DIFFERENTIABLE STRUCTURES ON THE 15-SPHERE AND PONTRJAGIN CLASSES OF CERTAIN MANIFOLDS

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Introduction. No manifold had been known which can carry two distinct differentiable structures until the recent important contribution due to J. Milnor [7] concerning the 7-sphere appeared.

In connection with his work, there are several problems, for example, about the existence of any other manifold with such property, about the topological invariance of the Pontrjagin classes of manifolds, etc.; some of them will be discussed in the present note.

First in § 1 and § 2, it will be shown that his method is applicable also for the case of 15-sphere to prove existence of many distinct differentiable structures. Secondly in § 3 we shall give some examples of differentiable manifolds which are all of the same homotopy type while any homotopy equivalence between them does not preserve their Pontrjagin classes.*) In addition we shall obtain the following result. Consider $2n$-manifolds X^{2n} whose homology groups $H_i(X^{2n}) = Z$ for $i = 0, n, 2n$ and $H_i(X^{2n}) = 0$ for $i \neq 0, n, 2n$. Known examples are the following: complex projective plane $(n = 2)$, quaternion projective plane $(n = 4)$ and Cayley projective plane $(n = 8)$. We shall show in § 4 that for $n = 4$ and 8 there exist several examples of such topological (triangulable) $2n$-manifolds with different homotopy types.

All manifolds considered in this note, with or without boundary, are to be differentiable of class C^∞ (unless otherwise stated) and orientable.

§ 1. Invariant $\lambda(M^{15})$

For every closed, oriented 15-manifold M^{15} satisfying the hypothesis

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*) After completing this note, I had an opportunity to notice Thom's remark in [16] and to read Dold's paper [15]. I understand that Dold has given already such examples. But I should like to preserve the original style of the present note, since it stands on a different view point. Cf. James and Whitehead [17], also Tamura [18].

(#) a) there exists a 16-manifold $\overline{B}^{16}$ with M^{15} as its boundary,[1]
and
b) $H^i(M^{15})=0$[2] for $i=3, 4, 7, 8, 11, 12$,

we will define a residue class $\lambda(M^{15})$ of integers modulo 381. This is to be an invariant of differentiable structure for such a 15-manifold, and will be defined as a function of the index τ and the Pontrjagin classes p_i of the open submanifold $B^{16}=\overline{B}^{16}-M^{15}$ of $\overline{B}^{16}$ with the induced differentiable structure. An orientation of B^{16} and that of $\overline{B}^{16}$ are chosen in such a way that they are consistent and the homological boundary of thereby oriented $\overline{B}^{16}$ is equal to the standard fundamental cycle of M^{15}. Then the *index* means that of the quadratic form defined by the cup-product over the group $H^8_*(B^{16}, R)$ with real coefficients (H_* means cohomology group with compact supports).

The hypothesis (#) implies that the inclusion homomorphism

$$\eta : H^{4i}(\overline{B}^{16}, M^{15}) \to H^{4i}(\overline{B}^{16})$$

is an isomorphism for $i=1, 2, 3$. This permits us to consider the i-th Pontrjagin class p_i of the manifold B^{16} as an element of $H^{4i}_*(B^{16})$ for $i=1, 2, 3$ (cf. [2]).

Let ν be the standard generator of $H^{16}_*(B^{16})$ which is dual to the orientation of B^{16}. Then $\lambda(M^{15})$ is defined by the following equation[3]

$$\lambda(M^{15})\nu = 3^4\cdot 5^2\cdot 7\tau\nu + 71p_3p_1 + 19p_2^2 - 22p_2p_1^2 + 3p_1^4 \pmod{381}.$$

THEOREM 1. *The residue class $\lambda(M^{15})$ modulo 381 does not depend on the choice of the manifold $\overline{B}^{16}$.*

