ON MODULES OF TRIVIAL COHOMOLOGY OVER A FINITE GROUP, II

(FINITELY GENERATED MODULES)

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Let G be a finite group. A (left) G-module A of G is said to be of trivial cohomology when $H^n(H, A) = 0$ for all rational integers n and for all subgroups H of G. The main purpose of the present note is to determine the structure of finitely generated G-modules of trivial cohomology, which turns out to be remarkably simple (See Theorem 1 and Corollary 3 below). We prove also an (easy) localization theorem for cohomological triviality.

However, first we recall a structural study of modules of trivial cohomology made in Part I (Illinois Math. Journ. 1 (1957), p. 36). It begins with considering a free G-module A_0 of which a given G-module A is a G-homomorphic image. Let A_1 be the kernel of the homomorphism. Then the G-module A is of trivial cohomology if and only if the G-module A_1 is so. *Having thus reduced the problem to the case of a (Z-)torsion-free (even Z-free) G-module,* we have, as we have shown in I,

PROPOSITION 0. A (Z-)torsion-free G-module A is of trivial cohomology, if and only if for each prime p (dividing the order $[G]$ of G) the residue-module A/pA is $Z(p)[H_p]$-free, where $Z(p)[H_p]$ denotes the group algebra of a p-Sylow subgroup H_p of G over the field $Z(p)$ of rational integers mod p.

Here "$Z(p)[H_p]$-free" may be replaced by "$Z(p)[H_p]$-projective" since $Z(p)[H_p]$ is primary. Moreover

PROPOSITION 0′. The condition in Proposition 0 may be replaced by that for every prime p (dividing $[G]$) A/pA is $Z(p)[G]$-projective.

Indeed, a $Z(p)[G]$-module is $Z(p)[G]$-projective if and only if it is $Z(p)[H_p]$-projective. For, since the index $[G : H_p]$ is inversible in $Z(p)$, any $Z(p)[G]$-module B is relatively projective with respect to the subring $Z(p)[H_p]$, (or, what

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amounts to the same, B is a $Z(p)[G]$-direct summand of the $Z(p)[G]$-module $B^* = Z(p)[G] \otimes_{Z(p)[H_p]} B \approx Z[(G/H_p)_L] \otimes_{Z(p)} B$ induced by B considered as $Z(p)[H_p]$-module, where on the left hand side of the isomorphism sign the operation of G is explained by the multiplication on $Z(p)[G]$ from left while in the right hand side the operation of G is explained by the operation on both factors $Z[(G/H_p)_L]$, B and the first factor $Z[(G/H_p)_L]$ denotes the vector space over $Z(p)$ spanned by the left cosets of H_p in G; the isomorphism is given by associating $\sigma_i \otimes_{Z(p)[H_p]} b$ in the left hand side to $\sigma_i H_p \otimes_{Z(p)} \sigma_i b$ in the right hand side, where $\{\sigma_i\}$ is a representative system of the left cosets of H_p in G). Indeed, if C is a $Z(p)[G]$-module having $Z(p)[G]$-submodule D such that there is a $Z(p)[H_p]$-submodule M with $C = D + M$ (direct), then, on denoting by π the projection of C onto M with respect to this direct decomposition, we have a direct decomposition $C = D + \rho C$ into $Z(p)[G]$-modules (indeed $D = (1-\rho)C$) by putting $\rho = [G : H_p]^{-1} \sum \sigma_i^{-1} \pi \sigma_i$, where $\{\sigma_i\}$ is as above (cf. [3], [4]).

1. Finitely generated modules of trivial cohomology

THEOREM 1. *A finitely generated* (Z-)*torsion-free G-module A is of trivial cohomology if and only if A is a direct summand of a free G-module, or, what is the same, if and only if A is $Z[G]$-projective, where $Z[G]$ is the group algebra of G over the ring Z of rational integers.*

As the "if" part is evident, we prove the "only if" part. Let, to do so, A be a finitely generated torsion-free G-module of trivial cohomology. Let p be any rational prime and H_p be a p-Sylow subgroup of G. By Proposition 0 the residue-module A/pA has an independent basis over $Z(p)[H_p]$. Let $a_1, \ldots, a_n$ be representatives in A of the basic elements. Denote, further, the quotient ring of Z with respect to p by Z_p. As A is Z-free, the tensor product $A_p = A \otimes_Z Z_p$ is Z_p-free and A may be looked upon as a G-submodule of A_p. We contend that $a_1, \ldots, a_n$ form an independent $Z_p[H_p]$-basis of A_p. Indeed, since A_p/pA_p is naturally isomorphic with A/pA, the residue-classes of $a_1, \ldots, a_n$ modulo pA_p form an independent $Z(p)[H_p]$-basis of A_p/pA_p, or, what amounts to the same, the $[H_p]n$ elements αa_i mod pA_p, α running over H_p, form an independent $Z(p)$-basis of A_p/pA_p. It follows then readily that the matrix of transformation from an independent Z-basis of A to our $[H_p]n$ elements αa_i has a determinant prime to p, whence inversible in Z_p. This shows that αa_i form an independent

Z_p-basis of A_p, or equivalently, a_i form an independent $Z_p[H_p]$-basis of A_p.

