a suitable locally compact space Ω such that Φ does not satisfy the continuity principle, but, for any compact set K, there exists an equilibrium measure μ_K. This example also shows that the continuity principle does not necessarily follow from the quasi continuity principle.

of open sets of $\mathfrak{G}$, and hence $\text{cap}(G_0)=0$. It follows immediately that, when Ω is of positive capacity, it holds that $\text{cap}(G)>0$ for any open set $G \supsetneqq G_0$.

Now we shall give the *proof of Theorem* 1.6. Let the restriction of U^{μ} to S_{μ} be continuous. Without loss of generality, we may assume that $(\Omega - S_{\mu}) - G_0$ is not empty. It is sufficient to prove that the restriction of U^{μ} to $F_0 = \Omega - G_0$ is continuous. Let P_0 be a point of $S_{\mu} \cap F_0$. If $\Phi(P_0, P_0)$ is finite, $U^{\mu}(P)$ is obviously continuous at P_0, considered as a function in Ω. Hence we suppose that $\Omega(P_0, P_0) = +\infty$. Let μ_n be the restriction of μ to $B_n = \{P;\ \Phi(P, P_0) > n\}$ $(n = 1, 2, \ldots)$. Then each potential U^{μ_n} is continuous on S_{μ} and it decreases uniformly to zero on S_{μ} as $n \to \infty$. We put $\varepsilon_n = \sup_{P \in S_{\mu}} U^{\mu_n}(P)$, then $\{\varepsilon_n\}$ decreases monotonously to zero. As $U^{\mu_n}(P) \leqq \varepsilon_n$ on $S_{\mu_n} \subset S_{\mu}$, it holds that $U^{\mu_n}(P) \leqq \varepsilon_n$ everywhere in F_0, since G_0 is the largest open set in Ω.

Consequently we have

$$\begin{aligned}\lim_{F_0 \ni P \to P_0} U^{\mu}(P) &\leqq \overline{\lim_{F_0 \ni P \to P_0}} U^{\mu_n}(P) + \lim_{F_0 \ni P \to P_0} U^{\mu-\mu_n}(P) \\ &\leqq \varepsilon_n + U^{\mu-\mu_n}(P_0) \leqq \varepsilon_n + U^{\mu}(P_0),\end{aligned}$$

and hence

$$\overline{\lim_{F_0 \ni P \to P_0}} U^{\mu}(P) \leqq U^{\mu}(P_0).$$

This shows that the restriction of U^{μ} to F_0 is upper semi-continuous at $P_0 \in F_0 \cap S_{\mu}$.

§ 2. Sequences of potentials

1. In the sections 2.1 and 2.2 we assume the quasi continuity principle and we consider a sequence $\{U^{\mu_n}\}$ of potentials of positive measures μ_n converging vaguely to μ.

First we shall prove

THEOREM 2.1.[9] *Every potential U^{μ} is quasi continuous in Ω.*

Proof. Since the set of points P such that $U^{\mu}(P) = +\infty$ is a G_{δ} set of outer capacity zero, there is no loss of generality in assuming that U^{μ} is finite in Ω. For any $\varepsilon > 0$ and for any positive integer n, by Lusin's theorem, there exists a compact set K_n such that $\mu(\Omega - K_n) < \frac{\varepsilon}{2.4^n}$ and U^{μ} is finite and continuous on K_n. Then the potential U^{μ_n} of the restriction μ_n of μ to K_n is

[9] For the Newtonian potentials this has been proved by Cartan [4], Proposition 5,

continuous on K_n, and hence, by our quasi continuity principle, U^{μ_n} is quasi continuous in Ω. Therefore, we have an open set G_n such that the restriction U^{μ_n} to $\Omega - G_n$ is continuous and $\text{cap}(G_n) \leqq \frac{\varepsilon}{2^{n+1}}$. Put

$$B_n = \left\{P \in \Omega - G_n \; ; \; U^{\mu}(P) - U^{\mu_n}(P) > \frac{1}{2^n}\right\}.$$

Then B_n is open in $\Omega - G_n$ and $B_n \cup G_n$ is open in Ω. Hence

$$\text{cap}(B_n \cup G_n) \leqq \text{cap}_i(B_n) + \text{cap}(G_n) \leqq \text{cap}_i(B_n) + \frac{\varepsilon}{2^{n+1}}.$$

We shall show the inequality $\text{cap}_i(B_n) \leqq \frac{\varepsilon}{2^{n+1}}$. For any compact subset $e \subset B_n$, let γ be admissible on e. Then

$$\frac{1}{2^n}\gamma(1) < \int (U^{\mu} - U^{\mu_n})\, d\gamma = \int_{\Omega - K_n} U^{\gamma} d\mu \leqq \mu(\Omega - K_n) < \frac{\varepsilon}{2.4^n},$$

whence we have $\gamma(1) < \frac{\varepsilon}{2^{n+1}}$ and $\text{cap}_i(e) \leqq \frac{\varepsilon}{2^{n+1}}$. Thus we have seen that $\text{cap}_i(B_n) \leqq \frac{\varepsilon}{2^{n+1}}$ and so $\text{cap}(B_n \cup G_n) \leqq \frac{\varepsilon}{2^n}$. Hence we see that $\text{cap}(G_\varepsilon) \leqq \varepsilon$, where $G_\varepsilon = \bigcup_n (B_n \cup G_n)$. Then it follows that the restriction of U^{μ} to $\Omega - G_\varepsilon$ is continuous, because it holds that $0 \leqq U^{\mu}(P) - U^{\mu_n}(P) \leqq \frac{1}{2^n}$ at every point P of $\Omega - G_\varepsilon$ and $U^{\mu_n}(P)$ is continuous in $\Omega - G_\varepsilon$.

THEOREM 2.2.[10] *Suppose that μ_n $(n = 1, 2, \ldots)$ are positive measures on a compact set such that $U^{\mu_n}(P) \leqq M < +\infty$ in Ω and that $\{\mu_n\}$ converges vaguely to μ. Then we have, for any potential $U^{\nu} \leqq 1$, $\lim_n \int U^{\mu_n} d\nu = \int U^{\mu}\, d\nu$.*

Proof. By Theorems 1.5 we have $U^{\mu}(P) \leqq \underline{\lim}_n U^{\mu_n}(P)$ everywhere in Ω. Hence we have

$$\int U^{\mu}\, d\nu \leqq \int \underline{\lim}\, U^{\mu_n} d\nu \leqq \underline{\lim} \int U^{\mu_n} d\nu.$$

We shall show $\overline{\lim} \int U^{\mu_n} d\nu \leqq \int U^{\mu} d\nu$. Since U^{ν} is quasi continuous in Ω by Theorem 2.1, we can find, for any $\varepsilon > 0$, an open set G_ε such that $\text{cap}(G_\varepsilon) \leqq \varepsilon$ and the restriction of U^{ν} to $\Omega - G_\varepsilon$ is continuous. Put

$$f = \begin{cases} U^{\nu} & \text{on} \quad \Omega - G_\varepsilon \\ 0 & \text{in} \quad G_\varepsilon. \end{cases}$$

[10] Cf. Brelot [3], Lemma 5.