Let $\overline{B}_1^{16}$, $\overline{B}_2^{16}$ be two manifolds with boundary M^{15}. Then $C^{16}=\overline{B}_1^{16}\cup\overline{B}_1^{16}$ is a closed 16-manifold which possesses a differentiable structure compatible with that of $\overline{B}_1^{16}$ and $\overline{B}_2^{16}$. Choose that orientation of C^{16} to be consistent with the orientation of $\overline{B}_1^{16}$ (and therefore consistent with the negative orientation of $\overline{B}_2^{16}$). Then the proof of Theorem 1 will be proceeded similarly as in the case of the invariant $\lambda(M^7)$ (See Milnor [7]) by making use of the Hirzebruch's index formula [5]:

$$3^4\cdot 5^2\cdot 7\tau(C^{16})\nu = 381p_4 - 71p_3p_1 - 19p_2^2 + 22p_2p_1^2 - 3p_1^4,$$

1) A 15-manifold is not always the boundary of a 16-manifold. See Dold [3].
2) Integer coefficients are to be understood.
3) As Milnor remarked, for every $n=4k-1$ a residue class $\lambda(M^n)$ modulo $s_k\cdot\mu(L_k)$ could be defined similarly. (See [5], p. 14.)

where ν is the standard generator of $H^{16}(C^{16})$ and p_i's are the Pontrjagin classes of C^{16}. Therefore the proof will be omitted here.

The following property of the invariant λ is clear.

LEMMA 1. *If the orientation of M^{15} is reversed, then $\lambda(M^{15})$ is multiplied by* -1.

As a consequence we have

COROLLARY 1. *If $\lambda(M^{15}) \neq 0$, then M^{15} possesses no orientation-reversing diffeomorphism*[4] *onto itself.*

§ 2. Examples of 15-manifolds

Consider 7-sphere bundles over the 8-sphere with the rotation group $SO(8)$ as structural group. The equivalence classes of such bundles are in one-one correspondence[5] with elements of the 7th homotopy group $\pi_7(SO(8))$ of the stractural group. This homotopy group is known to be isomorphic to $Z+Z$, and a specific isomorphism between these groups is obtained as follows.[6] For each $(h, j) \in Z+Z$, let $f_{h,j} : S^7 \to SO(8)$ be defined by $f_{h,j}(u) \cdot v = u^h v u^j$ for $v \in R^8$. Cayley number multiplication is understood on the right.[7]

Let ι be the standard generator for $H^8(S^8)$ and denote by $\xi_{h,j}$ the sphere bundle corresponding to $\{f_{h,j}\} \in \pi_7(SO(8))$.

LEMMA 2. *The Pontrjagin class $p_2(\xi_{h,j})$ equals $\pm 6(h-j)\iota$.*

(The proof will be given at the last of this section.)

For each odd integer k, let M_k^{15} be the total space of the bundle $\xi_{h,j}$, where h and j are determined by the equation $h+j=1$, $h-j=k$. This manifold M_k^{15} has a natural differentiable structure and orientation, which will be described as follows.

Let the base space S^8 be imbedded in R^9 by the equation

$$|s|^2 + \left(\sigma - \frac{1}{2}\right)^2 = \frac{1}{4} \text{ (or } |s|^2 = \sigma(1-\sigma)), \qquad (0 \leqq \sigma \leqq 1),$$

4) A diffeomorphism f is a homeomorphism such that both f and f^{-1} are differentiable.

5) See [10] § 18.

6) See [9]. By making use of the fibration of Spin (7) by G_2 over S^7, it can be proved that $\{f_{1,-1}\}$ generates $\pi_7(SO(7))$. See Toda, Saito and Yokota [19].

7) The division algebra of Cayley numbers is not associative, but it is known that any subalgebra generated by two elements is associative. Cf. Dikson, Linear Algebras, Cambridge Tract, 1914.

where s, $|s|$ and σ denotes a Cayley number, its norm and a real number respectively, and (s, σ) forms a coordinate system of R^9. In S^8 let V_1, V_0 be the complements of $(0, 0)$ and $(0, 1)$ respectively. Consider two spaces $V_1 \times S^7$ and $V_0 \times S^7$, and identify the two copies of subset $(V_1 \cap V_0) \times S^7$ under the diffeomorphism

$$(s, \sigma\ ;\ t)_1 \to (s, \sigma\ ;\ t')_0, \qquad t' = s^h t s^j / |s|$$

(using Cayley multiplication). The constructed space can be considered as M_k^{15} and has the natural differentiable structure.

