ON THE DIMENSION OF MODULES AND ALGEBRAS, VIII.

DIMENSION OF TENSOR PRODUCTS

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The questions concerning the dimension of the tensor product $\Lambda\otimes_K\Gamma$ of two K-algebras have turned out to be surprisingly difficult. In this paper we follow a method using spectral sequences (§§1-3) which in some concrete cases yields complete results (§§4-5). In particular, complete results are obtained when Γ is a ring of matrices, triangular matrices, polynomials or rational functions, so that in the first three cases $\Lambda\otimes_K\Gamma$ is respectively the ring of matrices, triangular matrices or polynomials with coefficients in the arbitrary algebra Λ.

Similar techniques yield additivity theorems for the dimensions associated with a tower of three algebras when one of the extensions is special.

At the end (§§6-8) we venture into the domain of semi-primary rings, where the behavior no longer seems to be controlled by spectral sequences. The key result here is Proposition 11 dealing with the case when Γ is semi-simple.

We adhere throughout to the setting and notation of H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton 1956. References to this work are indicated as follows [C-E, V, 4.1.2] meaning Chapter V, Proposition 4.1.2. Other references are made by number referring to the bibliography at the end of the paper.

We have concentrated our attention on the functor Ext and the resulting notions of dimension. All the results have analogues for Tor and the weak dimension. With very few exceptions we have not bothered to state those analogues explicitly. They supply a series of exercises for the willing reader.

Received February 26, 1957. Presented in part to the American Mathematical Society, Sept. 1, 1955 and April 21, 1956.

[1] Work done while S. Eilenberg was engaged under contract AF 18 (603)-67, A. Rosenberg and D. Zelinsky were under contract NSF-G-1357 and during the tenure of a Fulbright grant for D. Zelinsky.

§ 1. Associativity formulas

We shall reexamine here the spectral sequences of [C-E, XVI, § 4] and obtain them under slightly more general conditions.

We begin with the situation described by the symbol $(A_{\Lambda\text{-}\Gamma},\ {}_{\Lambda}B_{\Sigma},\ C_{\Gamma\text{-}\Sigma})$ where Λ, Γ and Σ are K-algebras, K any commutative ring. We consider the functor[2]

$$T(A,\ C) = \mathrm{Hom}_{\Lambda\otimes\Gamma}(A,\ \mathrm{Hom}_{\Sigma}(B,\ C)) = \mathrm{Hom}_{\Gamma\otimes\Sigma}(A\otimes_{\Lambda}B,\ C).$$

Let X be a $\Lambda\otimes\Gamma$-projective resolution of A and let Y be a $\Gamma\otimes\Sigma$-injective resolution of C. Then we have the double complex

$$T(X,\ Y) = \mathrm{Hom}_{\Lambda\otimes\Gamma}(X,\ \mathrm{Hom}_{\Sigma}(B,\ Y)) = \mathrm{Hom}_{\Gamma\otimes\Sigma}(X\otimes_{\Lambda}B,\ Y).$$

We have

$$H(\mathrm{Hom}_{\Sigma}(B,\ Y)) = H(\mathrm{Hom}_{\Gamma\otimes\Sigma}(\Gamma\otimes B,\ Y)) = \mathrm{Ext}_{\Gamma\otimes\Sigma}(\Gamma\otimes B,\ C)$$
$$H(X\otimes_{\Lambda}B) = H(X\otimes_{\Lambda\otimes\Gamma}(\Gamma\otimes B)) = \mathrm{Tor}^{\Lambda\otimes\Gamma}(A,\ \Gamma\otimes B).$$

This implies

$$H_{II}(T(X,\ Y)) = \mathrm{Hom}_{\Lambda\otimes\Gamma}(X,\ \mathrm{Ext}_{\Gamma\otimes\Sigma}(\Gamma\otimes B,\ C))$$
$$H_{I}(T(X,\ Y)) = \mathrm{Hom}_{\Gamma\otimes\Sigma}(\mathrm{Tor}^{\Lambda\otimes\Gamma}(A,\ \Gamma\otimes B),\ Y).$$

Consequently

$$H_{I}H_{II}(T(X,\ Y)) = \mathrm{Ext}_{\Lambda\otimes\Gamma}(A,\ \mathrm{Ext}_{\Gamma\otimes\Sigma}(\Gamma\otimes B,\ C))$$
$$H_{II}H_{I}(T(X,\ Y)) = \mathrm{Ext}_{\Gamma\otimes\Sigma}(\mathrm{Tor}^{\Lambda\otimes\Gamma}(A,\ \Gamma\otimes B),\ C).$$

Thus we obtain the spectral sequences

$$\mathrm{Ext}^{p}_{\Lambda\otimes\Gamma}(A,\ \mathrm{Ext}^{q}_{\Gamma\otimes\Sigma}(\Gamma\otimes B,\ C)) \underset{p}{\Longrightarrow} R^{n}T(A,\ C),$$
$$\mathrm{Ext}^{q}_{\Gamma\otimes\Sigma}(\mathrm{Tor}^{\Lambda\otimes\Gamma}_{p}(A,\ \Gamma\otimes B),\ C) \underset{q}{\Longrightarrow} R^{n}T(A,\ C),$$

valid without any assumptions.

Now assume that

$$\Gamma \text{ is } K\text{-flat, i.e. } \mathrm{Tor}^{K}_{r}(\Gamma,\ X) = 0$$

for all $r > 0$ and all K-modules X.

We shall prove that we then have the natural isomorphisms

[2] Unadorned $\otimes$ and Hom are always taken over K.

(i) $$\operatorname{Ext}^q_{\Gamma\otimes\Sigma}(\Gamma\otimes B, C) \approx \operatorname{Ext}^q_\Sigma(B, C)$$

(ii) $$\operatorname{Tor}^{\Lambda\otimes\Gamma}_p(A, \Gamma\otimes B) \approx \operatorname{Tor}^\Lambda_p(A, B).$$

Thus the spectral sequences become

(1) $$\operatorname{Ext}^p_{\Lambda\otimes\Gamma}(A, \operatorname{Ext}^q_\Sigma(B, C)) \underset{p}{\Rightarrow} R^n T(A, C)$$

(2) $$\operatorname{Ext}^q_{\Gamma\otimes\Sigma}(\operatorname{Tor}^\Lambda_p(A, B), C) \underset{q}{\Rightarrow} R^n T(A, C).$$

These are the spectral sequences given in [C-E, XVI, § 4] but under the stronger assumption that Γ is K-projective.

Before we establish (i) and (ii) we shall prove a lemma that will also be useful later.

LEMMA 1. *In the situation* $({}_\Lambda B_\Sigma, {}_{\Gamma-\Sigma}C)$ *assume B is Λ-flat and C is $\Gamma\otimes\Sigma$-flat. Then $B\otimes_\Sigma C$ is $\Lambda\otimes\Gamma$-flat.*

Proof. We must show that the functor $T(A) = A\otimes_{\Lambda\otimes\Gamma}(B\otimes_\Sigma C)$ is exact for $A_{\Lambda-\Gamma}$. Since $T(A) = (A\otimes_\Lambda B)\otimes_{\Gamma\otimes\Sigma}C$, T is the composite $T = VU$ of the exact functors $U(A) = A\otimes_\Lambda B$ and $V(A') = A'\otimes_{\Gamma\otimes\Sigma}C$.

Now assume that Γ is K-flat. Applying the lemma to $({}_K\Gamma_K, {}_{\Sigma-K}\Sigma)$ we find that $\Gamma\otimes\Sigma$ is Σ-flat. Consequently, the change of rings $\Sigma\to\Gamma\otimes\Sigma$ implies by [C-E, VI, 4.1.3] the isomorphism (i). Similarly $\Lambda\otimes\Gamma$ is Λ-flat and the change of rings $\Lambda\to\Lambda\otimes\Gamma$ implies by [C-E, V, 4.1.2] the isomorphism (ii).

Quite analogously starting from the associativity rule

$$T(A, C) = A\otimes_{\Lambda\otimes\Gamma}(B\otimes_\Sigma C) = (A\otimes_\Lambda B)\otimes_{\Gamma\otimes\Sigma}C$$

in the situation $(A_{\Lambda-\Gamma}, {}_\Lambda B_\Sigma, {}_{\Gamma-\Sigma}C)$ we obtain the spectral sequences

$$\operatorname{Tor}^{\Lambda\otimes\Gamma}_p(A, \operatorname{Tor}^{\Gamma\otimes\Sigma}_q(\Gamma\otimes B, C)) \underset{p}{\Rightarrow} L_nT(A, C),$$
$$\operatorname{Tor}^{\Gamma\otimes\Sigma}_q(\operatorname{Tor}^{\Lambda\otimes\Gamma}_p(A, B\otimes\Gamma), C) \underset{q}{\Rightarrow} L_nT(A, C).$$

Under the assumption that Γ is K-flat these reduce to

(1a) $$\operatorname{Tor}^{\Lambda\otimes\Gamma}_p(A, \operatorname{Tor}^\Sigma_q(B, C)) \underset{p}{\Rightarrow} L_nT(A, C),$$

(2a) $$\operatorname{Tor}^{\Gamma\otimes\Sigma}_q(\operatorname{Tor}^\Lambda_p(A, B), C) \underset{q}{\Rightarrow} L_nT(A, C).$$

