CYCLES ON ALGEBRAIC VARIETIES

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In the present note, applying the theory of harmonic integrals, we shall show some results on cycles on algebraic varieties and give a new birational invariant.

NOTATIONS:

$\mathbf{V}$: a non-singular algebraic variety of (complex) dimension n in a projective space,

$\mathbf{V}_1(\mathbf{V}_2)$: the first (second) component of $\mathbf{V}\times\mathbf{V}$,

$\delta(\mathbf{V})$: the diagonal sub-manifold of $\mathbf{V}\times\mathbf{V}$,

$\mathbf{W}_r$: a generic hyper-plane section of (complex) dimension r of $\mathbf{V}$,

Q, R, C: the fields of rational, real, complex numbers respectively,

$H_r(\mathbf{V}, Q)$, $H_r(\mathbf{V}, R)$, $H_r(\mathbf{V}, C)$: the r-th homology groups of $\mathbf{V}$ over Q, R and C respectively,

$H^r(\mathbf{V}, Q)$, $H^r(\mathbf{V}, R)$, $H^r(\mathbf{V}, C)$: the r-th cohomology groups of $\mathbf{V}$ over Q, R, C respectively,

$H_{p,q}(\mathbf{V}, *)$: the subgroup of $H_{p+q}(\mathbf{V}, *)$ consisting of all the classes of type (p, q),

$H^{p,q}(\mathbf{V}, *)$: the subgroup of $H^{p+q}(\mathbf{V}, *)$ consisting all the classes of type (p, q),

$\mathfrak{H}_r(\mathbf{V}, Q)$: the subgroup of $H_{2r}(\mathbf{V}, Q)$ consisting of all the classes containing algebraic cycles,

B_r: the degree of $H_r(\mathbf{V}, Q)$,

$\{\Gamma_r^1, \ldots, \Gamma_r^{B_r}\}$: a base of $H_r(\mathbf{V}_1, Q)$,

$\{\Delta_r^1, \ldots, \Delta_r^{B_r}\}$: the base of $H_r(\mathbf{V}_2, Q)$ corresponding to $\{\Gamma_r^1, \ldots, \Gamma_r^{B_r}\}$,

$\{\Gamma_r^{1+}, \ldots, \Gamma_r^{B_r+}\}$: the base of $H_{2n-r}(\mathbf{V}_1, Q)$ such that $I(\Gamma_r^i\Gamma_r^{j+})=\delta_{ij}$ $i, j=1, 2, \ldots, B_r$,

$\{\Delta_r^{1+}, \ldots, \Delta_r^{B_r+}\}$: the base of $H_{2n-r}(\mathbf{V}_2, Q)$ corresponding to $\{\Gamma_r^{1+}, \ldots, \Gamma_r^{B_r+}\}$,

α_X, $\alpha_Y^{1\times 2}$, α_Z^1, α_U^2: the harmonic forms on $\mathbf{V}$, $\mathbf{V}\times\mathbf{V}$, $\mathbf{V}_1$, $\mathbf{V}_2$ corresponding

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to cycles X, Y, Z, U on $\mathbf{V}$, $\mathbf{V}\times\mathbf{V}$, $\mathbf{V}_1$, $\mathbf{V}_2$ by means of Hodge's theorem respectively,

$\Omega^{(p,q)}$: the period matrix of harmonic forms of type (p, q) on $\mathbf{V}_1$ with period cycles $\Gamma_r^1, \ldots, \Gamma_r^{B_r}$ such that $p+q=r\leqq n$, $p\leqq q$,

$\Omega^{(n-q,n-p)}$: the period matrix of harmonic forms of type $(n-q, n-p)$ with period cycles $\Gamma_r^{1+}, \ldots, \Gamma_r^{B_r+}$ such that $p+q=r$, $p\leqq q$.

$<\alpha, X> = \int_X \alpha,$

$<\alpha, \beta>_M = \int_M \alpha\wedge\beta,$

$Z\approx 0$: Z is homologous zero over Q.