Now, since $[G : H_p]$ is inversible in Z_p, every $Z_p[G]$-module is relatively projective with respect to $Z_p[H_p]$; the proof is the same as was made above in context of Proposition 0'. As our A_p has been seen to be $Z_p[H_p]$-free, this implies that A_p is $Z_p[G]$-projective. Since this is the case for every rational prime p, Theorem 1 now follows from the following lemma which is of interest and significance by itself.

LEMMA 2. A finitely generated G-module A is $Z[G]$-projective if, and only if, for every rational prime p the tensor product $A_p = A \otimes_Z Z_p$ is $Z_p[G]$-projective, where Z_p is the ring of quotients of Z with respect to p. (More precisely, we have $\dim_{Z[G]} A = \sup_p \dim_{Z_p[G]} A_p$).

This lemma may be proved as follows by an argument of Serre [6]; cf. also [1], VII, Exer. 11 (Observe, however, that the lemma itself is not contained in [6], nor in [1]). Let, thus,

$$0 \leftarrow A \leftarrow F_0 \leftarrow F_1 \leftarrow \dots \qquad \text{(exact)}$$

be a resolution of A consisting of finitely generated free G-modules F_i. Setting $(F_i)_p = F_i \otimes_Z Z_p$ we obtain a resolution

$$0 \leftarrow A_p \leftarrow (F_0)_p \leftarrow (F_1)_p \leftarrow \dots \qquad \text{(exact)}$$

of A_p by $Z_p[G]$-free modules $(F_i)_p$. As each F_i has an independent finite basis over $Z[G]$, we see, for any G-module C, that the $Z[G]$-module $\mathrm{Hom}_{Z[G]}(F_i, C)$ is simply a direct sum of a finite number of copies of C and hence the $Z_p[G]$-module $(\mathrm{Hom}_{Z[G]}(F_i, C))_p = \mathrm{Hom}_{Z[G]}(F_i, C) \otimes_Z Z_p$ is isomorphic to $\mathrm{Hom}_{Z_p[G]}((F_i)_p, C_p)$ (which is the direct sum of the same finite number of copies of C_p), C_p being $C \otimes_Z Z_p$. We have then readily $\mathrm{Ext}^i_{Z[G]}(A, C) \otimes_Z Z_p \approx \mathrm{Ext}^i_{Z_p[G]}(A_p, C_p)$. Now, if A_p is $Z_p[G]$-projective, for every p, then the right hand side vanishes for $i > 0$, and the same must be the case case for the left hand side. $\mathrm{Ext}^i_{Z[G]}(A, C)$ being finitely generated in case C is so, this implies $\mathrm{Ext}^i_{Z[G]}(A, C) = 0$ for $i > 0$ whenever C is finitely generated. It follows that A must be $Z[G]$-projective. The converse is rather evident.

Theorem 1 being thus proved, we may apply it to the kernel of an epimorphism of a free G-module to a given module, to obtain:

COROLLARY 3. *A finitely generated G-module is of trivial cohomology if and* *only if it is a residue-module of a finitely generated free G-module modulo a $Z[G]$-projective submodule.*

Each of the following two propositions, in which Z_p denotes as above the ring of quotients of Z with respect to p, can readily be seen from a portion of our proof to Theorem 1:

PROPOSITION 4. Let A be a Z_p-(or $Z_p[G]$-) finitely generated (Z- or Z_p-) torsion-free $Z_p[G]$-module (the operation of the elements of Z_p being commutative with the operation of the elements of G). Each of the following conditions i), ii), iii) is necessary and sufficient for A to be of trivial cohomology: i) A is $Z_p[G]$-projective; ii) A is $Z_p[H_p]$-projective (where H_p is a p-Sylow subgroup of G); iii) A is $Z_p[H_p]$-free.

(Assume that ii) is the case. Then A is of trivial cohomology and hence satisfies iii), as well as i), by our proof to Theorem 1.)

PROPOSITION 5. Let A be a (Z-, or $Z[G]$-) finitely generated (Z-) torsion free G-module. A is of trivial cohomology if and only if $A_p = A \otimes_Z Z_p$ is $Z_p[G]$-projective for every prime p (dividing $[G]$). Alternative ways of stating the condition can be seen from Proposition 4.

The following proposition may be of interest in view of the (probably) open question whether every finitely generated $Z[G]$-projective module is $Z[G]$-free (cf. [1], p. 241):

PROPOSITION 6. Let A be a finitely generated $Z[G]$-projective module. Then the Z-rank of A is a multiple of the order $[G]$.