Then f is upper semi-continuous. Hence we have a continuous function g such that

$$g \geqq f \quad \text{and} \quad \int g d\mu \leqq \int f d\mu + \varepsilon = \int_{\Omega - G_\varepsilon} U^\nu d\mu + \varepsilon.$$

Then we see

$$\lim \int_{\Omega - G_\varepsilon} U^\nu d\mu_n = \lim \int f d\mu_n \leqq \lim \int g d\mu_n = \int g d\mu$$
$$\leqq \int_{\Omega - G_\varepsilon} U^\nu d\mu + \varepsilon \leqq \int U^\nu d\mu + \varepsilon.$$

On the other hand it is easily seen that $\mu_n(G_\varepsilon) \leqq M\varepsilon$ and $\int_{G_\varepsilon} U^\nu d\mu_n \leqq M\varepsilon$. In fact, for any compact set $e \subset G_\varepsilon$, the measure $\frac{1}{M}\mu'_n$ is admissible on e, where μ'_n is the restriction of μ_n to e. Hence

$$\frac{1}{M}\mu'_n(1) \leqq \operatorname{cap}(G_\varepsilon) \leqq \varepsilon.$$

Therefore we have

$$\overline{\lim} \int U^\nu d\mu_n \leqq \int U^\nu d\mu + (M+1)\varepsilon.$$

Consequently we obtain

$$\overline{\lim} \int U^{\mu_n} d\nu = \overline{\lim} \int U^\nu d\mu_n \leqq \int U^\nu d\mu = \int U^\mu d\nu.$$

2. The following theorem plays an important role in § 3.

THEOREM 2.3.[11] *Let μ_n $(n = 1, 2, \ldots)$ be measures on a compact set such that the potentials U^{μ_n} are uniformly bounded in Ω. If $\{\mu_n\}$ converges vaguely to μ, we have*

$$\lim U^{\mu_n} = U^\mu$$

quasi everywhere in Ω.

Proof. By Theorem 1.5 we have $U^\mu(P) \leqq \underline{\lim}\, U^{\mu_n}(P)$ everywhere in Ω. Hence it is sufficient to prove that $U^\mu(P) \geqq \overline{\lim}\, U^{\mu_n}(P)$ quasi everywhere in Ω. We put

$$V_{n,m}(P) = \min\,(U^{\mu_n}(P), \ldots, U^{\mu_m}(P)) \qquad \text{for} \quad m \geqq n,$$

[11] Cf. Kishi [11].

and

$$V_n(P) = \inf(U^{\mu_n}(P),\ U^{\mu_{n+1}}(P),\ \ldots).$$

Then the sequence $\{V_{n,m}(P)\}$ $(m = n, n+1, \ldots)$ decreases to $V_n(P)$ as $m \to \infty$ and the sequence $\{V_n(P)\}$ $(n = 1, 2, \ldots)$ increases to $V(P) = \underline{\lim}_n U^{\mu_n}(P)$ as $n \to \infty$. For any $\varepsilon' > 0$, we have an open set $G_{\varepsilon'}$ such that $\mathrm{cap}(G_{\varepsilon'}) \leqq \varepsilon'$ and each $U^{\mu_n}(P)$ and $U^{\mu}(P)$ are continuous on $\Omega - G_{\varepsilon'}$ by Theorem 2.1. For any positive number ε, we put

$$E_{n,m}(\varepsilon) = \{P;\ V_{n,m}(P) - U^{\mu}(P) > \varepsilon\}$$

and

$$E^{\varepsilon'}_{n,m}(\varepsilon) = \{P \in \Omega - G_{\varepsilon'};\ V_{n,m}(P) - U^{\mu}(P) > \varepsilon\}.$$

Then it is obvious that each $E^{\varepsilon'}_{n,m}(\varepsilon)$ is open in $\Omega - G_{\varepsilon'}$ and $E^{\varepsilon'}_{n,m}(\varepsilon) \cup G_{\varepsilon'}$ is open in Ω. Hence we easily obtain the inequalities

$$\text{(1)} \qquad \mathrm{cap}_e(E_{n,m}(\varepsilon)) \leqq \mathrm{cap}(E^{\varepsilon'}_{n,m}(\varepsilon) \cup G_{\varepsilon'}) \\ \leqq \mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon)) + \mathrm{cap}(G_{\varepsilon'}) \leqq \mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon)) + \varepsilon'.$$

We shall prove that $\lim_m \mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon)) = 0$. We can see immediately that $E^{\varepsilon'}_{n,m+1}(\varepsilon) \subset E^{\varepsilon'}_{n,m}(\varepsilon)$ and $E^{\varepsilon'}_{n,m+1}(\varepsilon) \subset E^{\varepsilon'}_{n,m}\left(\frac{\varepsilon}{2}\right)$. In fact, if $P^{(k)} \in E^{\varepsilon'}_{n,m+1}(\varepsilon)$ tends to P_0 as $k \to \infty$, then it follows that $P_0 \in \Omega - G_{\varepsilon'}$ and that $\lim_k V_{n,m+1}(P^{(k)}) = V_{n,m+1}(P)$ and $\lim_k U^{\mu}(P^{(k)}) = U^{\mu}(P_0)$. If $\lim_m \mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon)) = \alpha > 0$, we have, for any $m \geqq n$, an admissible measure $\gamma_{n,m}$ on a compact subset $e_{n,m}$ of $E^{\varepsilon'}_{n,m}(\varepsilon)$ such that $\gamma_{n,m}(e_{n,m}) \geqq \frac{\alpha}{2}$. Since $\mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon)) \geqq \mathrm{cap}_i(E^{\varepsilon'}_{n,n}(\varepsilon)) \geqq \gamma_{n,m}(e_{n,m})$, the total masses of $\gamma_{n,m}$ are uniformly bounded, and by Theorem 1.1, we can take out a subsequence $\{\gamma_{n,m'}\}$ of $\{\gamma_{n,m}\}$ such that $\{\gamma_{n,m'}\}$ converges vaguely to a positive measure γ_n, whose total mass is obviously not smaller than $\frac{\alpha}{2}$. S_{γ_n} is contained in $E^{\varepsilon'}_{n,m}\left(\frac{\varepsilon}{2}\right)$ for every sufficiently large m; otherwise there would be a point $P_0 \in S_{\gamma_n} - E^{\varepsilon'}_{n,m_0}\left(\frac{\varepsilon}{2}\right)$ for some m_0, and hence we could find a neighborhood N of P_0 such that $N \cap E^{\varepsilon'}_{n,m_0+1}(\varepsilon) = \varphi$. Then $\gamma_n(N) > 0$ and $\gamma_{n,m'}(N) = 0$ for every $m' \geqq m_0 + 1$, which is absurd. Since $S_{\gamma_n} \subset E^{\varepsilon'}_{n,m}\left(\frac{\varepsilon}{2}\right)$, we have