$\delta(\Gamma)$: the cycle on $\delta(\mathbf{V})$ corresponding by the natural correspondence to a cycle Γ on $\mathbf{V}$,

$\delta_1^{-1}(X)$: a cycle on $\mathbf{V}_1$ corresponding by the natural correspondence to a cycle X on $\delta(\mathbf{V})$,

$(A)_{\alpha\beta} = (a_{ij})_{\alpha\beta} = a_{\alpha\beta}$,

$I(X\cdot Y; \delta(\mathbf{V}))$: Kronecker index of the intersection of cycles X, Y of $\delta(\mathbf{V})$ along to $\delta(\mathbf{V})$.

Lemma 1. *Let C be a cycle of dimension $2r$. Then*

$$^t(I(C\times \Delta_r^{i+}\delta(\Gamma_r^{j+})) = (I(C\Gamma_r^{i+}\Gamma_r^{j+})).$$

Proof. By virtue of intersection theory,[1]

$$\delta(\Gamma_r^{j+}) \approx \sum_{q=0}^{r}\sum_{\mu,\nu}\lambda_{\mu\nu}^{q}(\Gamma_r^{j+})\,\Gamma_{q-r}^{\mu}\times\Delta_{2n-q}^{\nu},$$

where

$$^t\lambda^q(\Gamma_r^{j+}) = (-1)^{(2n-q)r}(I(\Gamma_q^{\mu}\Gamma_q^{\nu+}))^{-1}(I(\Gamma_r^{j+}\Gamma_q^{\mu}\Gamma_{2n+r-q}^{\nu}))(I(\Gamma_{q-r}^{\mu}\Gamma_{q-r}^{\nu+}))^{-1}.$$

Since

$$^t\lambda^{2n-r}(\Gamma_r^{j+}) = (-1)^r(I(\Gamma_r^{\mu+}\Gamma_r^{\nu}))^{-1}(I(\Gamma_r^{j+}\Gamma_{2n-r}^{\mu}\Gamma_{2r}^{\nu})(I(\Gamma_{2n-2r}^{\mu}\Gamma_{2n-2r}^{\nu+}))^{-1}.$$

we have

$$\begin{aligned} I(C\times\Delta_r^{i+}\cdot\delta(\Gamma_r^{j+})) &= I(C\times\Delta_r^{i+}\cdot\sum_{q=0}^{r}\sum_{\mu,\nu}\lambda_{\mu\nu}^{q}(\Gamma_r^{j+})\,\Gamma_{q-r}^{\mu}\times\Delta_{2n-q}^{\nu}) \\ &= \sum_{\mu,\nu}\lambda_{\mu,\nu}^{2n-r}(\Gamma_r^{j+})\,I(C\Gamma_{2n-2r}^{\mu})\,I(\Gamma_r^{i+}\Delta_r^{\nu}) \end{aligned}$$

[1] See S. Lefschetz, Topoloyg (New York), 1930.

$$= (-1)^r \sum_{\alpha,\beta} I(C\Gamma^{\alpha}_{2n-2r})\{{}^t(I(\Gamma^{\mu}_{2n-2r}\Gamma^{\nu+}_{2n-2r})^{-1}$$
$$ {}^t(I(\Gamma^{j+}_r \Gamma^{\mu}_{2n-r}\Gamma^{\nu}_{2r}))^t(I(\Gamma^{\mu+}_r\Gamma^{\nu}_r))^{-1}\}_{\alpha,\beta} I(\Gamma^{i+}_r\Gamma^{\beta}_r)$$
$$= \sum_{\alpha,\beta} I(C\Gamma^{\alpha}_{2n-2r})\{(I(\Gamma^{\mu+}_{2n-2r}\Gamma^{\nu}_{2n-2r}))^{-1}$$
$$(I(\Gamma^{j+}_r\Gamma^{\mu}_{2r}\Gamma^{\nu}_{2n-r})(I(\Gamma^{\mu}_r\Gamma^{\nu+}_r))^{-1}\}_{\alpha,\beta} I(\Gamma^{\beta}_r\Gamma^{i+}_r)$$
$$= I(\Gamma^{j+}_r C\Gamma^{i+}_r)$$
$$= I(C\Gamma^{j+}_r\Gamma^{i+}_r).$$

This proves our lemma.