Now define a function $f : M_k^{15} \to R$ by

$$(s, \sigma\ ;\ t)_1 \to \sqrt{\sigma}\,\Re(t),$$
$$(s, \sigma\ ;\ t')_0 \to \Re(\bar{s}t')/\sqrt{1-\sigma},$$

where $\Re(t)$ denotes the real part of t and $\bar{s}$ denotes the conjugate of s. It is easily verified that f has only two critical points (namely $(0, 1\ ;\ \pm 1)_1$) and that these are non-degenerate. Thus the manifold M_k^{15} satisfies the condition (H) stated in §2 of the paper [7], and therefore by Theorem 2 in [7] (cf. also [8]) we obtain

LEMMA 3. *The manifold M_k^{15} is homeomorphic to the 15-sphere S^{15}.*

Associated with each 7-sphere bundle $M_k^{15} \to S^8$, there is an 8-cell bundle $\rho_k : \overline{B}_k^{16} \to S^8$. The total space $\overline{B}_k^{16}$ of this bundle is a differentiable manifold with boundary M_k^{15}. The cohomology group $H_*^8(B_k^{16})$ is generated by the element $\alpha = \rho_k^*(\iota)$, where ι denotes the standard generator of $H^8(S^8)$. Choose orientation for M_k^{15} and B_k^{16} so that the index $\tau(B_k^{16})$ will be $+1$.

The tangent bundle of B_k^{16} is the Whitney sum of (1) the bundle of vectors tangent to the fibre, and (2) the bundle of vectors normal to the fibre. The first bundle (1) is induced (under ρ_k) from $\xi_{h,j}$, and therefore has the Pontrjagin class $p_2 = \rho_k^*(\pm 6(h-j)\iota) = \pm 6k\alpha$. The second is induced from the tangent bundle of S^8, and therefore has second Pontrjagin class zero. Thus we have $p_2(B_k^{16}) = \pm 6k\alpha$.

This and Lemma 3 give

LEMMA 4. *The invariant $\lambda(M_k^{15})$ is the residue class mod* 381 *of* $78(1-k^2)$.

Combining the above lemmas we have:

THEOREM 2. *For $k^2 \neq l^2$ mod 127 the manifolds M_k^{15} and M_l^{15} are homeo-*

morphic but not diffeomorphic.

(For $k = \pm 1$ the manifold M_k^{15} is diffeomorphic to S^{15}; but it is not known whether this is true for any other k.)

COROLLARY 2. *There exist such differentiable structures on S^{15} that cannot be extended throughout R^{16}.*

Proof of Lemma 2. It is clear that the Pontrjagin class $p_2(\xi_{h,j})$ is a linear function of h and j. Furthermore it is known that it is independent of the orientation of the fibre. But if the orientation of S^7 is reversed (for example replace t by $\bar{t}$), then $\xi_{h,j}$ is replaced by $\xi_{-j,-h}$. This shows that $p_2(\xi_{h,j})$ is given by an expression of the form $c(h-j)\iota$. Here c is a constant determined by $c\cdot\iota = p_2(\xi_{1,0})$ (and therefore $c\cdot\alpha = p_2(B_1^{16})$). In order to evaluate the constant c, we will note that the maniford B_1^{16} is diffeomorphic to the Cayley projective plane Π with a 16-cell removed.[8] The Pontrjagin class $p_2(\Pi)$ is known to be six times of a generator of $H^8(\Pi)$ (See Hirzebruch's announcement in [6], also Borel and Hirzebruch [1]). Therefore the constant c must be ± 6. This proves Lemma 2.

§ 3. Certain types of 16-manifolds

Some examples of 16-(respectively 8-)manifolds of the same homotopy type will be constructed and it will be shown that any homotopy equivalence between them does not preserve their second (respectively first) Pontrjagin classes. These can be done by parallel methods for the respective cases, and therefore we shall treat here mainly the case of the 16-manifolds.

Associated with each of the 7-sphere bundles M_k^{15}, there is an 8-sphere bundle whose total space $\tilde{B}_k^{16}$ is a closed 16-manifold. These 16-manifolds $\tilde{B}_k^{16}$ will serve as the examples mentioned above.