$$\text{(2)} \qquad \frac{\alpha\varepsilon}{4} \leqq \frac{\varepsilon}{2}\gamma_n(1) \leqq \int (V_{n,m} - U^{\mu})\,d\gamma_n \leqq \int (U^{\mu_m} - U^{\mu})\,d\gamma_n$$

fore very sufficiently large m. On the other hand, we have $\lim_m \int U^{\mu_m} d\gamma_n = \int U^{\mu} d\gamma_n$

by Theorem 2.2. This contradicts (2). Consequently, we see that $\lim_m \mathrm{cap}_i (E^{\varepsilon'}_{n,m}(\varepsilon)) = 0$. Therefore, from (1), we can conclude that $\lim_m \mathrm{cap}_e (E_{n,m}(\varepsilon)) \leqq \varepsilon'$. Thus we obtain $\lim_m \mathrm{cap}_e (E_{n,m}(\varepsilon)) = 0$.

Now we choose a sequence $\{\varepsilon_k\}$ of positive numbers such that $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_k > \varepsilon_{k+1} > \dots \to 0$ and we put

$$E_n(\varepsilon_k) = \{P;\ V_n(P) - U^\mu(P) > \varepsilon_k\}$$
$$E(\varepsilon_k) = \{P;\ V(P) - U^\mu(P) > \varepsilon_k\}$$

and

$$E = \{P;\ V(P) - U^\mu(P) > 0\}.$$

Then, since it is immediately seen that $E_n(\varepsilon_k) \subset E_{n,m}(\varepsilon_k)$ and $E(\varepsilon_k) = \bigcup_n E_n(\varepsilon_k)$, we have $\mathrm{cap}_e (E_n(\varepsilon_k)) = 0$ and $\mathrm{cap}_e (E(\varepsilon_k)) = 0$. Then, from $E = \bigcup_k E(\varepsilon_k)$, we get $\mathrm{cap}_e (E) = 0$, that is, $U^\mu(P) \geqq V(P)$ quasi everywhere in Ω.

3. THEOREM 2.4. *Suppose that there exists an equilibrium measure of every compact set and that μ_n $(n = 1, 2, \dots)$ are measures on a compact set such that the potentials $U^{\mu_n} \leqq 1$ in Ω. If $\{\mu_n\}$ converges vaguely to μ, we have*

$$\underline{\lim}_n U^{\mu_n} = U^\mu$$

quasi everywhere in Ω.

Proof. This follows immediately from Theorem 1.6 and 2.3.

4. When Φ satisfies the continuity principle, the uniform boundedness of U^{μ_n} $(n = 1, 2, \dots)$ is dispensable to assert that $U^\mu = \underline{\lim}_n U^{\mu_n}$ quasi everywhere in Ω.

For the purpose, first we shall show the following

LEMMA 2.1.[12] *Let μ_n $(n = 1, 2, \dots)$ be measures on a compact set such that $\{\mu_n\}$ converges vaguely to μ. Then, it holds that*

$$U^\mu = \underline{\lim}_n U^{\mu_n}$$

nearly everywhere in Ω.

Proof. We put

$$E = \{P;\ V(P) - U^\mu(P) > 0\},$$

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12) Cf. Brelot [3] and Ohtsuka [18].

where $V=\underline{\lim}_n U^{\mu_n}$. If $\mathrm{cap}_i(E) > \alpha > 0$, then we can find an admissible measure γ on a compact set in E such that $\gamma(1) \geqq \alpha$. By Lusin's theorem, we have a restriction γ' of γ to a suitable compact subset of S_γ such that $\gamma'(1) \geqq \frac{\alpha}{2}$ and $U^{\gamma'}$ is continuous on $S_{\gamma'}$. Then, by the continuity principle, $U^{\gamma'}$ is continuous in Ω. For this potential $U^{\gamma'}$, we get

$$0 < \int (V - U^{\mu})\, d\gamma' \leqq \underline{\lim} \int U^{\mu_n} d\gamma' - \int U^{\mu} d\gamma' = 0,$$

which is impossible. Thus our lemma is established.

By our lemma, we have immediately

LEMMA 2.2. *Let μ_n be measures on a compact set. If $\{\mu_n\}$ converges vaguely to μ and a potential $U^{\gamma} \leqq 1$ in Ω, then it holds that $\int U^{\mu} d\gamma = \int V d\gamma$, where $V = \underline{\lim}_n U^{\mu_n}$.*

THEOREM 2.5.[13] *Suppose that Φ satisfies the continuity principle. If μ_n $(n = 1, 2, \ldots)$ are measures on a compact set and $\{\mu_n\}$ converges vaguely to μ, then we have*

$$U^{\mu} = \underline{\lim}_n U^{\mu_n}$$

quasi everywhere in Ω.

Proof. We proceed in the same way as in the proof of Theorem 2.3. If $\lim_m \mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon)) = \alpha > 0$, then there exists an admissible measure γ_n, for which the inequality

$$\frac{\alpha\varepsilon}{4} \leqq \int (V_{n,m} - U^{\mu})\, d\gamma_n$$

holds for every sufficiently large m. Here, letting m tend to infinity, we have

$$(3) \qquad \frac{\alpha\varepsilon}{4} \leqq \int (V_n - U^{\mu})\, d\gamma_n \leqq \int (V - U^{\mu})\, d\gamma_n.$$

The last integral of (3) is equal to zero by Lemma 2.2, which is absurd. Consequently, we have $\lim_m \mathrm{cap}_i(E^{\varepsilon'}_{n,m}(\varepsilon)) = 0$. Then, we can prove, by the same argument as in the proof of Theorem 2.3, that $V = U^{\mu}$ quasi everywhere in Ω.

[13] See also Choquet [7].