For, with any prime p, A/pA is $Z(p)[H_p]$-free. Hence the $Z(p)$-rank of A/pA is a multiple of $[H_p]$. But the Z-rank of A is clearly equal to the $Z(p)$-rank of A/pA. Thus the Z-rank of A is a multiple of $[H_p]$. Since this is the case for every p, we have the assertion.

2. A localization theorem

Propositions 0, 0′ and 5 have evidently the effect of localization with respect to the property of cohomological triviality, while Lemma 2 is naturally a localization lemma for projectivity (or projective dimension in general). In stating local properties also in terms of cohomological triviality, in connection of Propositions 0, 0′, we have

PROPOSITION 0″. A torsion free G-module A is of trivial cohomology if and only if the G-module A/pA is of trivial cohomology for every prime p (dividing $[G]$).

(This is, however, merely an easy and rather trivial portion of the content of Proposition 0 and the main feature of the latter lies in that its structural local condition is implied by the present local condition.)

Contrary to that these Propositions 0, 0′, 0″ and 5 are for torsion-free modules only (though they have, except Proposition 0″, merits to be structural), the following localization theorem is for general modules:

THEOREM 7. A G-module A is of trivial cohomology if and only if $A_p = A \otimes_Z Z_p$ is of trivial cohomology for every prime p (dividing $[G]$), where Z_p is the ring of quotients of Z with respect to p.

To prove this, we construct a free G-module A_0 of which the given G-module A is a G-homomorphic image and denote the kernel of the homomorphism by A_1. As Z_p is (Z-)torsion-free, we have $\mathrm{Tor}_1^Z(A, Z_p) = 0$ and, therefore $0 \to A_1 \otimes_Z Z_p \to A_0 \otimes_Z Z_p \to A \otimes_Z Z_p \to 0$ (exact), for any prime p. So, for every p, the cohomological triviality of $A \otimes_Z Z_p$ is equivalent to that of $A_1 \otimes_Z Z_p$. Since $(A_1 \otimes_Z Z_p)/p(A_1 \otimes_Z Z_p) \approx A_1/pA_1$ and $(A_1 \otimes_Z Z_p)/q(A_1 \otimes_Z Z_p) = 0$ for $(q, p) = 1$, the G-module $A_1 \otimes_Z Z_p$ is of trivial cohomology if and only if A_1/pA_1 is so, by Proposition 0″. But, that this is the case for every p (dividing $[G]$) is equivalent, again by Proposition 0″, to that A_1 is of trivial cohomology, which is in turn equivalent to that A is so. (Of course we could use either of Proposition 0, 0′ instead of Proposition 0″.)

Remark. In Propositions 4, 5 and Theorem 7 (as well as in Lemma 2) we could replace Z_p by the ring of rational p-adic integers.

Remark. In the present note we have used only a small portion of Part I. Indeed, since we do not need to make dimension shifting in proving Proposition 0 (as well as Propositions 0′, 0″) the dimension shifting portion of our proof in Part I could be eliminated for our present purpose. Thus, what we have made use of, beyond the reduction (to torsion free modules and) to modules B with $pB = 0$, is Lemma 8 in Part I in which the $Z(p)[G]$-free structure is derived from $H^{-1}(G, B) = 0$ ($pB = 0$) for a p-group G. As an alternative, we shall here derive the same structure from $H^{-2}(G, G) = 0$ ($pB = 0$), G being a p-group.

Indeed, since $pB = 0$ we have $H^{-2}(G, B)$ (not only $= \mathrm{Tor}_1^Z(Z, B)$ but) $= \mathrm{Tor}_1^{Z(p)}(Z(p), B)$; this can readily be seen either directly by reducing the standard complex, say, modulo p or by a change of rings formula ([1], VI, 4.1.1). As $Z(p)[G]$ is primary, G being a p-group, $\mathrm{Tor}_1^{Z(p)}(Z(p), B) = 0$ implies, by a syzygy theorem ([2]), that B is $Z(p)[G]$-projective and, therefore, has the desired $Z(p)[G]$-free structure ([5]).

Added in proofs: Another way of formulating Corollary 3 is, as S. Eilenberg points out, to say that a finitely generated G-module A is of trivial cohomology if and only if $\mathrm{l.dim}_{Z[G]} A \leqq 1$, and the same holds with the last condition replaced by $\mathrm{l.dim}_{Z[G]} A \leqq \infty$. He also points out that in proving Lemma 2 we had better to make explicit the ***natural*** isomorphism $\mathrm{Ext}^i_\Lambda(A, C) \otimes \Gamma \approx \mathrm{Ext}^i_{\Lambda\otimes\Gamma}(A\otimes\Gamma, C\otimes\Gamma)$ ($\otimes$ standing for $\otimes_K$) for a left Noetherian K-algebra Λ, a K-flat K-algebra Γ, a finitely generated (left) Λ-module A and a (left) Λ-module C.

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