§ 2. Spectral sequences

We apply the spectral sequences (1) and (2) to the K-algebras $(\Gamma, \Gamma^*, \Lambda^*)$ in the situation $(\Gamma_{\Gamma-\Gamma^*}, {}_\Gamma B_{\Lambda^*}, C_{\Lambda^*-\Gamma^*})$, and under the assumption that Γ is K-flat (thus also Γ^* is K-flat). The spectral sequence (2) then collapses and gives

$\mathrm{Ext}^n_{\Gamma^*\otimes\Lambda^*}(B, C) \approx R^nT(A, C)$. The spectral sequence (1) thus becomes

$$\mathrm{Ext}^p_{\Gamma\otimes\Gamma^*}(\Gamma, \mathrm{Ext}^q_{\Lambda^*}(B, C)) \underset{p}{\Longrightarrow} \mathrm{Ext}^n_{\Gamma^*\otimes\Lambda^*}(B, C).$$

The functors Ext are treated here as functors of right modules. Replacing all rings by their opposites and reverting to left modules we obtain

$$\text{(I)} \qquad H^p(\Gamma, \mathrm{Ext}^q_\Lambda(B, C)) \underset{p}{\Longrightarrow} \mathrm{Ext}^n_{\Lambda\otimes\Gamma}(B, C)$$

in the situation $({}_{\Lambda-\Gamma}B, {}_{\Lambda-\Gamma}C)$. This is the *first fundamental spectral sequence,* valid under the assumption that Γ is K-flat.

We now consider the triple of K-algebras (K, Γ^e, Λ^e) and assume that Γ is K-flat. Then Γ^* also is K-flat and Lemma 1 (applied to $\Lambda = \Gamma = \Sigma = K$) shows that $\Gamma^e = \Gamma\otimes\Gamma^*$ is also K-flat. Thus the spectral sequences (1) and (2) may be used. We shall apply them to the situation $(\Gamma_{\Gamma^e}, \Lambda_{\Lambda^e}, A_{\Lambda^e-\Gamma^e})$. Then (2) collapses and gives

$$\mathrm{Ext}^q_{\Gamma^e\otimes\Lambda^e}(\Gamma\otimes\Lambda, A) \approx R^nT(\Gamma, A).$$

Thus the spectral sequence (1) becomes

$$\mathrm{Ext}^p_{\Gamma^e}(\Gamma, \mathrm{Ext}^q_{\Lambda^e}(\Lambda, A)) \underset{p}{\Longrightarrow} \mathrm{Ext}^n_{\Gamma^e\otimes\Lambda^e}(\Gamma\otimes\Lambda, A).$$

Again replacing right Ext by left Ext and replacing $\Gamma\otimes\Lambda$ by $\Lambda\otimes\Gamma$ we obtain the spectral sequence

$$\text{(II)} \qquad H^p(\Gamma, H^q(\Lambda, A)) \underset{p}{\Longrightarrow} H^n(\Lambda\otimes\Gamma, A)$$

valid for any two sided $\Lambda\otimes\Gamma$-module A, under the assumption that Γ is K-flat. This is the *second fundamental spectral sequence.*

PROPOSITION 2. *If Λ and Γ are K-algebras and Γ is K-flat then*

(1) $\mathrm{l.dim}_{\Lambda\otimes\Gamma} B \leqq \dim\Gamma + \mathrm{l.dim}_\Lambda B \qquad ({}_{\Lambda-\Gamma}B)$

(2) $\mathrm{l.inj.dim}_{\Lambda\otimes\Gamma} C \leqq \dim\Gamma + \mathrm{l.inj.dim}_\Lambda C \qquad ({}_{\Lambda-\Gamma}C)$

(3) $\mathrm{l.gl.dim}\,\Lambda\otimes\Gamma \leqq \dim\Gamma + \mathrm{l.gl.dim}\,\Lambda$

(4) $\dim\Lambda\otimes\Gamma \leqq \dim\Gamma + \dim\Lambda.$

If further Γ is K-projective and contains a direct K-summand K' isomorphic with K then[3]

[3] The *finitistic left global dimension,* $\mathrm{f.l.gl.dim}\,\Lambda$, is defined as the supremum of all $\mathrm{l.dim}_\Lambda B$, for left Λ-modules B such that $\mathrm{l.dim}_\Lambda B < \infty$.

(5) $$\text{f. l. gl. dim}\, \Lambda \leqq \text{f. l. gl. dim}\, \Lambda \otimes \Gamma \leqq \dim \Gamma + \text{f. l. gl. dim}\, \Lambda$$

(6) $$\text{l. gl. dim}\, \Lambda \leqq \text{l. gl. dim}\, \Lambda \otimes \Gamma$$

(7) $$\dim \Lambda \leqq \dim \Lambda \otimes \Gamma.$$

Proof. Inequalities (1), (2), (3) follow from (I) while (4) follows from (II) (cf. beginning of § 5).

When Γ is K-projective, we can apply [C-E, VI, 4. 1. 4] and [C-E, VI, 4. 1. 3] to the change of rings $\Lambda \to \Lambda \otimes \Gamma$ to get

(8) $$\text{l. dim}_\Lambda B \leqq \text{l. dim}_{\Lambda \otimes \Gamma} B \qquad ({}_{\Lambda \text{-} \Gamma} B)$$

(9) $$\text{l. dim}_\Lambda B' \geqq \text{l. dim}_{\Lambda \otimes \Gamma} B' \otimes \Gamma \qquad ({}_\Lambda B').$$

Assuming Γ has a K-direct summand isomorphic to K, $B' \otimes \Gamma$ will have a Λ-direct summand isomorphic to B', so that $\text{l. dim}_\Lambda B' \leqq \text{l. dim}_\Lambda B' \otimes \Gamma$. Combining this with (8) applied to $B = B' \otimes \Gamma$ and with (9) we obtain

(10) $$\text{l. dim}_\Lambda B' = \text{l. dim}_{\Lambda \otimes \Gamma} B' \otimes \Gamma \qquad ({}_\Lambda B').$$

From (10) we get (6) and the left inequality in (5). If B is a $\Lambda \otimes \Gamma$-module of finite dimension then (8) shows that B is also finite dimensional over Λ so that the right inequality in (5) follows from (1).

By (10), $\dim \Lambda = \text{l. dim}_{\Lambda^e} \Lambda = \text{l. dim}_{\Lambda^e \otimes \Gamma} \Lambda \otimes \Gamma$ and by (8) $\text{l. dim}_{\Lambda^e \otimes \Gamma} \Lambda \otimes \Gamma \leqq \text{l. dim}_{\Lambda^e \otimes \Gamma^e} \Lambda \otimes \Gamma = \dim \Lambda \otimes \Gamma$. This proves (7) and completes the proof of Proposition 2.

Remark 1. All the hypotheses of Proposition 2 are fulfilled when Γ is K-free or when Γ is a K-projective, supplemented K-algebra.

Remark 2. The inequalities (1)-(4) can also be proved by induction, without using spectral sequences, following the method given in [5, Proposition 3].

Remark 3. Taking $\Lambda = K$ in (3) we find

$$\text{l. gl. dim}\, \Gamma \leqq \dim \Gamma + \text{l. gl. dim}\, K.$$

This generalizes [C-E, IX, 7. 6] where K was assumed semi-simple.

Remark 4. The statement (4) is contained in [C-E, IX, 7. 4] when both Λ and Γ are assumed K-projective.

Remark 5. Assuming Λ semi-simple in (I) we find

$$H^n(\Gamma, \text{Hom}_\Lambda(B, C)) \approx \text{Ext}^n_{\Lambda \otimes \Gamma}(B, C)$$

in the situation $({}_{\Lambda\text{-}\Gamma}B,\ {}_{\Lambda\text{-}\Gamma}C)$ provided Γ is K-flat. This generalizes [C-E, IX, 4.3]. If further, $\dim \Gamma = 0$ then $\Lambda\otimes\Gamma$ is semi-simple.