§ 3. Capacitability

1. In this section we shall prove that every borelian and analytic set contained in a compact set are capacitable. We assume first that any compact set K is metrisable. Then K is a compact G_δ set and it can be concluded that any compact set K is capacitable. We also assume that there exists an equilibrium measure μ_K of any compact set K. By our assumption we conclude that, for any open set G contained in a compact set, there exists an equilibrium measure μ_G such that $\mu_G(1) = \text{cap}\,(G)$, $U^{\mu_G}(P) \leqq 1$ everywhere in Ω and $U^{\mu_G}(P) = 1$ quasi everywhere in G. By this fact we can prove that, if a sequence $\{X_n\}$ of arbitrary subsets contained in a fixed compact set K increases monotously to X, then it holds that $\lim_n \text{cap}_e\,(X_n) = \text{cap}_e\,(X)$. Using Choquet's method, we can see that every borelian and analytic set contained in a compact set are capacitable.

2. We assume the following two conditions:

1° Any compact subset K is metrisable.

2° There always exists an equilibrium measure μ_K of any compact subset K, that is, there exists a positive measure μ_K on K such that $\mu_K(1) = \text{cap}_i\,(K)$, $U^{\mu_K}(P) \leqq 1$ everywhere in Ω and $U^{\mu_K}(P) = 1$ nearly everywhere on K.

THEOREM 3.1. *If a decreasing sequence K_n of compact subsets converges to K, then we have* $\lim_n \text{cap}_i\,(K_n) = \text{cap}_i\,(K)$.

Proof. This theorem is proved without our two assumptions. For any $\varepsilon > 0$ and for each n, there exists an admissible measure on K_n such that $\text{cap}_i\,(K_n) - \varepsilon \leqq \mu_n(1)$. As the total masses $\mu_n(1)$ of μ_n are uniformly bounded, we can take out a subsequence $\{\mu_{n'}\}$ of $\{\mu_n\}$ which converges vaguely to μ_0. We see that S_{μ_0} is contained in K. In fact, if there exists a point $P_0 \in S_{\mu_0} - K$, then we can find a relatively compact neighborhood $\omega(P_0)$ such that $\overline{\omega(P_0)} \cap K = \emptyset$. As K is contained in $\Omega - \overline{\omega(P_0)}$, there exists a sufficiently large n_0 such that, for all $n \geqq n_0$, $\Omega - \overline{\omega(P_0)} \supset K_{n_0} \supset K$. Now let f be a continuous function in Ω such that $0 \leqq f(P) \leqq 1$, $f(P) = 1$ in $\omega(P_0)$ and $f(P) = 0$ on K_{n_0}. Then we see that

$$0 < \int f d\mu_0 = \lim_{n'} \int f d\mu_{n'} = 0,$$

which is impossible. Hence μ_0 is admissible on K and $\mu_0(1) \leqq \text{cap}_i\,(K)$. Thus we have seen that $\lim_n \mathrm{cap}_i(K_n) - \varepsilon \leqq \mathrm{cap}_i(K)$, and hence $\lim_n \mathrm{cap}_i(K_n) \leqq \mathrm{cap}_i(K)$.

To prove the capacitability of compact set we prove the following

LEMMA 3.1. *If a compact set K is a G_δ set, that is, $K = \bigcap_{n=1}^{\infty} G'_n$, then there exists a sequence $\{G_n\}$ of open sets such that $G'_n \supset G_n$, $G_n \supset \overline{G_{n+1}} \supset G_{n+1} \ldots$, $\overline{G_n}$ are compact and $K = \bigcap_{n=1}^{\infty} G_n$.*

Proof. There is no loss of generality in assuming that each G'_n is relatively compact. For each G'_n, there exists a continuous function f_n in Ω such that $0 \leqq f_n \leqq 1$ in Ω, $f_n(P) = 0$ on K and $f_n(P) = 1$ on $\Omega - G'_n$. We put

$$f(P) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(P),$$

then f is continuous in Ω and $0 \leqq f \leqq 1$ and $K = \{P;\ f(P) = 0\}$. The open sets $G_n = \left\{P;\ f(P) < \frac{1}{2^{n-1}}\right\}$ $(n = 1, 2, \ldots)$ satisfy the conditions required in our lemma. In fact, $K = \cap G_n$ and $\overline{G_n} \subset \left\{P;\ f(P) \leqq \frac{1}{2^{n-1}}\right\}$ and hence $\overline{G_{n+1}} \subset G_n$. In $\Omega - G'_n$ it holds that $f_m(P) = 1$ for all $m \geqq n$ and hence $f(P) \geqq \sum_{m=n}^{\infty} \frac{1}{2^m} = \frac{1}{2^{n-1}}$. Consequently we have $G'_n \supset G_n$.

LEMMA 3.2. *Every compact G_δ set K is capacitable.*

Proof. By Lemma 3.1, we can choose a sequence $\{G_n\}$ of open sets such that $G_n \supset \overline{G_{n+1}} \supset G_{n+1} \supset \ldots$, each $\overline{G_n}$ is compact and $K = \cap G_n$. Then we have by Theorem 3.1 that

$$\mathrm{cap}_i(K) = \lim \mathrm{cap}_i(\overline{G_n}) \geqq \lim \mathrm{cap}_i(G_n) \geqq \mathrm{cap}_e(K).$$

THEOREM 3.2. *Every compact set is capacitable.*

Proof. Since, by our assumption 1°, every compact set is a G_δ set, this follows immediately from Lemma 3.2.

THEOREM 3.3. *An equilibrium potential U^{μ_K} of a compact set K has the following properties: $\mu_K(1) = \mathrm{cap}(K)$, $U^{\mu_K}(P) \leqq 1$ in Ω and $U^{\mu_K}(P) = 1$ quasi everywhere on K.*

Proof. It is sufficient to show that $U^{\mu_K}(P) = 1$ quasi everywhere on K. We put

$$E = \{P \in K;\ U^{\mu_K}(P) < 1\}$$

and

$$E_n = \left\{P \in K;\ U^{\mu_K}(P) \leqq 1 - \frac{1}{n}\right\} \qquad (n = 1, 2, \ldots).$$

Then each E_n is compact and $E = \cup E_n$. Since $\mathrm{cap}_i(E) = 0$, we have $\mathrm{cap}_i(E_n) = 0$ and hence, by Theorem 3.2,

$$\mathrm{cap}_e(E) \leqq \sum_{n=1}^{\infty} \mathrm{cap}_e(E_n) = 0.$$

The following theorem is very useful to estimate the outer capacity of a subset of Ω.

THEOREM 3.4. *Suppose that* $U^{\mu}(P) \leqq 1$ *in* Ω *and* $U(P) = 1$ *quasi everywhere in* X. *Then we have* $\mathrm{cap}_e(X) \leqq \mu(1)$.