Consider, in general, 8-sphere bundles over the 8-sphere with rotation group $SO(9)$ as structural group. The equivalence classes of such bundles are in one-one correspondence with elements of $\pi_7(SO(9))$. This group is known to be isomorphic[6] to Z. Let $i : SO(8) \to SO(9)$ be a natural injection map, then the induced homomorphism $i_* : \pi_7(SO(8)) \to \pi_7(SO(9))$ is onto and the kernel of i_* is generated by $\{f_{1,1}\}$ (in the notation of § 2, cf. [10] § 23).

[8] This fact is proved by using the expression by matrices of points of Π. See [4].

Let $\tilde{f}_{h,j} : S^7 \to SO(9)$ be defined by $\tilde{f}_{h,j} = i \circ f_{h,j}$, then $\{\tilde{f}_{1,0}\}$ is a generator of $\pi_7(SO(9))$ and we have $\{\tilde{f}_{h,j}\} = (h-j)\{\tilde{f}_{1,0}\}$. Denote by $\eta_{h,j}$ the 8-sphere bundle corresponding to $\{\tilde{f}_{h,j}\}$. Since the structural group of $\eta_{h,j}$ is reduced to $SO(8)$, we have $p_2(\eta_{h,j}) = p_2(\xi_{h,j})$.

For each odd integer k let $\tilde{B}_k^{16}$ be the total space of the bundle $\eta_{h,j}$, where h and j are determined by the equation $h+j=1$, $h-j=k$. This manifold has a natural differentiable structure and orientation, which will be described as follows. Let (s, σ), (t, τ) with $|s|^2 = \sigma(1-\sigma)$, $|t|^2 = \tau(1-\tau)$ be the coordinates of the base S^3 and the fibre S^8 respectively (See §2). Consider two spaces $V_1 \times S^8$ and $V_0 \times S^8$, and identify the two copies of subset $(V_1 \cap V_0) \times S^8$ under the diffeomorphism

$$(s, \sigma\ ;\ t, \tau)_1 \to (s, \sigma\ ;\ t', \tau')_0, \qquad t' = s^h t s^j/|s|, \qquad \tau' = \tau.$$

The constructed space is considered as $\tilde{B}_k^{16}$ and has the natural differentiable structure. There are two natrual cross sections $(s, \sigma\ ;\ 0, 0)$, $(s, \sigma\ ;\ 0, 1)$. The part $\left(\tau \leqq \frac{1}{2}\right)$ and the part $\left(\tau \geqq \frac{1}{2}\right)$ of the manifold $\tilde{B}_k^{16}$ are just regarded as two copies of $\overline{B}_k^{16}$ previously constructed.

LEMMA 5. *The manifold $\tilde{B}_k^{16}$ is considered as the sum $\overline{B}_k^{16} \cup \overline{B}_k^{16}$ of two copies of $\overline{B}_k^{16}$ with identification of the corresponding points on their boundaries M_k^{15}. The differentiable structure is compatible with that of each $\overline{B}_k^{16}$. An orientation of $\tilde{B}_k^{16}$ is consistent with that of the one of $\overline{B}_k^{16}$ and consistent with the negative orientation of the other $\overline{B}_k^{16}$.*

Let $\eta_0, \eta_1 : \overline{B}_k^{16} \to \tilde{B}_k^{16}$ be the above inclusion maps, then there are natrual injection homomorphisms $\eta_i^* : H_*^8(\overline{B}_k^{16}) \to H^8(\tilde{B}_k^{16})$, $i = 0, 1$. It is easy to see that $p_2(\tilde{B}_k^{16}) = \eta_0^*(p_2(\overline{B}_k^{16})) + \eta_1^*(p_2(\overline{B}_k^{16}))$. It follows from Lemma 2:

LEMMA 6. $p_2(\tilde{B}_k^{16}) = 6k\alpha_0 + 6k\alpha_1$, *where $\alpha_i = \eta_i^*(\alpha)$ are generators of $H^8(\tilde{B}_k^{16})$.*

We shall prove the following theorem in the next section:

THEOREM 3. *The manifolds $\tilde{B}_k^{16}$ and $\tilde{B}_l^{16}$ have the same homotopy type if and only if $k \equiv \pm l \mod 240$.***)

From this theorem and Lemma 6 we have

**) Cf. [15], [17].

THEOREM 4. *The second Pontrjagin class of a* (16-)*manifold is not, in general, a homotopical invariant.****)

As for the first Pontrjagin class, we can construct 8-manifolds $\tilde{B}_k^8$ similarly as $\tilde{B}_k^{16}$ which are 4-sphere bundles over the 4-sphere associated with the 3-sphere bundles M_k^7 which was treated in Milnor's paper [7]. We can obtain similarly (see the next section).