LEMMA 2. *If a cycle X of dimension r on $\delta(\mathbf{V})$ is not homologous to zero over Q on $\delta(\mathbf{V})$. Then it is not homologous to zero over Q on $\mathbf{V}\times\mathbf{V}$, too.*

Proof. Let $\{\omega_1, \dots, \omega_{B_r}\}$ be a base of harmonic forms of degree r on $\mathbf{V}_1$. Then they can be considered as harmonic forms on $\mathbf{V}\times\mathbf{V}$ and on $\delta(\mathbf{V})$ and they are linearly independent on $\mathbf{V}\times\mathbf{V}$ and on $\delta(\mathbf{V})$. Therefore, by d'Rham's theorem our assertion is ture.

LEMMA 3. *Let C be a cycle of dimension $2r$. Then*

$$C\times\Delta^{j+}_r\cdot\delta(\mathbf{V}) \approx \sum_k I(C\times\Delta^{j+}_r\cdot\delta(\Gamma^{k+}_r))\cdot\delta(\Gamma^k_r).$$

Proof. By Lemma 2 $H(\delta(\mathbf{V}), C)$ is inbedded in $H(\mathbf{V}, C)$. Hence $I((C\times\Delta^{j+}_r\cdot\delta(\mathbf{V}))\delta(\Gamma^{k+}_r)\ ;\ \delta(\mathbf{V}) = I(C\times\Delta^{j+}_r\cdot\delta(\Gamma^k_r))$. Therefore

$$C\times\Delta^{j+}_r\delta(\mathbf{V}) \approx \sum_k I(C\times\Delta^{j+}_r\delta(\Gamma^{k+}_r))\delta(\Gamma^k_r).$$

PROPOSITION 1. Let C be a cycle of type $(r\mp s, r\pm s)$ with complex coefficients. Then

$$\Lambda(C)\,\Omega^{(n-q\pm s,\, n-p\mp s)} = \Omega^{(p,q)}(I(C\Gamma^{i+}_r\Gamma^{j+}_r)),$$

with a matrix $\Lambda(C)$, where $p+q=r<n$.

Proof. Let $\{\alpha_1, \dots, \alpha_l\}$ be a minimum base of harmonic forms of type (p, q) on $\mathbf{V}_1$. We denote by the same notations $\alpha_1, \dots \alpha_l$ the harmonic forms on $\mathbf{V}\times\mathbf{V}$ induced by $\alpha_1, \dots, \alpha_l$. Then we have

$$(<\alpha_i,\ \delta_1^{-1}(C\times\Delta^{j+}_r\cdot\delta(\mathbf{V}))>)$$
$$= (<\alpha_i,\ C\times\Delta^{j+}_r\delta(\mathbf{V})>)$$
$$= (<\alpha_i,\ \sum_k I(C\times\Delta^{j+}_r\cdot\delta(\Gamma^{k+}_r))\delta(\Gamma^k_r)>)$$
$$= (<\alpha_i,\ \sum_k I(C\times\Delta^{j+}_r\delta(\Gamma^{k+}_r))\Gamma^k_r>)$$
$$= (<\alpha_i,\ \Gamma^j_r>)^t(I(C\times\Delta^{j+}_r\delta(\Gamma^{k+}_r))$$
$$= \Omega^{(p,q)}(I(C\Gamma^{i+}_r\Gamma^{j+}_r)).$$

On the other hand

$$
\begin{aligned}
&(<\alpha_i,\ C\times\Delta_r^{j+}\delta(\mathbf{V})>)\\
&=(<\alpha_i,\ \alpha^{1\times 2}_{C\times\Delta_r^{j+}\delta(\mathbf{V})}>_{V\times V})\\
&=(<\alpha_i,\ \alpha^1_C\wedge\alpha^2_{\Delta_r^{j+}}\wedge\alpha^{1\times 2}_{\delta(\mathbf{V})}>_{V\times V})\\
&=(<\alpha_i\wedge\alpha^1_C\wedge\alpha^{1\times 2}_{\delta(\mathbf{V})},\ \alpha^2_{\Delta_r^{j+}}>_{V\times V})\\
&=(<\int_C\alpha_i\wedge\alpha^{1\times 2}_{\delta(\mathbf{V})},\ \Delta_r^{j+}>).
\end{aligned}
$$

The type of the form

$$\int_C\alpha_i\wedge\alpha^{1\times 2}_{\delta(\mathbf{V})}$$

is $(p,\ q)+(n,\ n)-(r\mp s,\ r\pm s)=(n-q\pm s,\ n-p\mp s)$.