Proof. Putting

$$Y = \{P;\ U^{\mu}(P) = 1\}$$

and

$$E = \{P \in X;\ U^{\mu}(P) < 1\},$$

we have $\mathrm{cap}_e(E) = 0$ and $X = (X \cap Y) \cup E$, and hence

$$\mathrm{cap}_e(X) \leqq \mathrm{cap}_e(X \cap Y) + \mathrm{cap}_e(E) = \mathrm{cap}_e(X \cup Y) \leqq \mathrm{cap}_e(Y).$$

Thus it is sufficient to show that $\mathrm{cap}_e(Y) \leqq \mu(1)$. We put

$$Y_n = \left\{P;\ U^{\mu}(P) > 1 - \frac{1}{n}\right\} \qquad (n = 2, 3, \ldots),$$

then each Y_n is open and $\cap Y_n = Y$, hence $\lim_n \mathrm{cap}(Y_n) \geqq \mathrm{cap}_e(Y)$. For any $\varepsilon > 0$ and each n, there exists an admissible measure μ_n on a compact subset of Y_n such that $\mathrm{cap}(Y_n) - \varepsilon \leqq \mu_n(1)$. Here it follows that

$$\left(1 - \frac{1}{n}\right)\mu_n(1) < \int U^{\mu} d\mu_n = \int U^{\mu_n} d\mu \leqq \mu(1).$$

Consequently we have

$$\mathrm{cap}_i(Y_n) - \varepsilon \leqq \frac{n}{n-1}\mu(1)$$

and hence $\lim_n \mathrm{cap}(Y_n) \leqq \mu(1)$ and $\mathrm{cap}_e(Y) \leqq \mu(1)$.

Analogously to Theorem 3.4, we have

THEOREM 3.5. *Suppose that* $U^{\mu}(P) \leqq 1$ *in* Ω *and* $U^{\mu}(P) = 1$ *nearly everywhere in* X. *Then the inequality* $\mathrm{cap}_i(X) \leqq \mu(1)$ *holds.*

THEOREM 3.6, *For any relatively compact set G, there exists an equilibrium measure μ_G such that $\mu_G(1) = \mathrm{cap}(G)$, $U^{\mu_G}(P) \leqq 1$ everywhere in Ω and $U^{\mu_G}(P) = 1$ quasi everywhere in G.*

Proof. We can find an increasing sequence $\{K_n\}$ of compact sets such that $\cup K_n = G$. For each K_n, there exists an equilibrium potential U^{μ_n} which is, by Theorem 3.3, equal to 1 quasi everywhere on K_n. We can choose a subsequence $\{\mu_{n'}\}$ of $\{\mu_n\}$ which converges vaguely to μ_0, because $\mu_n(1) \leqq \mathrm{cap}(G) < +\infty$. It is seen that μ_0 is an equilibrium measure of G. In fact, by Theorem 1.5, $U^{\mu_0}(P) \leqq \lim U^{\mu_{n'}}(P) \leqq 1$ everywhere in Ω and $\mu_0(1) = \lim \mu_{n'}(1) = \lim \mathrm{cap}(K_n) \leqq \mathrm{cap}(G)$. We shall show that $U^{\mu_0}(P) = 1$ quasi everywhere in G. Since for any $n' \geqq n_0'$, $U^{\mu_{n'}}(P) = 1$ quasi everywhere on $K_{n_0'}$, we have that $\underline{\lim}\, U^{\mu_{n'}}(P) = 1$ quasi everywhere on $K_{n_0'}$, and hence, by Theorem 2.4, $U^{\mu_0}(P) = 1$ quasi everywhere on $K_{n_0'}$ and $U^{\mu_0}(P) = 1$ quasi everywhere in G. Consequently, by Theorem 3.4, we get $\mathrm{cap}(G) \leqq \mu_0(1)$.

THEOREM 3.7. *For an arbitrary subset X contained in a compact set, there exists an equilibrium measure μ_X such that $\mu_X(1) = \mathrm{cap}_e(X)$, $U^{\mu_X}(P) \leqq 1$ everywhere in Ω and $U^{\mu_X}(P) = 1$ quasi everywhere in X.*

Proof. For a given X, there exists a sequence $\{G_n\}$ of open sets such that $G_n \supset X$ and $\mathrm{cap}_e(X) = \lim\limits_n \mathrm{cap}(G_n)$, where we may suppose that each G_n is relatively compact. For each G_n, there exists an equilibrium measure μ_n by Theorem 3.6. The total masses $\mu_n(1)$ being uniformly bounded, a subsequence $\{\mu_{n'}\}$ of $\{\mu_n\}$ converges vaguely to μ_0. Obviously it follows that $U^{\mu_0}(P) \leqq 1$ everywhere in Ω and, by Theorem 2.4, we see that $U^{\mu_0}(P) = 1$ quasi everywhere in X. We have also

$$\mu_0(1) = \lim_{n'} \mu_{n'}(1) = \lim_{n} \mathrm{cap}(G_n) = \mathrm{cap}_e(X).$$

THEOREM 3.8. *If an increasing sequence $\{X_n\}$ of arbitrary subsets converges to a relatively compact subset X, then we have $\lim \mathrm{cap}_e(X_n) = \mathrm{cap}_e(X)$.*

Proof. By Theorem 3.7, there exists an equilibrium measure μ_n of each X_n. We can choose a subsequence $\{\mu_{n'}\}$ of $\{\mu_n\}$ which converges vaguely to μ_0. Obviously $U^{\mu_0}(P) \leqq 1$ everywhere in Ω. Since, by Theorem 2.4, $U^{\mu_0}(P) = \underline{\lim}\, U^{\mu_{n'}}(P)$ quasi everywhere in Ω, we see that $U^{\mu_0}(P) = 1$ quasi everywhere in X. Hence, by Theorem 3.4, we obtain that

$$\text{cap}_e(X) \leqq \mu_0(1) = \lim \mu_n(1) = \lim \text{cap}_e(X_n) \leqq \text{cap}_e(X).$$

3.[14] Now we consider a product space $E \times F$ of a compact subset E of Ω and a compact auxiliary space F. We define a capacity $c(K)$ of a compact subset $K \subset E \times F$ by $c(K) = \text{cap}\,(\text{pr. } K)$, where pr. K means the canonical projection of K on E. The definitions of the inner capacity $c_i(X)$ and the outer capacity $c_e(X)$ of an arbitrary subset $X \subset E \times F$ are evident.

THEOREM 3.9.

1° *For any subset* $X \subset E \times F$, $\text{cap}_i\,(\text{pr. } X) \geqq c_i(X)$.

2° *For an open set* $G \subset E \times F$, $\text{cap}\,(\text{pr. } G) = c(G)$.

3° *For any subset* $X \subset E \times E$, $\text{cap}_e\,(\text{pr. } X) = c_e(X)$.