§ 3. A new spectral sequence

Consider ring homomorphisms

$$\Lambda \xrightarrow{\psi} \Gamma \xrightarrow{\varphi} \Omega$$

(here Λ, Γ and Ω are not assumed to be algebras), and let K be a left Λ-module such that

$$\operatorname{Tor}_r^{\Lambda}(\Gamma, K) = 0 \qquad \text{for} \quad r > 0. \tag{1}$$

Assume further that we are given a left Γ-isomorphism

$$\alpha : \Omega \approx \Gamma \otimes_{\Lambda} K = {}_{(\psi)}K. \tag{2}$$

Apply the change of rings given by φ to the situation $({}_{\Omega}B,\ {}_{\Gamma}\Omega_{\Omega},\ {}_{\Gamma}C)$. There results [C-E, XVI, § 5, case 4] the spectral sequence

$$\operatorname{Ext}_{\Omega}^{p}(B, \operatorname{Ext}_{\Gamma}^{q}(\Omega, C)) \underset{p}{\Longrightarrow} \operatorname{Ext}_{\Gamma}^{n}(B, C). \tag{3}$$

In view of (2) we have

$$\operatorname{Ext}_{\Gamma}^{q}(\Omega, C) \approx \operatorname{Ext}_{\Gamma}^{q}(\Gamma\otimes_{\Lambda}K, C). \tag{4}$$

We now apply the change of rings given by ψ to the situation $({}_{\Lambda}K,\ {}_{\Gamma}\Gamma_{\Lambda},\ {}_{\Gamma}C)$. There results [C-E, XVI, § 5, case 3] the spectral sequence

$$\operatorname{Ext}_{\Gamma}^{q}(\operatorname{Tor}_{p}^{\Lambda}(\Gamma, K), C) \underset{q}{\Longrightarrow} \operatorname{Ext}_{\Lambda}^{n}(K, C)$$

which in view of (1) collapses to

$$\operatorname{Ext}_{\Gamma}^{q}(\Gamma\otimes_{\Lambda}K, C) \approx \operatorname{Ext}_{\Lambda}^{q}(K, C). \tag{5}$$

Combining (4) with (5) and substituting into (3) we obtain the spectral sequence

$$\operatorname{Ext}_{\Omega}^{p}(B, \operatorname{Ext}_{\Lambda}^{q}(K, C)) \underset{p}{\Longrightarrow} \operatorname{Ext}_{\Gamma}^{n}(B, C), \qquad ({}_{\Omega}B,\ {}_{\Gamma}C). \tag{6}$$

The operators of Ω on $\operatorname{Ext}_{\Lambda}^{q}(K, C)$ are defined using (4) and (5) and depend upon α.

As a first application consider a homomorphism

$$\psi : \Lambda \to \Gamma$$

of supplemented K-algebras. Assume that

$$(1') \qquad \mathrm{Tor}_r^{\Lambda}(\Gamma, K) = 0 \qquad \text{for} \quad r > 0$$

which is certainly true if Γ is Λ-flat. Let $I(\Lambda)$ be the kernel of the supplementation $\Lambda \to K$ and assume that the left ideal $\Gamma \cdot I(\Lambda)$ of Γ is two-sided. Define the K-algebra $\Omega = \Gamma / \Gamma \cdot I(\Lambda)$ with its supplementation induced by that of Γ, and let $\varphi : \Gamma \to \Omega$ be the natural map. From the exact sequence $0 \to I(\Lambda) \to \Lambda \to K \to 0$, there follows the exact sequence $\Gamma \otimes_\Lambda I(\Lambda) \to \Gamma \to \Gamma \otimes_\Lambda K \to 0$. Thence results an isomorphism $\Omega \approx \Gamma \otimes_\Lambda K$ of left Γ-modules. Thus the spectral sequence (6) applies, and yields the spectral sequence of [C-E, XVI, Thm. 6.1] from which are derived the Hochschild-Serre spectral sequences for groups and Lie algebras.

The new application of the spectral sequence (6) that interests us here deals with the case of a K-algebra Λ such that K itself is an L-algebra. We shall assume that

$$(1'') \qquad \mathrm{Tor}_r^{K \otimes_L K}(\Lambda \otimes_L \Lambda^*, K) = 0 \qquad \text{for} \quad r > 0$$

or equivalently

$$(1''\mathrm{a}) \qquad H_r^L(K, \Lambda \otimes_L \Lambda^*) = 0 \qquad \text{for} \quad r > 0.$$

The superscript L indicates that we take the homology groups of K regarded as an L-algebra. The condition (1$''$) holds if Λ is K-flat, for by Lemma 1 applied to the situation $({}_K\Lambda_L, {}_{K-L}\Lambda^*)$, with K and L treated as L-algebras, $\Lambda \otimes_L \Lambda^*$ is then $K \otimes_L K$-flat.

We now consider the L-algebra homomorphisms

$$K \otimes_L K \xrightarrow{\psi} \Lambda \otimes_L \Lambda^* \xrightarrow{\varphi} \Lambda \otimes_K \Lambda^*$$

where ψ is induced by the natural map $K \to \Lambda$ while $\varphi(\lambda_1 \otimes_L \lambda_2^*) = \lambda_1 \otimes_K \lambda_2^*$. We define a left $\Lambda \otimes_L \Lambda^*$-isomorphism

$$(2'') \qquad \alpha : \Lambda \otimes_K \Lambda^* \approx (\Lambda \otimes_L \Lambda^*) \otimes_{K \otimes_L K} K$$

by setting

$$\alpha(\lambda_1 \otimes_K \lambda_2^*) = (\lambda_1 \otimes_L \lambda_2^*) \otimes 1.$$

Its inverse β is given by $\beta[(\lambda_1 \otimes_L \lambda_2^*) \otimes k] = \lambda_1 k \otimes_K \lambda_2^*$. Thus all the conditions above are met and the spectral sequence (6) yields

$$(6'') \qquad \mathrm{Ext}^p_{\Lambda \otimes_K \Lambda^*}(B, H^q_L(K, C)) \underset{p}{\Rightarrow} \mathrm{Ext}^n_{\Lambda \otimes_L \Lambda^*}(B, C)$$

in the situation $({}_{\Lambda\otimes_K\Lambda^*}B,\ {}_{\Lambda\otimes_L\Lambda^*}C)$. Taking $B=\Lambda$ and replacing C by A we obtain the *third fundamental spectral sequence*

$$\text{(III)}\qquad H_K^p(\Lambda,\ H_L^q(K,\ A)) \underset{p}{\Longrightarrow} H_L^n(\Lambda,\ A), \qquad ({}_{\Lambda\otimes_L\Lambda^*}A)$$

valid under the assumption (1″), or in particular if Λ is K-flat.

A remark is needed concerning the operators of $\Lambda\otimes_K\Lambda^*$ upon $H_L^q(K,\ A)$. Since $H_L^q(K,\ A)=\mathrm{Ext}^q_{K\otimes_L K}(K,\ A)$ it follows from general principles that $H_L^q(K,\ A)$ will have operators of the same type as $C=\mathrm{Hom}_{K\otimes_L K}(K,\ A)$. If A is a $\Lambda\otimes_L\Lambda^*$-module, there results that C is a $\Lambda\otimes_L\Lambda^*$-module. However the operators of K on C may be derived using either the K-module structure of A or of K. It follows that the left K-operators of C coincide with the right K-operators of C. Thus C is a $\Lambda\otimes_K\Lambda^*$-module. The same follows for $H_L^q(K,\ A)$.

PROPOSITION 3. *Let Λ be a K-algebra satisfying* (1″) *and let K be an L-algebra. Then*[4]

$$L\text{-dim}\,\Lambda \leqq L\text{-dim}\,K + K\text{-dim}\,\Lambda.$$

If further Λ is K-projective and contains a K-direct summand K' isomorphic with K (as a K-module) then

$$L\text{-dim}\,K \leqq L\text{-dim}\,\Lambda$$

(*cf.* [10, Theorem 5]).

Proof. The first inequality follows directly from the spectral sequence (III). To prove the second inequality assume $H_L^n(K,\ A)\neq 0$ for some ${}_{K\otimes_L K}A$. Then $\mathrm{Ext}^n_{K\otimes_L K}(K,\ A)\neq 0$. Since K' is a direct K-summand of Λ it follows also that K' is a $(K\otimes_L K)$-direct summand of Λ and K and K' are $(K\otimes_L K)$-isomorphic. Thus $\mathrm{Ext}^n_{K\otimes_L K}(\Lambda,\ A)\neq 0$. Now consider the change of rings given by $\varphi: K\otimes_L K\to\Lambda\otimes_L\Lambda^*$. Since Λ is K-projective it follows that $\Lambda\otimes_L\Lambda^*$ is $K\otimes_L K$-projective, and thus by [C-E, VI, 4.1.4] we have $\mathrm{Ext}^n_{\Lambda\otimes_L\Lambda^*}(\Lambda,\ {}^{(\varphi)}A)\neq 0$. Consequently $H_L^n(\Lambda,\ {}^{(\varphi)}A)\neq 0$.

PROPOSITION 4. *Let Λ be a K-algebra and let K be an L-algebra. If L-dim $K=0$ then*

$$L\text{-dim}\,\Lambda \leqq K\text{-dim}\,\Lambda$$

[4] L-dim Λ means dim Λ where Λ is considered as an L-algebra, i.e., L-dim Λ $=$ l. dim$_{\Lambda\otimes_L\Lambda^*}\Lambda$.

and for every $\Lambda\otimes_L\Lambda^*$-module A we have

$$H_L^n(\Lambda, A)\approx H_K^n(\Lambda, A')$$

where A' is the submodule of A consisting of all elements a satisfying $ka=ak$ for all $k\in K$. (*Cf.* [8, Theorem 10.1].)

Proof. Since L-dim $K=0$, condition (1″a) holds and the spectral sequence (III) may be applied. It collapses to the isomorphism

$$H_K^n(\Lambda, H_L^0(K, A))\approx H_L^n(\Lambda, A)$$

which is precisely the required result since $H_L^0(K, A)=\mathrm{Hom}_{K\otimes_L K}(K, A)=A'$.