THEOREM 3′. *The manifolds $\tilde{B}_k^8$ and $\tilde{B}_l^8$ have the same homotopy type if and only if $k \equiv \pm l$ mod 24.***)

THEOREM 4′. *The first Pontrjagin class of an* (8-)*manifold is not, in general, a homotopical invariant.****)

COROLLARY 3. *Either* (*a*) *the Hurewicz's conjecture*[9] *is negative for the above cases, or* (*b*) *the first Pontrjagin class of a closed* 8-*manifold and the second Pontrjagin class of a closed* 16-*manifold are not topological invariants.*

§ 4. Homotopy types of the manifolds $\tilde{B}_k^{16}$.

In this section we shall prove Theorem 3 (and Theorem 3′), and give also an interesting side-result.

We need some preparation. Let (x, y) denote the coordinate system of R^{16} and let (s, σ) denote that of R^9, where x, y, s are Cayley numbers and σ a real number. The 15-sphere S^{15} in R^{16} is defined by the equation $|x|^2+|y|^2=1$, and S^8 in R^9 by the equation $|s|^2=\sigma(1-\sigma)$ as above.

Consider the map $\tilde{g}_{h,j} : S^{15} \to S^8$ for any pair of integers h, j which is defined by $\tilde{g}_{h,j}(x, y) = (|x|^{1-h-j} x^h \bar{y} x^j, |y|^2)$. Let the map $g_{h,j} : S^7 \times S^7 \to S^7$ be defined by $g_{h,j}(u, v) = f_{h,j}(u)\bar{v}$ (as for $f_{h,j}$ see § 2), then $\tilde{g}_{h,j}$ is no other than the so-called Hopf construction of $g_{h,j}$. The J-homomorphism: $\pi_7(SO(8)) \to \pi_{15}(S^8)$ in the sense of G. W. Whitehead [12] is known to be onto in this case and maps $\{f_{h,j}\}$ to $-\{\tilde{g}_{h,j}\}$. It follows easily

LEMMA 7. *$\tilde{g}_{h,j}$ represents the element $(h+j)\sigma_8 - j \cdot E(\tau_7)$ of $\pi_{15}(S^8)$, where σ_8 is represented by the Hopf fibre map $\tilde{g}_{1,0}$, and $E(\tau_7)$ is the image of a generator τ_7 of $\pi_{14}(S^7)$ by the suspension homomorphism and is represented by $\tilde{g}_{1,-1}$.*

LEMMA 8. *a*) $(-\iota_8) \circ \{\tilde{g}_{h,j}\} = \{\tilde{g}_{j,h}\}$, *where ι_8 is the standard generator of $\pi_8(S^8)$.* *b*) $[\iota_8, \iota_8] = 2\sigma_8 - E(\tau_7)$, *the left side denotes the Whitehead product.*

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9) By the Hurewicz's conjecture we mean that two manifolds of the same homotopy type would be homeomorphic.

***) Cf. [18].

This is a known result (cf. Toda [11]), but we shall give here a simple proof. Let $\kappa : S^{15} \to S^{15}$ be defined by $\kappa(x, y) = (\bar{x}, \bar{y})$ and $\kappa' : S^8 \to S^8$ by $\kappa'(s, \sigma) = (\bar{s}, \sigma)$. Then we have $\kappa' \circ \tilde{g}_{h,j} \circ \kappa = \tilde{g}_{j,h}$. Since κ' reverses the orientation of S^8, we obtain a). b) follows immediately from a) and Theorem 5. 15 of G. W. Whitehead [14].

Now we return to our purpose. Let $\varphi : S^{15} \to M_k^{15}$ be an orientation preserving homeomorphism, of which existence is assured by Lemma 3, and denote by ρ_k the projection $M_k^{15} \to S^8$ as in the preceding section. We shall determine the element of $\pi_{15}(S^8)$ represented by the composition map $\rho_k \circ \varphi$.