These assertions are easily verified. See also Benzécri [1].

THEOREM 3.10. *If a subset $X \subset E \times F$ is capacitable, then* pr. X *is capacitable.*

Proof. By Theorem 3.9, we have

$$\text{cap}_i\,(\text{pr. } X) \geqq c_i(X) = c_e(X) = \text{cap}_e\,(\text{pr. } X).$$

THEOREM 3.11. *Every compact subset K of $E \times F$ is capacitable.*

Proof. This follows from the definition of $c(K)$ and Theorems 3.2 and 3.9.

THEOREM 3.12. *If a decreasing sequence $\{K_n\}$ of compact subsets of $E \times F$ converges to K, then we have* $\lim c(K_n) = c(K)$.

Proof. This follows immediately from the equalities $c(K_n) = \text{cap}\,(\text{pr. } K_n)$ and Theorem 3.1.

THEOREM 3.13. *If an increasing sequence $\{X_n\}$ of subsets of $E \times F$ converges to X, then we have* $\lim\limits_n c_e(X_n) = c_e(X)$.

Proof. This is an immediate consequence of Theorems 3.8 and 3.9.

By Theorems 3.11, 12 and 13, we can prove

THEOREM 3.14. *Every $K_{\sigma\delta}$ set of $E \times F$ is capacitable.*

This theorem can be proved in the same way as in [1].

4. THEOREM 3.15. *Every K-borelian and more generally K-analytic set in a compact set of Ω are capacitable.*

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[14] In this section 3.3 we apply Choquet-Benzecri's method [1] and [5].

Proof. This follows from Theorems 3.10 and 3.14 and the fact that every K-borelian subset and K-analytic subset in a compact set of a Hausdorff space E are the canonical projection on E of a $K_{\sigma\delta}$ of the product space of E and a compact auxiliary space.[15]

THEOREM 3.16. *If a K-analytic set, contained in a K_σ set, is of inner capacity zero, then it is of outer capacity zero.*

Proof. This is an immediate consequence of Theorems 1.4 and 3.15.

§ 4. Function $m^*(P, U^\mu)$

1. In this section we assume that every open set is of positive capacity. We associate the function $m^*(P, U^\mu)$ with every potential U^μ of positive measure μ and consider the behavior of this function. We shall say that a constant k is an *E-lower bound* of U^μ in an open set ω, if a set of points $P \in \omega$ such that $U^\mu(P) < k$ is of inner capacity zero. It is easily seen that, for any open set ω, there exists the maximum of E-lower bounds of U^μ in ω, which we denote by $m^*(U^\mu, \omega)$. Then the function $m^*(P, U^\mu)$ is defined as follows:

$$m^*(P, U^\mu) = \sup m^*(U^\mu, \omega(P)) \quad \text{for all open neighborhoods of } P.$$

It is well-known[16] that $m^*(P, U^\mu)$ is lower semi-continuous in Ω and $U^\mu(P) \leqq m^*(P, U^\mu)$ everywhere in Ω, as U^μ is lower semi-continuous. When $m^*(P, U^\mu)$ is continuous in an open set ω, U^μ is called *E-continuous* in ω.

THEOREM 4.1. *If U^μ is E-continuous in ω, then the equality $m^*(P, U^\mu) = M(P, U^\mu)$ holds everywhere in ω, where $M(P, U^\mu)$ means the upper limit function of U^μ.*

Proof. We put $\varphi(P) = m^*(P, U^\mu)$, then $\varphi(P)$ is continuous in ω, and hence $\varphi(P) = M(P, \varphi)$ everywhere in ω. Since it holds that $\varphi(P) \geqq U^\mu(P)$ everywhere in ω, we have

$$\varphi(P) = M(P, \varphi) \geqq M(P, U^\mu) \geqq m^*(P, U^\mu) = \varphi(P)$$

everywhere in ω. Consequently we get $m^*(P, U^\mu) = M(P, U^\mu)$.

THEOREM 4.2. *Suppose that there exists a continuous function ψ in an open*

15) Cf. Choquet [5].
16) Cf. Hahn [9].

set ω which differs from U^μ at most at a set of points of inner capacity zero. Then U^μ is E-continuous in ω.

Proof. Let P_0 be a point of ω. We shall show that, for every neighborhood $\omega(P_0) \subset \omega$, the equality $m^*(U^\mu, \omega(P_0)) = \inf_{P \in \omega(P_0)} \psi(P)$ holds. Put $k = m^*(U^\mu, \omega(P_0))$, then the set of points $P \in \omega(P_0)$ such that $U^\mu(P) < k$ is of inner capacity zero, hence $\psi(P) \geqq k$ at every point P of $\omega(P_0)$, because every open set is of positive capacity. Therefore, we obtain $\inf_{P \in \omega(P_0)} \psi(P) \geqq m^*(U^\mu, \omega(P_0))$.

Now let k be a constant which is smaller than $\inf_{P \in \omega(P_0)} \psi(P)$. Then, since $\psi = U^\mu$ nearly everywhere in $\omega(P_0)$, k is an E-lower bound, and we get $k \leqq m^*(U^\mu, \omega(P_0))$. This shows that $\inf_{P \in \omega(P_0)} \psi(P) \leqq m^*(U^\mu, \omega(P_0))$. Consequently we have that $m^*(P_0, U^\mu) = \psi(P_0)$ at every point $P_0 \in \omega$, and that U^μ is E-continuous in ω.

Conversely we can prove

THEOREM 4.3. *If U^μ is E-continuous in Ω, then there exists a continuous function ψ in Ω which coincides with U^μ nearly everywhere in Ω.*

To prove this theorem, we shall prove

THEOREM 4.4. *If U^μ is E-continuous in Ω, then $m^*(P, U^\mu) = U^\mu(P)$ nearly everywhere in Ω.*

Proof. We put

$$E_n = \left\{P;\ m^*(P, U^\mu) > U^\mu(P) + \frac{1}{n}\right\} \qquad (n = 1, 2, \ldots)$$

and

$$E_\infty = \{P;\ m^*(P, U^\mu) > U^\mu(P)\}.$$

Then $E_\infty = \cup E_n$. It is sufficient to show that $\mathrm{cap}_i(E_n) = 0$. We can easily see that $E_n \subset S_\mu$. For each $P_0 \in S_\mu$, we can take a neighborhood $\omega_n'(P_0)$ such that

$$m^*(P, U^\mu) + \frac{1}{2n} > m^*(P, U^\mu) \quad \text{at every point} \quad P \in \omega_n'(P_0)$$

and a neighborhood $\omega_n''(P_0)$ such that

$$m^*(U^\mu, \omega_n''(P_0)) + \frac{1}{2n} > m^*(P_0, U^\mu),$$

hence we have

$$m^*(U^\mu,\ \omega_n(P_0)) + \frac{1}{n} > m^*(P,\ U^\mu) \text{ at every point}$$
$$P \in \omega_n(P_0) = \omega'_n(P_0) \cap \omega''_n(P_0).$$