Hence

$$(<\alpha_i,\ C\times\Delta_r^{j+}\delta(\mathbf{V})>)=\Lambda(C)\,\Omega^{(n-q\pm s,\,np-\mp s)}$$

with a matrix $\Lambda(C)$. Therefore

$$\Omega^{(p,q)}(I(C\Gamma_r^{i+}\Gamma_r^{j+}))=\Lambda(C)\,\Omega^{(n-q\pm s,\,n-p\mp s)}.$$

LEMMA 4. *Let $r\leqq n$. Then $(I(\mathbf{W}_r\Gamma_r^{i+}\Gamma_r^{j+}))$ is non-singular.*

Proof. Since $\{\Gamma_r^{1+},\ldots,\Gamma_r^{B_r+}\}$ is a base of $H_{2n-r}(\mathbf{V},Q)$, by virtue of theory of harmonic integral on a Hodge variety,[2] $\{\mathbf{W}_r\Gamma_r^{1+},\ldots,\mathbf{W}_r\Gamma_r^{B_r+}\}$ is a base of $H_r(\mathbf{V},Q)$. Hence $(I(\mathbf{W}_r\Gamma_r^{i+}\Gamma_r^{j+}))$ is non-singular.

THEOREM 1. *Let $r\leqq n$. Let C be a cycle of type (r,r). Then*

$$\Omega^{(r)}(I(C\Gamma_r^{i+}\Gamma_r^{j+}))(I(\mathbf{W}_r\Gamma_r^{i+}\Gamma_r^{j+}))^{-1}=\begin{pmatrix}\Lambda_0(C)&&&\\&\Lambda_1(C)&&\\&&\ddots&\\&&&\Lambda_{[r/2]}(C)\end{pmatrix}\Omega^{(r)},$$

where

$$\Omega^{(r)}=\begin{cases}\begin{pmatrix}\Omega^{(r,0)}\\\Omega^{(r-2,2)}\\\vdots\\\Omega^{(1,r-1)}\end{pmatrix} & \textit{for odd } r,\\[2ex]\begin{pmatrix}\Omega^{(r,0)}\\\Omega^{(r-1,1)}\\\vdots\\\Omega_{(r/2,\,r/2)}\end{pmatrix} & \textit{for even } r.\end{cases}$$

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[2] See J. Igusa, On Picard varieties § II, 6, Proposition 3 American Journal, **74**, 1-22 (1952).

This is an immediate consequence from Proposition 1.

THEOREM 2. *Let r be an odd integer less than n. Let $\{s_1, \dots, s_l\}$ be a base of the module of rational matrices $S=(s_{ij})$ such that*

$$\sum_{i,j} s_{ij}\Gamma_r^{i+}\Gamma_r^{j+} \approx 0.$$

Let $K_{2r}(\mathbf{V}, Q)$ be the sub-module of $H_{2r}(\mathbf{V}, Q)$ consisting of Z such that $I(Z\Gamma_r^{i+}\Gamma_r^{j+})=0$ $i, j=1, 2, \dots, B_r$. Then there exists an isomorphism from

$$H_{r,r}(\mathbf{V}, Q)/H_{r,r}(\mathbf{V}, Q)\cap K_{2r}(\mathbf{V}, Q)$$

onto the module of rational matrices M satisfying

i) $\Omega^{(r)}M = \Lambda\Omega^{(r)}$ *with a matrix Λ,*

where

$$\Omega^{(r)} = \begin{cases} \begin{pmatrix} \Omega^{(r,0)} \\ \Omega^{(r-2,2)} \\ \vdots \\ \Omega^{(1,r-1)} \end{pmatrix} & \text{for odd } r, \\ \begin{pmatrix} \Omega^{(r,0)} \\ \Omega^{(r-1,1)} \\ \vdots \\ \Omega_{(r/2,r/2)} \end{pmatrix} & \text{for even } r. \end{cases}$$

ii) $S_pS_\nu M(I(\mathbf{W}_r\Gamma_r^{i+}\Gamma_r^{j+}))=0$ $\nu=1, 2, \dots, l.$