THEOREM 5. *Let Γ be an integral domain which is a K-algebra and let Λ be the field of quotients of Γ, also treated as a K-algebra. Then*

$$H^n(\Lambda, A)\approx H^n(\Gamma, A)$$

for any two-sided Λ-module A. In particular

$$\dim\Lambda\leqq\dim\Gamma.$$

Proof. First we note the natural isomorphism

$$\Lambda\otimes_\Gamma\Lambda\approx\Lambda$$

given by $\frac{\gamma_1}{\gamma_2}\otimes\frac{\gamma_2'}{\gamma_2}\to\frac{\gamma_1\gamma_1'}{\gamma_2\gamma_2'}$. This implies

(i) $$H_\Gamma^r(\Lambda, C)=0 \quad \text{for } r>0$$

(ii) $$H_\Gamma^0(\Lambda, C)=C$$

for any $\Lambda\otimes_\Gamma\Lambda$-module C i.e. for any Λ-module C.

Since Λ is Γ-flat ([C-E, VII, §2, (1)]), we may apply the spectral sequence (III) with (L, K, Λ) replaced by (K, Γ, Λ). In view of (i), this spectral sequence collapses and gives

$$H_\Gamma^0(\Lambda, H_K^n(\Gamma, A))\approx H_K^n(\Lambda, A)$$

which in view of (ii) yields the result.

M. Auslander has proven a similar theorem for more general rings of quotients (unpublished).

§4. Applications

Given a K-algebra Γ, the following properties of Γ may be considered:

(P_1) For every K-algebra Λ

$$\text{f. l. gl. dim } \Lambda \otimes \Gamma = \dim \Gamma + \text{f. l. gl. dim } \Lambda$$

and

$$\text{l. gl. dim } \Lambda \otimes \Gamma = \dim \Gamma + \text{l. gl. dim } \Lambda.$$

(P_2) For every K-algebra Λ

$$\dim \Lambda \otimes \Gamma = \dim \Gamma + \dim \Lambda.$$

(P_3) If K is an L-algebra then

$$L\text{-}\dim \Gamma = K\text{-}\dim \Gamma + L\text{-}\dim K.$$

If Γ is commutative, we may also consider

(P_4) If Λ is a Γ-algebra satisfying

$$H_r^K(\Gamma, \Lambda \otimes_K \Lambda^*) = 0 \qquad \text{for } r > 0$$
$$\Gamma\text{-}\dim \Lambda < \infty$$

then

$$K\text{-}\dim \Lambda = K\text{-}\dim \Gamma + \Gamma\text{-}\dim \Lambda.$$

Note that we have proved that the first condition in (P_4) is satisfied whenever Λ is Γ-flat. Note also that (P_1) implies $\text{l. gl. dim } \Gamma = \dim \Gamma + \text{l. gl. dim } K$. Adding $\text{l. gl. dim } K$ to both sides of the equalities in (P_1) we obtain

$$\text{l. gl. dim } K + \text{f. l. gl. dim } \Lambda \otimes \Gamma = \text{l. gl. dim } \Gamma + \text{f. l. gl. dim } \Lambda$$
$$\text{l. gl. dim } K + \text{l. gl. dim } \Lambda \otimes \Gamma = \text{l. gl. dim } \Gamma + \text{l. gl. dim } \Lambda.$$

THEOREM 6. *Let $\Gamma = K[x_1, \ldots, x_n]$ be the algebra of polynomials in indeterminates $x_1, \ldots, x_n$ with coefficients in K. Then* $\dim \Gamma = n$ *and Γ has properties* (P_1), (P_2), (P_3) *and* (P_4).

THEOREM 7. *Let $\Gamma = K(x_1, \ldots, x_n)$ be the algebra of rational functions in indeterminates $x_1, \ldots, x_n$ with coefficients in the field K. Then* $\dim \Gamma = n$ *and Γ has properties* (P_2), (P_3) *and* (P_4).

THEOREM 8. *Let $\Gamma = T_n(K)$, $n > 1$ be the algebra of all $n \times n$ "triangular" matrices (k_{ij}) with entries in K satisfying $k_{ij} = 0$ for $i < j$. Then* $\dim \Gamma = 1$ *and Γ has properties* (P_1), (P_2) *and* (P_3).

The proofs of Theorems 6, 7, and 8 will be given in the next section.

THEOREM 9. *Let $\Gamma = M_n(K)$ be the algebra of all $n \times n$ matrices with entries in K. Then* $\dim \Gamma = 0$ *and Γ has properties* (P_1), (P_2), *and* (P_3).

Theorem 9 can also be proved in the spirit of section 5 but, since $\dim \Gamma = 0$ [C-E, IX, 7.8] and Γ is K-free, this theorem is already implied by Propositions 2 and 3.

Theorem 9 has already been proved by Harada, [7, Theorems 1 and 2].

Remark 1. For $\Gamma = K(x_1, \dots, x_n)$, property (P_1) actually fails. Indeed Γ being a field we have $\text{l.gl.dim}\, \Gamma = 0$, while (P_1) would imply $\text{l.gl.dim}\, \Gamma = n$.

Remark 2. Note that $\Lambda \otimes K[x_1, \dots, x_n] = \Lambda[x_1, \dots, x_n]$, $\Lambda \otimes T_n(K) = T_n(\Lambda)$ and $\Lambda \otimes M_n(K) = M_n(\Lambda)$. Furthermore, since every ring is an algebra over its own center, Theorems 6, 8 and 9 imply results on $\Lambda[x_1, \dots, x_n]$, $T_n(\Lambda)$ and $M_n(\Lambda)$ for arbitrary rings Λ. For example,

$$\text{l.gl.dim}\, \Lambda[x_1, \dots, x_n] = n + \text{l.gl.dim}\, \Lambda.$$

Remark 3. If K is an L-algebra we may use the device in Remark 2 to write $K[x_1, \dots, x_n] = L[x_1, \dots, x_n] \otimes_L K$, $T_n(K) = T_n(L) \otimes_L K$ and $M_n(K) = M_n(L) \otimes_L K$. Thus in those cases, (P_3) follows from (P_2).

§ 5. Proofs of Theorems 6, 7 and 8

The proofs are based on the "maximum term principle" of spectral sequences. This well known principle may be stated as follows: *Let*

$$E_2^{p,q} \underset{p}{\Longrightarrow} H^n$$

and let (p_0, q_0) be a pair of indices such that

$$E_2^{p,q} = 0 \quad \textit{if} \quad p > p_0 \quad \textit{or} \quad q > q_0. \tag{1}$$

Then we may conclude that

$$H^n = 0 \quad \textit{for} \quad n > p_0 + q_0, \tag{2}$$

$$H^{p_0 + q_0} \approx E_2^{p_0, q_0}. \tag{3}$$

First we note that (1) implies $E_r^{p,q} = 0$ for all $r \geqq 2$, if $p > p_0$ or $q > q_0$. Thus, because of convergence conditions, this also holds for $r = \infty$. Again, because of convergence conditions, (2) holds, and moreover $H^{p_0+q_0} \approx E_\infty^{p_0, q_0}$. For every $r \geqq 2$ the homomorphisms

$$E_r^{p_0 - r, q_0 + r - 1} \xrightarrow{d_r} E_r^{p_0, q_0} \xrightarrow{d_r} E_r^{p_0 + r, q_0 - r + 1}$$

are zero and therefore $E_{r+1}^{p_0, q_0} = E_r^{p_0, q_0}$. Thus $E_2^{p_0, q_0} = E_\infty^{p_0, q_0}$ and everything is proved.

In each of the cases when Γ is $K[x_1, \ldots, x_n]$ or $K(x_1, \ldots, x_n)$ or $T_n(K)$ we shall exhibit a class $\mathfrak{A}$ of special Γ^e-modules with some of the properties (Q), (Q_1)-(Q_4) listed below. More specifically we shall show that

$\Gamma = K[x_1, \ldots, x_n]$ satisfies (Q) with $p = n$, and (Q_1)-(Q_4),
$\Gamma = K(x_1, \ldots, x_n)$ satisfies (Q) with $p = n$, and (Q_2)-(Q_4),
$\Gamma = T_n(K)$ satisfies (Q) with $p = 1$, and (Q_1)-(Q_3).

(Q) $\dim \Gamma = p < \infty$ and $H^p(\Gamma, A) \approx A$ for $A \in \mathfrak{A}$,

(Q_1) If Λ is a K-algebra and $\text{f.l.gl.dim}\, \Lambda = q < \infty$, then there exist $\Lambda \otimes \Gamma$-modules B, C such that $\text{l.dim}_\Lambda B = q$ and

$$0 \neq \text{Ext}_\Lambda^q (B, C) \in \mathfrak{A}.$$

(Q_2) If Λ is a K-algebra and $\dim \Lambda = q < \infty$ then there exists a $(\Lambda \otimes \Gamma)^e$-module A such that

$$0 \neq H^q(\Lambda, A) \in \mathfrak{A}.$$

(Q_3) If K is an L-algebra and L-dim $K = q < \infty$ then there exists a $\Gamma \otimes_L \Gamma^*$-module A such that

$$0 \neq H_L^q(K, A) \in \mathfrak{A}.$$

(Q_4) If Λ is a Γ-algebra satisfying $H_r^K(\Gamma, \Lambda \otimes_K \Lambda^*) = 0$ for $r > 0$ and Γ-dim $\Lambda = q < \infty$, there exists a $\Lambda \otimes_L \Lambda^*$-module $A \in \mathfrak{A}$ such that

$$H_\Gamma^q(\Lambda, A) \neq 0.$$

From (Q) and (Q_i) we prove (P_i). Since each of our Γ's is K-free we may use Propositions 2 and 3 at will.