As $m^*(U^\mu, \omega_n(P_0))$ is an E-lower bound of U^μ in $\omega_n(P_0)$, we get $U^\mu(P) \geqq m^*(U^\mu, \omega_n(P_0))$ nearly everywhere in $\omega_n(P_0)$. Consequently, $U^\mu(P) + \frac{1}{n} \geqq m^*(P, U^\mu)$ nearly everywhere in $\omega_n(P_0)$, and hence $\text{cap}_i(E_n \cap \omega_n(P_0)) = 0$. Since S_μ is compact, we have $\text{cap}_i(E_n) = 0$ and $\text{cap}_i(E_\infty) = 0$, that is, $m^*(P, U^\mu) \leqq U^\mu(P)$ nearly everywhere in Ω.

Here we shall give the *proof of Theorem* 4.3.[17]

By Theorem 4.4, we can take a continuous function $\psi(P) = m^*(P, U^\mu)$, which coincides with U^μ nearly everywhere in Ω.

We shall say that Φ is equal to zero at infinity, if, for any fixed point $P \in \Omega$, $\Phi(P, Q)$ tends to zero as Q tends to ω^*, where ω^* is the Alexandroff's point of Ω.

If a point $P_0 \in S_\mu$ has such a property that, for any neighborhood $\omega(P_0)$ of P_0, $\text{cap}_i(\omega(P_0) \cap S_\mu) > 0$, then we say that P_0 belongs to $\tilde{S}_\mu$.

THEOREM 4.5. *Suppose that Φ is equal to zero at infinity, and the restriction of U^μ to S_μ is continuous at $P_0 \in S_\mu$. If U^μ coincides with a continuous function ψ nearly everywhere in a neighborhood $\omega(P_0)$ of P_0, then U^μ is continuous at P_0.*

Proof. Since U^μ coincides with ψ nearly everywhere in $\omega(P_0)$, the function $m^*(P, U^\mu)$ is continuous in $\omega(P_0)$ by Theorem 4.2, and U^μ is bounded in a neighborhood $\omega'(P_0)$ of P_0 such that $\omega'(P_0) \subset \overline{\omega'(P_0)} \subset \omega(P_0)$. We assert that P_0 belongs to $\tilde{S}_\mu$. In fact, if $P_0 \in S_\mu - \tilde{S}_\mu$, then we have a neighborhood $\omega''(P_0) \subset \overline{\omega''(P_0)} \subset \omega'(P_0)$ such that $\text{cap}_i(\overline{\omega''(P_0)} \cap S_\mu) = 0$. On the other hand, the potential $U^{\mu'}$ of the restriction μ' of μ to $\overline{\omega''(P_0)}$ is bounded in Ω, because $U^{\mu'} \leqq U^\mu$ and U^μ is bounded in $\omega'(P_0)$ and Φ is equal to zero at infinity. Consequently, we have $\mu' \equiv 0$, which contradicts the assumption that $P_0 \in S_\mu$.

Now, for any $\varepsilon > 0$, there exist two neighborhoods $\omega_1(P_0)$ and $\omega_2(P_0)$ of P_0 such that

$$m^*(P_0,\ U^\mu) - \varepsilon \leqq m^*(U^\mu,\ \omega_1(P_0))$$

and

$$U^\mu(P) < U^\mu(P_0) + \varepsilon \text{ at every point } P \text{ of } \omega_2(P) \cap S_\mu.$$

[17] When S_μ is separable, this has been proved in Hahn [9], p. 175.

Then, in $\tilde{\omega}(P_0) \cap S_\mu$, where $\tilde{\omega}(P_0) = \omega_1(P_0) \cap \omega_2(P_0)$, we have

$$m^*(P_0, U^\mu) - \varepsilon \leqq m^*(U^\mu, \tilde{\omega}(P_0))$$

and

$$U^\mu(P) < U^\mu(P_0) + \varepsilon.$$

It follows that $m^*(U^\mu, \tilde{\omega}(P_0)) \leqq U^\mu(P_0) + \varepsilon$. In fact, if $m^*(U^\mu, \tilde{\omega}(P_0)) > U^\mu(P_0) + \varepsilon$, then $S_\mu \cap \tilde{\omega}(P_0)$ is contained in the set $\{P \in \tilde{\omega}(P_0)\,;\ U^\mu(P) < m^*(U^\mu, \tilde{\omega}(P_0))\}$, which contradicts the fact $P_0 \in \tilde{S}_\mu$. Thus we have seen that

$$m^*(P_0, U^\mu) - \varepsilon \leqq m^*(U^\mu, \tilde{\omega}(P_0)) \leqq U^\mu(P_0) + \varepsilon,$$

and hence $m^*(P_0, U^\mu) = U^\mu(P_0)$.

Since $m^*(P_0, U^\mu)$ is continuous at P_0, there exists a neighborhood $\omega'(P_0)$ of P_0 such that

$$m(P_0, U^\mu) + \varepsilon > m^*(P, U^\mu) \quad \text{at every point} \quad P \in \omega'(P_0).$$

Hence $U^\mu(P_0) + \varepsilon > m^*(P, U^\mu) \geqq U^\mu(P)$ at every point $P \in \omega'(P_0)$. This means that $U^\mu(P)$ is upper semi-continuous at P_0.

COROLLARY. *Suppose that Φ is equal to zero at infinity, and the restriction of U^μ to S_μ is continuous. If there exists a continuous function ψ in Ω such that $\psi = U^\mu$ nearly everywhere in Ω, then U^μ is continuous in Ω.*

2. Now we consider a potential $U^\mu(P)$, which coincides with $m^*(P, U^\mu)$ everywhere in Ω.

THEOREM 4.6. *A potential $U^\mu(P)$ coincides with $m^*(P, U^\mu)$ everywhere in Ω if and only if it has the following property: at every point $P_0 \in \Omega$,*

$$\mathrm{cap}_i\,(\{P \in \omega(P_0)\,;\ U^\mu(P) \leqq h\}) > 0$$

for any neighborhood $\omega(P_0)$ and $h > U^\mu(P_0)$.[18]

Proof. Suppose that $U^\mu(P_0) < m^*(P_0, U^\mu)$, then there exist a neighborhood $\omega(P_0)$ of P_0 and a positive number ε such that

$$U^\mu(P_0) < m^*(U^\mu, \omega(P_0)) - \varepsilon \leqq m^*(U^\mu, \omega(P_0)) \leqq m^*(P_0, U^\mu).$$

Then the set $\{P \in \omega(P_0)\,;\ U^\mu(P) \leqq m^*(U^\mu, \omega(P_0)) - \varepsilon\}$ is of inner capacity zero.