For this purpose, define the following map $\varphi' : S^{15} \to M_k^{15}$ by

$$\varphi'(x, y) = \begin{cases} (\sqrt{2(2|y|^2-1)}\,x,\ 2|y|^2-1\ ;\ y/|y|)_1 & \text{for } |y|^2 > \frac{1}{2}, \\ (0, 0\ ;\ 2x^h y x^j)_0 & \text{for } |y|^2 = \frac{1}{2}, \\ ((\sqrt{2(1-2|y|^2)}/|x|)\,x^h y x^j,\ 1-2|y|^2\ ;\ 1)_1 & \text{for } |y|^2 < \frac{1}{2}, \end{cases}$$

where h and j are the integers determined by the equation $h+j=1$, $h-j=k$. Thus defined map φ' is obviously continuous and, we may consider, of degree 1.

Since φ' is homotopic to φ, we have only to consider the map $\rho_k \circ \varphi' : S^{15} \to S^8$, which is defined by

$$\rho_k \circ \varphi'(x, y) \begin{cases} (\sqrt{2(2|y|^2-1)}\,x,\ 2|y|^2-1) & \text{for } |y|^2 \geqq \frac{1}{2}, \\ ((\sqrt{2(1-2|y|^2)}/|x|)\,x^h y x^j,\ 1-2|y|^2) & \text{for } |y|^2 \leqq \frac{1}{2}. \end{cases}$$

Denote by E^8 the closed spherical 8-cell defined in R^8 by $|x| \leqq 1$. The boundary $(E^8 \times E^8)^\cdot$ of $E^8 \times E^8$ is homeomorphic to S^{15}. A specific homeomorphism f is defined by

$$f(q, r) = \begin{cases} \left(q/\sqrt{2},\ \sqrt{1-\frac{1}{2}|q|^2}\cdot r\right) & \text{on } E^8 \times \dot{E}^8, \\ \left(\sqrt{1-\frac{1}{2}|r|^2}\cdot q,\ r/\sqrt{2}\right) & \text{on } \dot{E}^8 \times E^8. \end{cases}$$

Further define two maps $\psi_1, \psi_0 : (E^8 \times E^8)^\cdot \to S^{15}$ by

$$\psi_1(q, r) = \begin{cases} (q,\ \sqrt{1-|q|^2}\,r)\ , & \text{on } E^8 \times \dot{E}^8, \\ (q, 0) & \text{on } \dot{E}^8 \times E^8, \end{cases}$$

$$\psi_0(q, r) = \begin{cases} (0, r) & \text{on } E^8 \times \dot{E}^8, \\ (\sqrt{1-|r|^2}q, r) & \text{on } \dot{E}^8 \times E^8. \end{cases}$$

The maps f, ψ_1, ψ_0 are all considered as of degree 1 with respective to the natural orientation of $(E^8 \times E^8)^{\cdot}$ and S^{15}.

Let μ, $\gamma : S^{15} \to S^{15}$ denote two maps of degree 1 and of degree -1 which are defined by $\mu(x, y) = (y, x)$ and $\gamma(x, y) = (x, \bar{y})$ respectively. And let $\chi : S^8 \to S^8$ be defined by $\chi(s, \sigma) = (s, 1-\sigma)$.

Set

$$F_1 = \chi \circ \tilde{g}_{0,0} \circ \gamma \circ \mu \circ \psi_1$$

and

$$F_0 = \chi \circ \tilde{g}_{h,j} \circ \gamma \circ \psi_0,$$

then F_1, F_0 are two maps of $(E^8 \times E^8)^{\cdot}$ into S^8 and satisfy the following conditions:

$$F_1(q, r) = \begin{cases} \rho_k \circ \varphi' \circ f(q, r) & \text{on } E^8 \times \dot{E}^8, \\ (0, 0) & \text{on } \dot{E}^8 \times E^8, \end{cases}$$

and

$$F_0(q, r) = \begin{cases} (0, 0) & \text{on } E^8 \times \dot{E}^8, \\ \rho_k \circ \varphi' \circ f(q, r) & \text{on } \dot{E}^8 \times E^8. \end{cases}$$