By Proposition 2 (5), $q = \text{f.l.gl.dim}\, \Lambda \leqq \text{f.l.gl.dim}\, \Lambda \otimes \Gamma$. Thus, if $q = \infty$, the finitistic global dimension statement in (P_1) is true. If $q < \infty$, apply the maximum term principle to the spectral sequence (I) to obtain

$$\text{Ext}_{\Lambda\otimes\Gamma}^{p+q} (B, C) \approx H^p(\Gamma, \text{Ext}_\Lambda^q (B, C)) \approx \text{Ext}_\Lambda^q (B, C) \tag{4}$$

if $\text{Ext}_\Lambda^q (B, C) \in \mathfrak{A}$. Now choose B and C as in (Q_1) to get $\text{l.dim}_{\Lambda\otimes\Gamma} B \geqq p + q$. But $\text{l.dim}_{\Lambda\otimes\Gamma} B \leqq p + q < \infty$ by Proposition 2 (1) so that $\text{f.l.gl.dim}\, \Lambda \otimes \Gamma \geqq \text{l.dim}_{\Lambda\otimes\Gamma} B \geqq p + q$. The converse inequality is Proposition 2 (3).

To prove the second half of (P_1) we note that by Proposition 2, (3), (6) we have

$$\text{l. gl. dim } \Lambda \leqq \text{l. gl. dim } \Lambda \otimes \Gamma \leqq \dim \Gamma + \text{l. gl. dim } \Lambda$$

and this disposes of the case $\text{l.gl.dim } \Lambda = \infty$. If $\text{l.gl.dim } \Lambda < \infty$ then $\text{f.l.gl.dim } \Lambda = \text{l. gl. dim } \Lambda$ and the part of (P_1) already proved shows that

$$\dim \Gamma + \text{l. gl. dim } \Lambda = \text{f. l. gl. dim } \Lambda \otimes \Gamma \leqq \text{l. gl. dim } \Lambda \otimes \Gamma$$

which combined with the inequality above yields the desired equality.

To prove (P_2), write $\dim \Lambda = q$. Once again if $q = \infty$, Proposition 2 (4) implies the conclusion of (P_2). If $q < \infty$, apply the maximum term principle to the spectral sequence (II) to obtain

$$(5) \qquad H^{p+q}(\Lambda \otimes \Gamma, A) \approx H^q(\Lambda, A) \quad \text{if} \quad H^q(\Lambda, A) \in \mathfrak{A}.$$

If A is chosen as in (Q_2), then $\dim \Lambda \otimes \Gamma \geqq p+q$. The converse inequality is Proposition 2 (4).

As for (P_3), Proposition 3 (with $\Lambda = \Gamma$) allows us to restrict our attention to the case $L\text{-dim } K = q < \infty$. Apply the maximum term principle to the spectral sequence (III) to obtain

$$(6) \qquad H_L^{p-q}(\Gamma, A) \approx H_L^q(K, A) \quad \text{if} \quad H_L^q(K, A) \in \mathfrak{A}.$$

Choosing A as in (Q_3) we get $L\text{-dim } \Gamma \geqq p+q$. The converse inequality is in Proposition 3.

In proving (P_4) we are assuming $\Gamma\text{-dim } \Lambda = q < \infty$ and $H_\Gamma^r(\Gamma, \Lambda \otimes_K \Lambda^*) = 0$ for $r > 0$. Apply the maximum term principle to the spectral sequence (III) with L replaced by K and K by Γ to obtain

$$(7) \qquad H_K^{p-q}(\Lambda, A) \approx H_\Gamma^q(\Lambda, A) \quad \text{if} \quad A \in \mathfrak{A}.$$

If A is chosen as in (Q_4) we have $K\text{-dim } \Lambda \geqq p+q$. The converse inequality is in Proposition 3.

We are thus reduced to producing the class $\mathfrak{A}$, proving (Q) and the appropriate (Q_i). We treat theorems 6 and 7 together, writing Γ_n for either $K[x_1, \ldots, x_n]$ (K a commutative ring) or $K(x_1, \ldots, x_n)$ (K a field). Now $\mathfrak{A}$ is to be the class of *symmetric* Γ^e-modules:

$$\mathfrak{A} = \{A \mid \gamma a = a\gamma \quad \text{for all} \quad \gamma \in \Gamma \quad \text{and} \quad a \in A\}.$$

Clearly it suffices to know $x_i a = a x_i$, for $i = 1, \ldots, n$. Here (Q_i) is obvious.

since any $\Lambda\otimes_{\Gamma}\Lambda^{*}$-module is symmetric by definition. Thus (Q) implies (7) and (P_4) directly.

A proof of (Q) for polynomial rings and rational function fields can be extracted from [C-E, VIII, § 4]. In the present situation, however, it is easy to give an inductive proof along the lines of [10, Theorem 6]: If $\Gamma_1 = K[x_1]$ we consider the sequence

$$0 \longrightarrow \Gamma_1\otimes\Gamma_1 \xrightarrow{\varphi} \Gamma_1\otimes\Gamma_1 \xrightarrow{\eta} \Gamma_1 \longrightarrow 0 \tag{8}$$

where φ is multiplication by $z = x_1\otimes 1 - 1\otimes x_1$ and η is the ring homomorphism defined by $\eta(x_1\otimes 1) = \eta(1\otimes x_1) = x_1$. The kernel of η is then the ideal generated by z. To see that the sequence is exact it must be shown that φ is a monomorphism i.e. that z is not a zero divisor. This is clear if we identify $\Gamma_1\otimes\Gamma_1$ with $K[x, y]$ using the identification $x_1\otimes 1\to x$, $1\otimes x_1 \to y$. The sequence (8) being a Γ_1^e-projective resolution of Γ_1 we have $\dim\Gamma_1 \leqq 1$. If A is any symmetric Γ_1^e-module then $\mathrm{Hom}_{\Gamma_1^e}(\Gamma_1, A) = \mathrm{Hom}_{\Gamma_1}(\Gamma_1, A) \approx A$. Further the map $A\to A$ induced by φ is zero. Thus $H^1(\Gamma_1, A)\approx A$.

Essentially the same proof can be used to prove (Q) for $\Gamma_1 = K(x_1)$, or we may use Theorem 5 and the property (Q) for the polynomial ring.

We now assume that Γ_n satisfies (Q) and hence also (7) and (P_4). If we take $\Lambda = \Gamma_{n+1} = \Gamma_n[x_{n+1}]$ (or $\Lambda = \Gamma_n(x_{n+1})$) in (7) and ($P_4$) and use (Q) for $n=1$, then (Q) follows for Γ_{n+1}. This proves (Q).

To prove (Q_1), for $\Gamma = K[x_1, \ldots, x_n]$ only, choose Λ-modules B and C so that $\mathrm{l.dim}_{\Lambda} B = q$ and $\mathrm{Ext}^q_{\Lambda}(B, C) \neq 0$. Convert B and C into $\Lambda\otimes\Gamma$-modules by setting $x_iB = 0$, $x_iC = 0$, $i = 1, \ldots, n$. Then $\mathrm{Ext}^q_{\Lambda}(B, C)$ is symmetric (i.e. is in $\mathfrak{A}$).

For (Q_2), let A be a Λ^e-module with $H^q(\Lambda, A)\neq 0$, $q = \dim\Lambda$. Since Γ is K-free, $H^q(\Lambda, A\otimes\Gamma)\neq 0$ and clearly $A\otimes\Gamma$ and $H^q(\Lambda, A\otimes\Gamma)$ both are symmetric as Γ^e-modules.

For (Q_3) choose a $K\otimes_L K^*$-module A such that $H^q_L(K, A)\neq 0$, $q = L\text{-}\dim K$. Since Γ has a two-sided K-direct summand isomorphic to K, $A\otimes_K\Gamma$ has a $K\otimes_L K^*$-direct summand isomorphic to A. Thus $H^q_L(K, A\otimes_K\Gamma)\neq 0$ and, as two-sided Γ-modules, $A\otimes_K\Gamma$ and $H^q_L(K, A\otimes_K\Gamma)$ are symmeric. This completes the proofs of Theorems 6 and 7.

We now consider $\Gamma = T_n(K)$. Here the class $\mathfrak{A}$ is the set of Γ^e-modules A satisfying

$$e_{ij}A = Ae_{kl} = 0 \quad \text{for} \quad (i, j) \neq (2, 2),\ (k, l) \neq (1, 1)$$

where the e's are matrix units. We deduce that e_{22} acts as identity operator on the left and e_{11} acts as the identity on the right.