Proof. Let $D_1, \dots, D_m$ be independent generators of $H_{r,r}(\mathbf{V}, Q)/H_{r,r}(\mathbf{V}, Q)\cap K_{2r}(\mathbf{V}, Q)$. Let φ be the linear mapping such that

$$\varphi(\sum_k a_k\mathbf{D}_k) = \sum_k a_k(I(\mathbf{D}_k\Gamma_r^{i+}\Gamma_r^{j+}))(I(\mathbf{W}_r\Gamma_r^{i+}\Gamma_r^{j+}))^{-1}$$

Then, by virtue of Theorem 1,

$$\Omega^{(r)}\varphi(\sum_k a_k\mathbf{D}_k) = \Lambda\Omega^{(r)}$$

with a matrix Λ.

On the other hand we get

$$\begin{aligned} S_pS_\nu\varphi(\sum_k a_k\mathbf{D}_k)(I(\mathbf{W}_r\Gamma_r^{i+}\Gamma_r^{j+})) &= S_pS_\nu(I(\sum_k a_k\mathbf{D}_k\Gamma_r^{i+}\Gamma_r^{j+})) \\ = \sum_k a_k\mathrm{I}(\mathbf{D}_k\sum_{i,j}s_{ij}^{(\nu)}\Gamma_r^{i+}\Gamma_r^{j+}) &= 0 \qquad \nu=1, 2, \dots, l. \end{aligned}$$

Conversely we assume that a rational matrix M satisfies the condition i), ii). From ii) it follows that there exists a cycle with rational coefficients C such that

$$(I(C\Gamma_r^{i+}\Gamma_r^{j+}))=M(I(\mathbf{W}_r\Gamma_r^{i+}\Gamma_r^{j+})).$$

We assume that C is not homologous to a cycle of type (r, r) modulo $K_{2r}(\mathbf{V}, Q)$. We put $\alpha_C=\alpha_{C_0}+(\alpha_{C_1}+\alpha_{C_1'})+\ldots+(\alpha_{C_r}+\alpha_{C_r'})$, where

$$\begin{array}{lll}\alpha_{C_\nu} \text{ is of type } & (r-\nu,\ r+\nu) & \nu=0, 1, \ldots, r,\\ \alpha_{C'_\mu} \text{ is of type } & (r+\nu,\ r-\nu) & \mu=1, 2, \ldots, r\end{array}$$

and C_ν, C'_μ are cycles with complex coefficients corresponding to harmonic forms α_{C_ν}, $\alpha_{C'_\mu}$ by means of Hodge's theorem respectively. Then, since C is real, necessalily we get $\alpha_{C'_\nu}=\overline{\alpha_{C_\nu}}$. By virtue of the assumption on C, there exists ν_0 such that

$$(I((C_{\nu_0}+C'_{\nu_0})\Gamma_r^{i+}\Gamma_r^{j+}))\neq 0.$$

On the other hand from Proposition 1, putting

$$T(C_\nu+C'_\nu)\Omega^{(r)}=\Omega^{(r)}(I((C_\nu+C'_\nu)\Gamma_r^{i+}\Gamma_r^{j+}))(I(\mathbf{W}_r\Gamma_r^{i+}\Gamma_r^{j+}))^{-1},$$

we have that for any i, j at most one i, j-element of $T(C_0)$, $T(C_1+C_1')$, $\ldots$, $T(C_r+C_r')$ does not vanish. From $(I((C_{\nu_0}+C'_{\nu_0}(\Gamma_r^{i+}\Gamma_r^{j+}))\neq 0$ we see that $T(C_{\nu_0}+C'_{\nu_0})\neq 0$. By virtue of Proposition 1 $T(C_{\nu_0}+C'_{\nu_0})$ varies of the type of integrants. This is a contradiction to our assumption. Therefore our theorem is proved.