Denote the 8-cell $E^8 \times 1$ in $E^8 \times \dot{E}^8$ by E_1^8 and $1 \times E^8$ in $\dot{E}^8 \times E^8$ by E_0^8. Let $f_1 : (E_1^8, \dot{E}_1^8) \to (S^8, p_*)$ and $f_0 : (E_0^8, \dot{E}_0^8) \to (S^8, p_*)$ be the restriction of F_1 and F_0 respectively, where p_* denotes the south pole $(0, 0)$. Since $f_1(q, 1) = (\sqrt{1-|q|^2}q, 1-|q|^2)$ and $f_0(1, r) = (\sqrt{1-|r|^2}r, 1-|r|^2)$, both f_1 and f_0 represent the standard generator ι_8 of $\pi_8(S^8)$.

Now from the theorem of G. W. Whitehead [13] we have

$$\{\rho_k \circ \varphi' \circ f\} = \{F_1\} + \{F_0\} + [\{f_1\}, \{f_0\}].$$

Clearly F_1 represents the zero element and F_0 represents the element $-(-\iota_8) \circ \{\tilde{g}_{h,j}\}$, therefore by using Lemmas 7 and 8 we have

$$\{\rho_k \circ \varphi\} = \{\rho_k \circ \varphi'\} = -\{\tilde{g}_{j,h}\} + [\iota_8, \iota_8] = \sigma_8 - j \cdot E(\tau_7)$$

THEOREM 5. *The projection* $\rho_k : M_k^{15} \to S^8$ *represents the element* $\sigma_8 + \frac{1}{2}(k-1)E(\tau_7)$ *of* $\pi_{15}(S^8)$.

THEOREM 5'. *The projection* $\rho'_k : M_k^7 \to S^4$ *represents the element*

$\nu_1 + \frac{1}{2}(k-1)E(\omega_3)$ *of* $\pi_7(S^4)$, *where* ν_4 *is represented by the Hopf fibre map and* $E(\omega_3)$ *is the image of a generator* ω_3 *of* $\pi_6(S^3)$ *by the suspension homomorphism.*

Proof of Theorem 3. It is easily seen that the homotopy type of the manifold $\tilde{B}_k^{16}$ is the same as that of the reduced cell complex L_k constructed as follows. Consider the union of two copies of 8-sphere with only one point in common, which is denoted by $S^8 \vee S^8$. Attach a 16-cell e^{16} to $S^8 \vee S^8$ by such a map β_k of the boundary e^{16} into $S^8 \vee S^8$ that represents the element $\{\rho_k \circ \varphi\} + \{\rho_k \circ \varphi\}$ of $\pi_{15}(S^8) + \pi_{15}(S^8) \subset \pi_{15}(S^8 \vee S^8)$. The constructed cell complex $(S^8 \vee S^8) \cup e^{16}$ is the above mentioned complex L_k.

Since the homotopy type of L_k is determined by the element $\{\beta_k\}$ of $\pi_{15}(S^8 \vee S^8)$ and the order of the element $E(\tau_7)$ is known to be 120 (See [9]), Theorem 3 can be now proved easily. Theorem 3′ can be proved similarly.

Now we shall state the side-result. Consider the following hypothesis for a topological $2n$-manifold X^{2n}:

$$(*)\qquad \begin{aligned} H_i(X^{2n}) &= Z \qquad \text{for } i = 0,\ n,\ 2n, \\ H_i(X^{2n}) &= 0 \qquad \text{for } i \neq 0,\ n,\ 2n. \end{aligned}$$

Manifolds with the properties $(*)$ are known for $n = 2, 4, 8$ (complex, quaternion, Cayley projective planes respectively).

THEOREM 6. *There exist several topological* $2n$*-manifolds satisfying the hypothesis* $(*)$ *and having different homotopy types for the cases* $n = 4, 8$ *respectively.*

(The author does not know if these topological manifolds admit any differentiable structures. Cf. Problem 5 in [6].)

Let X_k^{2n} be the closed manifold obtained from $\overline{B}_k^{2n}$ by collapsing its boundary (a topological $(2n-1)$-sphere) to a point x_0 for $n = 4, 8$. Then the topological manifolds X_k^{2n} serve themselves as the examples of the manifolds stated in Theorem 6.

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