We examine the sequence

$$(9) \qquad 0 \longrightarrow X_1 \xrightarrow{\varphi} X_0 \xrightarrow{\eta} \Gamma \longrightarrow 0$$

of Γ^e-modules:

$$\begin{aligned} X_1 &= \sum \Gamma e_{ii} \otimes e^*_{i-1,i-1}\Gamma^* && 1 < i \leqq n \\ X_0 &= \sum \Gamma e_{ii} \otimes e^*_{ii}\Gamma^* && 1 \leqq i \leqq n \\ \varphi(e_{ii} \otimes e^*_{i-1,i-1}) &= e_{ii} \otimes e^*_{i,i-1} - e_{i,i-1} \otimes e^*_{i-1,i-1} \\ \eta(e_{ii} \otimes e^*_{ii}) &= e_{ii}. \end{aligned}$$

Clearly X_0 and X_1 are Γ^e-projective. To see that the sequence is exact we introduce the K-basis

$$c_{k,i,l} = e_{ki} \otimes e^*_{i-1,l} \qquad n \geqq k \geqq i > l \geqq 1$$

for X_1, the K-basis

$$d_{k,i,l} = e_{ki} \otimes e^*_{il} \qquad n \geqq k \geqq i \geqq l \geqq 1$$

for X_0 and the usual K-basis e_{kl}, $n \geqq k \geqq l \geqq 1$ for Γ. Then

$$\varphi(c_{k,i,l}) = d_{k,i,l} - d_{k,i-1,l} \qquad \eta(d_{k,i,l}) = e_{kl}.$$

Exactness is then obvious by fixing k and l. Thus (9) is a Γ^e-projective resolution of Γ and therefore $\dim \Gamma \leqq 1$.

If $A \in \mathfrak{A}$ we obtain an isomorphism $\mathrm{Hom}_{\Gamma^e}(X_1, A) \approx A$ by the correspondence $f \to f(e_{22} \otimes e^*_{11})$. Further the homomorphism $\mathrm{Hom}_{\Gamma^e}(X_0, A) \to \mathrm{Hom}_{\Gamma^e}(X_1, A)$ induced by φ is zero. Consequently,

$$H^1(\Gamma, A) = \mathrm{Ext}^1_{\Gamma^e}(\Gamma, A) = \mathrm{Hom}_{\Gamma^e}(X_1, A) \approx A,$$

for any $A \in \mathfrak{A}$. This proves (Q).

For (Q_1), choose Λ-modules B and C such that $\mathrm{l.dim}_\Lambda B = q$ and $\mathrm{Ext}^q(B, C) \neq 0$. Convert B and C into $\Lambda \otimes \Gamma$-modules by setting $e_{ij}B = 0$, $(i, j) \neq (1, 1)$ and $e_{kl}C = 0$ for $(k, l) \neq (2, 2)$. Then $\mathrm{Ext}^q_\Lambda(B, C) \in \mathfrak{A}$.

For (Q_2) choose a Λ^e-module A with $H^q(\Lambda, A) \neq 0$, $q = \dim \Lambda$ and convert A into a $(\Lambda \otimes \Gamma)^e$-module by setting $e_{ij}Ae_{kl} = 0$ for $(i, j) \neq (2, 2)$, $(k, l) \neq (1, 1)$. Then both A and $H^q(\Lambda, A)$ are in $\mathfrak{A}$.

For (Q_3) choose a $K \otimes_L K^*$-module A with $H_L^q(K, A) \neq 0$, $q = L\text{-}\dim K$. Again convert A into a $\Gamma \otimes_L \Gamma^*$-module by setting $e_{ij} A e_{kl} = 0$ for $(i, j) \neq (2, 2)$, $(k, l) \neq (1, 1)$. Then $H_L^q(K, A) \in \mathfrak{A}$. Thus the proof of Theorem 8 is complete.

If K is a field $\operatorname{l.gl.dim} T_n(K) = \dim T_n(K) = 1$ is already proved in [6].

Remark 1. In proving (Q_1) for $\Gamma = T_n(K)$ and $\Gamma = K[x_1, \ldots, x_n]$ we converted Λ-modules B and C into $\Lambda \otimes \Gamma$-modules ${}_\varepsilon B$, ${}_{\varepsilon'} C$ defined via supplementations $\varepsilon : \Gamma \to K$ and $\varepsilon' : \Gamma \to K$. Similarly in proving (Q_2) we converted (or could have converted) Λ^e-modules A into $(\Lambda \otimes \Gamma)^e$-modules ${}_\varepsilon A_{\varepsilon'}$.

Remark 2. The proof of property (P_1) really shows somewhat more: If A is any $\Lambda[x_1, \ldots, x_n]$-module satisfying $x_i A = 0$, $i = 1, \ldots, n$ (Λ any ring) then

$$\operatorname{l.dim}_{\Lambda[x_1, \ldots, x_n]} A = n + \operatorname{l.dim}_\Lambda A.$$

Similar remarks can be made about $T_n(\Lambda)$ and $M_n(\Lambda)$.

§ 6. Application of products

PROPOSITION 10. *If K is a field, Λ and Γ are any K algebras, and $({}_\Lambda B, {}_\Gamma B')$, we have*

$$\operatorname{l.dim}_\Lambda B + \operatorname{w.l.dim}_\Gamma B' \leqq \operatorname{l.dim}_{\Lambda \otimes \Gamma} B \otimes B' \leqq \operatorname{l.dim}_\Lambda B + \operatorname{l.dim}_\Gamma B' \tag{1}$$

$$\operatorname{l.gl.dim} \Lambda + \operatorname{w.gl.dim} \Gamma \leqq \operatorname{l.gl.dim} \Lambda \otimes \Gamma \leqq \operatorname{l.gl.dim} \Lambda + \dim \Gamma \tag{2}$$

$$\dim \Lambda + \operatorname{w.dim} \Gamma \leqq \dim \Lambda \otimes \Gamma \leqq \dim \Lambda + \dim \Gamma. \tag{3}$$

Proof. The right hand inequalities result from [C-E, XI, 3.2], Proposition 2 (3) and Proposition 2 (4), respectively. If $\operatorname{Ext}_\Lambda^p(B, C) \neq 0$ and $\operatorname{Tor}_q^\Gamma(C', B') \neq 0$ then since K is a field we have

$$\operatorname{Hom}(\operatorname{Tor}_q^\Gamma(C', B'), \operatorname{Ext}_\Lambda^p(B, C)) \neq 0.$$

Thus by [C-E, XI, 3.1] the $\perp$-product shows that

$$\operatorname{Ext}_{\Lambda \otimes \Gamma}^{p+q}(B \otimes B', \operatorname{Hom}(C', C)) \neq 0.$$

Hence $\operatorname{l.dim}_{\Lambda \otimes \Gamma} B \otimes B' \geqq p + q$, proving (1), and also the left hand inequality in (2).

Finally, (3) follows from (1) by replacing Λ, Γ, B, B' by $\Lambda^e, \Gamma^e, \Lambda, \Gamma$.

Remark 1. The analogue of Proposition 10 is equally true for weak dimensions ([C-E, XI, 3.2] no longer applies but may be replaced by the spectral sequences (1a), (2a) of section 1 with $\Sigma = K$). Then (1) and (3) become equalities

$$\text{w.l.dim}_{\Lambda\otimes\Gamma} B\otimes B' = \text{w.l.dim}_{\Lambda} B + \text{w.l.dim}_{\Gamma} B'$$
$$\text{w.dim} \Lambda\otimes\Gamma = \text{w.dim}\,\Lambda + \text{w.dim}\,\Gamma.$$

Remark 2. If in Proposition 10 we know that w.gl.dim $\Gamma = \dim \Gamma$, then (2) becomes

$$\text{l.gl.dim}\,\Lambda\otimes\Gamma = \dim\Gamma + \text{l.gl.dim}\,\Lambda$$

i.e. the second conclusion of (P_1) is valid. This is the case when Γ is semi-primary with (nilpotent) radical N and with $\Sigma = \Gamma/N$ separable and of finite degree over K ([4, Corollary 5 and Proposition 12] and [2, Corollary 9]).

Remark 3. If in Proposition 10 we know that w.dim $\Gamma = \dim\Gamma$ (i.e. that w.l.dim$_{\Gamma^e}\Gamma$ = l.dim$_{\Gamma^e}\Gamma$) then (3) becomes

$$\dim\Lambda\otimes\Gamma = \dim\Lambda + \dim\Gamma$$

i.e. Γ satisfies (P_2). This is the case if Γ^e is either left Noetherian or semi-primary (see [C-E, VI, Exer. 3] and [2, Corollary 8]). With the stronger hypotheses $[\Lambda : K] < \infty$ and $[\Gamma : K] < \infty$ the result was proved in [C-E, IX, 7.4].

Remark 4. In general it is not true that if K is a field

$$\dim\Lambda\otimes\Gamma = \dim\Lambda + \dim\Gamma.$$

For example, let Λ and Γ be locally separable algebras over K with $[\Lambda : K] = [\Gamma : K] = \aleph_0$. Then $\Lambda\otimes\Gamma$ has the same properties, and [10, Theorem 4] implies $\dim\Lambda = \dim\Gamma = \dim\Lambda\otimes\Gamma = 1$.

§ 7. Semi-simple algebras

PROPOSITION 11. *Let Σ be a semi-simple algebra over a field K with $[\Sigma : K] < \infty$ and let Λ be any K-algebra. If in the situation (${}_{\Lambda\text{-}\Sigma}B$, ${}_{\Lambda\text{-}\Sigma}C$) we have*

$$\text{Ext}^m_{\Lambda\otimes\Sigma}(B, C) \neq 0, \qquad \text{Ext}^m_{\Lambda}(B, C) = 0$$

for some $m > 0$ then

$$\text{Ext}^{m+1}_{\Lambda\otimes\Sigma}(B, C) \neq 0.$$

Proof. We begin by noting that Σ must be inseparable. For otherwise $\dim\Sigma = 0$ [8, Theorem 4.1] and so the spectral sequence (I) collapses to

$$\text{Ext}^m_{\Lambda\otimes\Sigma}(B, C) \approx H^0(\Sigma, \text{Ext}^m_{\Lambda}(A, B))$$

contradicting the hypothesis.