[18] Choquet [7] assumes this condition to state the result that if $U^{\mu_1} = U^{\mu_2}$ nearly everywhere in Ω, we have $U^{\mu_1} = U^{\mu_2}$ everywhere in Ω.

Now suppose that $U^{\mu}(P_0)=m^*(P_0,\ U^{\mu})$. Then, for any neighborhood $\omega(P_0)$ of P_0, we have $m^*(U^{\mu},\ \omega(P_0))\leqq U^{\mu}(P_0)$. Therefore, for any $h>U^{\mu}(P_0)$, the set $\{P\in\omega(P_0)\ ;\ U^{\mu}(P)\leqq h\}$ must be of inner capacity positive.

Hereafter, we assume that every compact set is separable.

THEOREM 4.7. *Suppose that there exists an equilibrium measure of every compact set. If, for any potential* U^{μ}, $U^{\mu}(P)=m^*(P,\ U^{\mu})$ *everywhere in* Ω, *then there exists an equilibrium potential* U^{μ_G} *for any open set contained in a compact set such that* $\mu_G(1)=\mathrm{cap}\,(G)$, $U^{\mu_G}(P)\leqq 1$ *everywhere in* Ω *and* $U^{\mu_G}(P)=1$ *everywhere in* G.

Proof. Let G be an open set contained in a compact set. There exists an increasing sequence $\{K_n\}$ of compact sets such that $K_n\subset G$ and $\cup K_n=G$. Let μ_n be an equilibrium measure of K_n, then a subsequence $\{\mu_{n'}\}$ of $\{\mu_n\}$ converges vaguely to μ_0. It is easily seen by Theorems 1.5, 2.4 and 3.5 that $U^{\mu_0}\leqq 1$ everywhere in Ω, $U^{\mu_0}=1$ nearly everywhere in G and $\mu_0(1)=\mathrm{cap}\,(G)$. Suppose that there exists a point $P_0\in G$ such that $U^{\mu_0}(P_0)<1$. Then there exist a neighborhood $\omega(P_0)\subset G$ and a positive number δ such that

$$U^{\mu_0}(P_0)=m^*(P_0,\ U^{\mu_0})<m^*(U^{\mu_0},\ \omega(P_0))+\delta<1.$$

Hence the set

$$\{P\in\omega(P_0)\ ;\ U^{\mu_0}(P)\leqq m^*(U^{\mu_0},\ \omega(P_0))+\delta\}$$

is of inner capacity positive by Theorem 4.7, which contradicts the fact that $U^{\mu_0}(P)=1$ nearly everywhere in G. Thus we have seen that $U^{\mu_0}(P)=1$ everywhere in G.

THEOREM 4.8. *Suppose that* Φ *satisfies Cartan's maximum principle*[19] *and that there exists a potential* U^{μ_G}, *for any open set* G, *such that* $U^{\mu_G}(P)=1$ *everywhere in* G, *and* $U^{\mu_G}(P)=1$ *nearly everywhere on* $S_{\mu_G}\subset\bar{G}$. *Then, for any potential* U^{μ}, *we have* $U^{\mu}(P)=m^*(P,\ U^{\mu})$ *everywhere in* Ω.

Proof. Suppose that there exist a constant h and a neighborhood $\omega(P_0)$ of P_0 such that $U^{\mu}(P_0)<h$ and the set

$$\{P\in\overline{\omega(P_0)}\ ;\ U^{\mu}(P)<h\}$$

[19] Cartan's maximum principle means: Let μ be a positive measure with finite energy and ν be an arbitrary positive measure. If $U^{\mu}\leqq U^{\nu}$ nearly everywhere on S_{μ}, then $U^{\mu}\leqq U^{\nu}$ everywhere in Ω.

is of inner capacity zero. Let $U^{\mu_{\omega}}$ be a potential assured by our assumption such that $U^{\mu_{\omega}}(P)=1$ everywhere in $\omega(P_0)$ and $U^{\mu_{\omega}}(P)=1$ nearly everywhere on $S_{\mu_{\omega}} \subset \omega(P_0)$. Then it holds that $U^{\mu}(P) \geqq hU^{\mu_{\omega}}(P)$ nearly everywhere on $\overline{\omega(P_0)} \supset S_{\mu_{\omega}}$ and hence, by Cartan's maximum principle, we see that $U^{\mu}(P) \geqq hU^{\mu_{\omega}}(P)$ everywhere in Ω and $U^{\mu}(P_0) \geqq hU^{\mu_{\omega}}(P_0) = h$, which is a contradiction.

3. Now we shall state an application of Theorem 4.7.

We shall say that a set E is *thin* at a point P_0 if there exists a positive measure ν such that

$$\varliminf_{E \ni P \to P_0} U^{\nu}(P) > U^{\nu}(P_0).$$

THEOREM 4.9. *Suppose that E is of outer capacity zero. Then E is thin at each point $P_0 \in E$, where $\Phi(P_0, P_0) = +\infty$.*

Proof. Let $P_0 \in E$ and $\Phi(P_0, P_0) = +\infty$. Put

$$B_n = \{P;\ \Phi(P_0, P) > n\} \qquad (n = 1, 2, \ldots).$$

Then $E_n = (B_{n-1} - B_n) \cap E$ $(B_0 = \Omega)$ is of outer capacity zero and there exists an open set G_n such that $E_n \subset G_n$, $G_n \subset B_{n-1} - B_{n+1}$ and $\operatorname{cap}(G_n) < \dfrac{1}{n(n+1)2^n}$. Let μ_n be an equilibrium measure of G_n and $\nu_n = n\mu_n$. Then $\nu_n(1) < \dfrac{1}{(n+1)2^n}$ and $U^{\nu_n}(P) = n$ everywhere in G_n by Theorem 4.8. We put $\nu = \sum \nu_n$. At a point $P \in (B_{n-1} - B_{n+1}) \cap E$, we have $U^{\nu}(P) \geqq U^{\nu_n}(P) = n$ and $U^{\nu}(P_0) = \sum U^{\nu_n}(P) < \sum (n+1)\nu_n(1) = 1$, and hence

$$\varliminf_{E \ni P \to P_0} U^{\nu}(P) > U^{\nu}(P_0).$$

This shows that E is thin at P_0.

Added in proofs: During the proofs of this paper, the author finds that Theorem 1.2 is established by Z. Semadeni and P. Zbijewski: Spaces of continuous functions (I), Studia Math, **16** (1957), 130-141.

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