The rest of the proof breaks up into three steps. In the first, we assume that Σ is simple and the center Z of Σ is a purely inseparable extension field of K (i.e. there exists an integer f such that $z^{p^f} \in K$ for all $z \in Z$, where $p \neq 0$ is the characteristic of K).

Consider the algebra $\Omega = \Sigma \otimes_Z \Sigma^*$. We have the natural K-algebra epimorphism $\Sigma \otimes_K \Sigma^* = \Sigma^e \to \Omega$ whose kernel M is the two-sided ideal in Σ^e generated by elements $z \otimes 1 - 1 \otimes z^*$ for $z \in Z$. Now, $(z \otimes 1 - 1 \otimes z^*)^{p^f} = z^{p^f} \otimes 1 - 1 \otimes z^{p^f *}$ and since $z^{p^f} \in K$ it follows that $(z \otimes 1 - 1 \otimes z^*)^{p^f} = 0$. Since the elements $z \otimes 1 - 1 \otimes z^*$ generate M and are in the center of Σ^e, it follows that every element of M is nilpotent. Since $[\Sigma^e : K] < \infty$ we have $[Z : K] < \infty$ and therefore M is nilpotent. There is therefore an integer k such that

$$M^k \neq 0, \qquad M^{k+1} = 0.$$

Since $\Omega = \Sigma \otimes_Z \Sigma^*$ is simple [1, Theorem 7.1 F] every left Ω-module D is a direct sum of simple Ω-modules and all simple Ω-modules are isomorphic. But Σ, being a simple ring, is a simple Ω-module. Thus D is a direct sum of copies of Σ as a left Ω-module. Consequently $D \otimes_\Sigma B$ is a direct sum of copies of B as a left $\Lambda \otimes \Sigma$-module. Since the functor Ext converts direct sums (in the first variable) into direct products [C-E, VI, 1.2] it follows that for each q the relations

$$\mathrm{Ext}^q_{\Lambda\otimes\Sigma}(D \otimes_\Sigma B, C) = 0$$
$$\mathrm{Ext}^q_{\Lambda\otimes\Sigma}(B, C) = 0$$

are equivalent if $D \neq 0$. In particular,

(1) $$\mathrm{Ext}^m_{\Lambda\otimes\Sigma}(M^k \otimes_\Sigma B, C) \neq 0.$$

Now assume

$$\mathrm{Ext}^{m+1}_{\Lambda\otimes\Sigma}(B, C) = 0.$$

Then from the above it follows that

$$\mathrm{Ext}^{m+1}_{\Lambda\otimes\Sigma}(M^i/M^{i+1} \otimes_\Sigma B, C) = 0$$

for all $i \geq 0$. Thus from the exact sequence $0 \to M^{i+1} \to M^i \to M^i/M^{i+1} \to 0$ (with $M^0 = \Sigma^e$) it follows that

$$\mathrm{Ext}^m_{\Lambda\otimes\Sigma}(M^i \otimes_\Sigma B, C) \to \mathrm{Ext}^m_{\Lambda\otimes\Sigma}(M^{i+1} \otimes_\Sigma B, C)$$

is an epimorphism. Consequently (1) implies that

$$\mathrm{Ext}^m_{\Lambda\otimes\Sigma}(\Sigma^e\otimes_\Sigma B,\ C) \neq 0.$$

Now, $\Sigma^e\otimes_\Sigma B\approx\Sigma\otimes_K B$. Further, if we consider the change of rings $\varphi : \Lambda\to\Lambda\otimes\Sigma$ we have

$$_{(\varphi)}B = (\Lambda\otimes\Sigma)\otimes_\Lambda B \approx \Sigma\otimes_K B.$$

Since $\Lambda\otimes\Sigma$ is Λ-projective it follows from [C-E, VI, 4.1.3] that $\mathrm{Ext}^m_{\Lambda\otimes\Sigma}(\Sigma\otimes_K B, C) \approx \mathrm{Ext}^m_\Lambda(B, C)$. Thus $\mathrm{Ext}^m_\Lambda(B, C)\neq 0$, a contradiction.

In the second step we assume that Σ is simple. Let Z be the center of Σ and let L be a maximal separable subfield of Z; then Z is a purely inseparable extension of L. Setting $\Lambda'=\Lambda\otimes_K L$ we have

$$\Lambda\otimes_K\Sigma=\Lambda'\otimes_L\Sigma.$$

Since L is separable and $[L : K]<\infty$ we have K-dim $L=0$ and the spectral sequence (I), applied with $\Gamma=L$, yields

$$\mathrm{Ext}^m_{\Lambda'}(B, C)\approx H^0(L,\ \mathrm{Ext}^m_\Lambda(B, C)).$$

Consequently $\mathrm{Ext}^m_{\Lambda'}(B, C)=0$. The conclusion now follows from step one.

The third step reduces the case of Σ semi-simple to the case of Σ simple by a trivial direct product argument.

COROLLARY 12. *If* $\mathrm{Ext}^m_{\Lambda\otimes\Sigma}(B, C)\neq 0$ *for some* $m>\mathrm{l.dim}_\Lambda B$, *then* $\mathrm{Ext}^q_{\Lambda\otimes\Sigma}(B, C)\neq 0$ *for all* $q\geqq m$.

COROLLARY 13. *If* $\mathrm{l.dim}_{\Lambda\otimes\Sigma}B>\mathrm{l.dim}_\Lambda B$ *then* $\mathrm{l.dim}_{\Lambda\otimes\Sigma}B=\infty$.

PROPOSITION 14. *Let Σ be an inseparable semi-simple algebra over a field K with $[\Sigma : K]<\infty$. Then*

$$H^n(\Sigma, \Sigma)\neq 0 \qquad \textit{for all} \quad n>0$$

(*cf.* [9, Theorem 11.1]).

Proof. Since $H^n(\Sigma, \Sigma)=\mathrm{Ext}^n_{\Sigma\otimes\Sigma^*}(\Sigma, \Sigma)$ it suffices, in view of Corollary 12, to prove the conclusion for $n=1$. By [C-E, IX, 5.3] we may assume that Σ is simple. Let Z be the center of Σ. It was proved by Hochschild [8, Lemma 4.1] that $H^1(Z, Z)\neq 0$. Since Z is a field, Σ is free as a left (or right) Z-module. It follows that as a $Z\otimes_K Z^*$-module Σ is isomorphic with a finite direct sum of copies of Z. Thus $H^1(Z, \Sigma)\neq 0$. We now consider the change of rings given by $\varphi : Z^e\to\Sigma^e$. Since Σ^e is Z^e-free we have [C-E, VI, 4.1.3]

$$0 \neq H^1(Z, \Sigma) = \mathrm{Ext}^1_{Z^e}(Z, \Sigma) \approx \mathrm{Ext}^1_{\Sigma^e}(\Sigma^e \otimes_{Z^e} Z, \Sigma).$$

Now

$$\Sigma^e \otimes_{Z^e} Z = (\Sigma \otimes \Sigma^*) \otimes_{Z \otimes Z} Z \approx (\Sigma \otimes_Z \Sigma^*) \otimes_Z Z \approx \Sigma \otimes_Z \Sigma^*.$$

We have seen in the first step of the proof of Proposition 11 that as a $\Sigma \otimes_Z \Sigma^*$-module $\Sigma \otimes_Z \Sigma^*$ is isomorphic with a direct sum of copies of Σ. A fortiori, this is true if we regard $\Sigma \otimes_Z \Sigma^*$ and Σ as Σ^e-modules. Consequently $\mathrm{Ext}^1_{\Sigma^e}(\Sigma, \Sigma) \neq 0$ as required.

§ 8. Semi-primary algebras

THEOREM 15. *Let Γ denote a semi-primary algebra over a field K, with (nilpotent) radical N and $\Sigma = \Gamma/N$ (semi-simple and) finite over K. If we assume*

$$\dim \Gamma > p = \mathrm{gl.}\dim \Gamma^{5)}$$

then there exists a two-sided Σ-module A such that

$$H^{p+1}(\Gamma, A) \neq 0.$$

For any such module A we have

$$H^q(\Gamma, A) \neq 0 \qquad \textit{for all} \quad q > p.$$

In particular, $\dim \Gamma = \infty$.

Proof. There exists a two-sided Γ-module A with $H^{p+1}(\Gamma, A) \neq 0$. From the consideration of the exact sequences $0 \to N^{i+1}A \to N^iA \to N^iA/N^{i+1}A \to 0$ it follows readily that $H^{p+1}(\Gamma, A') \neq 0$ for $A' = N^iA/N^{i+1}A$ for some $i \geqq 0$, (cf. [4, Proposition 3]). Applying the same argument to the right operators we find $H^{p+1}(\Gamma, A'') \neq 0$ for

$$A'' = A'N^j/A'N^{j+1} = N^iAN^j/(N^{i+1}AN^j + N^iAN^{j+1})$$

for some $i > 0$.

For any two-sided Σ-module A we have [4, Proposition 8]

$$H^q(\Gamma, A) \approx \mathrm{Ext}^q_{\Gamma \otimes \Sigma^*}(\Sigma, A).$$

Thus it follows from Corollary 12 that the relation $H^{p+1}(\Gamma, A) \neq 0$ implies $H^q(\Gamma, A) \neq 0$ for all $q > p$.

The last part of Theorem 15 appears already in [3].

[5] If Γ is semi-primary $\mathrm{l.gl.}\dim \Gamma = \mathrm{r.gl.}\dim \Gamma = \mathrm{w.gl.}\dim \Gamma$ [2, Corollary 9]. We use the notation $\mathrm{gl.}\dim \Gamma$ for this common number.

THEOREM 16. *Let K be a field, Λ any K-algebra, and Γ a semi-primary K-algebra with radical N and with $\Sigma = \Gamma/N$ finite over K. Then*

$$\text{f.l.gl.dim}\, \Lambda \otimes \Gamma = \text{f.l.gl.dim}\, \Lambda + \text{f.l.gl.dim}\, \Gamma.$$

Proof. In the situation $({}_\Gamma B')$, we have $\text{w.l.dim}_\Gamma B' = \text{l.dim}_\Gamma B'$ [2, Corollary 8]. Thus Proposition 10 (1) becomes an equality and shows

$$\text{f.l.gl.dim}\, \Lambda \otimes \Gamma \geqq \text{f.l.gl.dim}\, \Lambda + \text{f.l.gl.dim}\, \Gamma.$$

Before we prove the reverse inequality we pause for a lemma.

LEMMA 17. *Let $\varphi : \Xi \to \Omega$ be an arbitrary ring homomorphism, and in the situation $({}_\Xi B)$ assume that*

$$\text{Tor}_i^\Xi (\Omega, B) = 0 \qquad i > p.$$

Then there exists a left Ω-module B' such that

$$\text{Ext}_\Xi^m (B, C) \approx \text{Ext}_\Omega^{m-p} (B', C)$$

for all $m > p$ and all left Ω-modules C.

Proof. Consider an exact sequence of Ξ-modules

$$0 \to D \to X_{p-1} \to \dots \to X_0 \to B \to 0$$

with X_j projective for $j = 0, 1, \dots, p-1$. The iterated connecting homomorphism then yields isomorphisms

$$\begin{aligned} &(1) \qquad &\text{Ext}_\Xi^m (B, C) &\approx \text{Ext}_\Xi^{m-p} (D, C) \qquad &m > p \\ & &\text{Tor}_m^\Xi (\Omega, B) &\approx \text{Tor}_{m-p}^\Xi (\Omega, D) \qquad &m > p. \end{aligned}$$

Consequently $\text{Tor}_r^\Xi (\Omega, D) = 0$ for $r > 0$. Thus by [C-E, VI, 4.1.3] we have

$$\text{Ext}_\Xi^{m-p} (D, C) \approx \text{Ext}_\Omega^{m-p} ({}_{(\varphi)}D, C)$$

which combined with (1) yields the conclusion with $B' = {}_{(\varphi)}D$.

We now return to the proof of Theorem 16. Let $\text{l.dim}_{\Lambda\otimes\Gamma} B = m < \infty$. Let $p = \text{l.dim}_\Gamma B$. By inequality (8) of section 2 we have $p \leqq m$ and thus $p < \infty$. It follows that $\text{Tor}_i^\Gamma(\Sigma, B) = 0$ if $i > p$. Since K is a field [C-E, VI, 4.1.1] shows that $\text{Tor}_i^\Gamma (\Sigma, B) \approx \text{Tor}_i^{\Lambda\otimes\Gamma} (\Lambda \otimes \Sigma, B)$, so that we may apply Lemma 17 with $\Xi = \Lambda \otimes \Gamma$ and $\Omega = \Lambda \otimes \Sigma$ to obtain a $\Lambda \otimes \Sigma$-module B' such that

$$\text{Ext}_{\Lambda\otimes\Sigma}^{i-p} (B', C) \approx \text{Ext}_{\Lambda\otimes\Gamma}^i (B, C) \qquad ({}_{\Lambda\otimes\Sigma}C)$$

for all $i > p$.

This implies

$$p + \mathrm{l.dim}_{\Lambda\otimes\Sigma} B' \leqq \mathrm{l.dim}_{\Lambda\otimes\Gamma} B.$$

If $p + \mathrm{l.dim}_{\Lambda\otimes\Sigma} B' < s < \infty$ then it follows that $\mathrm{Ext}^s_{\Lambda\otimes\Gamma}(B, C) = 0$ for all $({}_{\Lambda\otimes\Sigma}C)$. Since the kernel of $\Lambda\otimes\Gamma \to \Lambda\otimes\Sigma$ is nilpotent it follows (cf. [4, Proposition 3]) that $\mathrm{Ext}^s_{\Lambda\otimes\Gamma}(B, C) = 0$ for all $({}_{\Lambda\otimes\Gamma}C)$. Thus $\mathrm{l.dim}_{\Lambda\otimes\Gamma} B < s$. We have thus proved that

$$p + \mathrm{l.dim}_{\Lambda\otimes\Sigma} B' = \mathrm{l.dim}_{\Lambda\otimes\Gamma} B.$$

Since $\mathrm{l.dim}_{\Lambda\otimes\Sigma} B' < \infty$ it follows from Corollary 13 that $\mathrm{l.dim}_{\Lambda\otimes\Sigma} B' = \mathrm{l.dim}_{\Lambda} B'$. Thus we have

$$p + \mathrm{l.dim}_{\Lambda} B' = \mathrm{l.dim}_{\Lambda\otimes\Gamma} B.$$

This implies

$$p + \mathrm{f.l.gl.dim}\, \Lambda \geqq \mathrm{f.l.gl.dim}\, \Lambda\otimes\Gamma$$

Since $p = \mathrm{l.dim}_{\Gamma} B < \infty$ we have $p \leqq \mathrm{f.l.gl.dim}\, \Gamma$. Thus

$$\mathrm{f.l.gl.dim}\, \Gamma + \mathrm{f.l.gl.dim}\, \Lambda \geqq \mathrm{f.l.gl.dim}\, \Lambda\otimes\Gamma$$

as desired.

COROLLARY 18. *Under the hypotheses of Theorem* 16,

$$\mathrm{l.gl.dim}\Lambda\otimes\Gamma = \mathrm{l.gl.dim}\, \Lambda + \mathrm{gl.dim}\, \Gamma$$

or

$$\mathrm{l.gl.dim}\, \Lambda\otimes\Gamma = \infty.$$

If Σ is separable the first alternative holds.

Proof. If either $\mathrm{l.gl.dim}\, \Lambda$ or $\mathrm{gl.dim}\, \Gamma$ is infinite, Proposition 2 (6) shows that $\mathrm{l.gl.dim}\, \Lambda\otimes\Gamma$ is, too. In the contrary case $\mathrm{l.gl.dim}\, \Lambda$ and $\mathrm{gl.dim}\, \Gamma$ are equal to $\mathrm{f.l.dim}\, \Lambda$ and $\mathrm{f.l.gl.dim}\, \Gamma$ respectively, which proves the first assertion of the Corollary.

If Σ is separable, and $\mathrm{gl.dim}\, \Gamma$ and $\mathrm{l.gl.dim}\, \Lambda$ are finite, then $\dim \Gamma = \mathrm{gl.dim}\, \Gamma$ [4, Corollary 5 and Proposition 12] and so $\mathrm{l.gl.dim}\, \Lambda\otimes\Gamma < \infty$, by Proposition 2 (3). Thus all three finitistic dimensions are equal to the corresponding global dimensions.

Remark 1. For a result overlapping Corollary 18 see [3, Theorem 2].

Remark 2. If $[\Sigma : K] = \infty$ the conclusions of Theorem 16 and Corollary 18 may fail. For example, let $\Lambda = \Gamma = K(x_1, \ldots, x_n)$ so that $\mathrm{gl.dim}\, \Lambda = \mathrm{gl.dim}\, \Gamma = 0$ but $\mathrm{l.gl.dim}\, \Lambda\otimes\Gamma = \dim \Gamma = n$ [C-E, IX, 7.5].

Remark 3. Under the hypotheses of Theorem 16, the ring Γ^e is semi-primary. Indeed the kernel of $\Gamma^e \to \Sigma^e$ is nilpotent, and Σ^e being of finite degree over K is semi-primary. It therefore follows from Remark 3 of section 6 that $\dim \Lambda \otimes \Gamma = \dim \Lambda + \dim \Gamma$.

Remark 4. If under the hypotheses of Theorem 16 we assume that $\text{gl.}\dim \Gamma$ is finite, a similar argument can prove several extensions of Proposition 11. For example, let $\text{l.}\dim_\Lambda B < \infty$. Then

$$\text{Ext}^m_{\Lambda\otimes\Gamma}(B, C) \neq 0 \quad \text{implies} \quad \text{Ext}^{m-1}_{\Lambda\otimes\Gamma}(B, C) \neq 0$$

if either

$$NC = 0 \quad \text{and} \quad m > \text{f.l.gl.}\dim \Lambda + \text{gl.}\dim \Gamma$$

or

$$NB = 0 \quad \text{and} \quad m > \text{l.}\dim_\Lambda B + \text{gl.}\dim \Gamma.$$

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