THEOREM 3. *Let $\{S_1, \ldots, S_l\}$ be a base of the module of rational matrices $S=(s_{ij})$ such that*

$$\sum_{i,j}s_{ij}\Gamma_1^{i+}\Gamma_1^{j+}\approx 0.$$

Let $K^*_{2n-2}(\mathbf{V}, Q)$ be the sub-module of $H_{2n-2}(\mathbf{V}, Q)$ consisting of Z such that $I(\mathbf{W}_2Z\Gamma_1^{i+}\Gamma_1^{j+})=0$ $i, j=1, 2, \ldots, B_1$.

Then there exists an isomorphism from

$$\mathfrak{H}_{n-1}(\mathbf{V}, Q)/\mathfrak{H}_{n-1}(\mathbf{V}, Q)\cap K^*_{2n-2}(\mathbf{V}, Q).$$

onto the module of rational matrices M satisfying

i) $\Lambda\Omega^{(1,0)}=\Omega^{(1,0)}M$ with a matrix Λ,

ii) $S_pS_\nu M(I(\mathbf{W}_1\Gamma_1^{i+}\Gamma_1^{j+}))=0$, $\nu=1, 2, \ldots, l$.

Proof. Let $\mathbf{D}_1, \ldots, \mathbf{D}_m$ be independent generators of $\mathfrak{H}_{n-1}(\mathbf{V}, Q)$. Then $\mathbf{D}_1\mathbf{W}_2, \ldots, \mathbf{D}_m\mathbf{W}_2$ are independent generators of $\mathfrak{H}_1(\mathbf{V}, Q)$.[3] On the other hand, by virtue of Lefschetz-Hodge's theorem,[4] $H_{1,1}(\mathbf{V}, Q) = \mathfrak{H}_1(\mathbf{V}, Q)$. Hence if we put

$$\varphi(\sum_k a_k \mathbf{D}_k) = \sum_k a_k (I(\mathbf{W}_2 \mathbf{D}_k \Gamma_1^{i+} \Gamma_1^{j+}))(I(\mathbf{W}_1 \Gamma_1^{i+} \Gamma_1^{j+}))^{-1}.$$

Then, by the strictly same reason in the proof of Theorem 3, φ gives our isomorphism.

We call the degree of $\mathfrak{H}_{n-1}(\mathbf{V}, Q)/\mathfrak{H}_{n-1}(\mathbf{V}, Q) \cap K^*_{2n-2}(\mathbf{V}, Q)$ the restricted Picard number of $\mathbf{V}$.

Then we get the following.

THEOREM 4. *Restricted Picard number is a birational invariant.*

Proof. Let $\mathbf{V}'$ be another non-singular algebraic variety, which is equivalent to $\mathbf{V}$ by a birational correspondence T. Then T induces isomorphisms from $H_1(\mathbf{V}, Q)$, $H^{(1,0)}(\mathbf{V}, C)$ onto $H_1(\mathbf{V}', Q)$, $H^{(1,0)}(\mathbf{V}', C)$ respectively.[5] We denote by f and f^* these isomorphisms.

We denote by $[H^1(\mathbf{V}, C)]$, $[H^1(\mathbf{V}', C)]$ the sub-rings generated by $H^1(\mathbf{V}, C)$, $H^1(\mathbf{V}', C)$ respectively. Then f^* induces an isomorphism from $[H^1(\mathbf{V}', C)]$ onto $[H^1(\mathbf{V}, C)]$, for f^* mapps $H^1(\mathbf{V}', C)$ onto $H^1(\mathbf{V}, C)$ and f^* induces a homomorphism from $[H'(V, C)]$, onto $[H'(V', C)]$.

On the other hand, since

$$\alpha_{\Gamma_1^{i+}} = f^*(\alpha'_{f(\Gamma_1^{i+})})$$

and

$$\alpha'_{f(\Gamma_1^{i+})} = \alpha'_{f(\Gamma_1^{i})+},$$

we have

$$\begin{aligned} \alpha_{\Gamma_1^{i+}\Gamma_1^{j+}} = \alpha_{\Gamma_1^{i+}} \wedge \alpha_{\Gamma_1^{j+}} &= f^*(\alpha'_{f(\Gamma_1^{i+})}) \wedge f^*(\alpha'_{f(\Gamma_1^{j+})}) \\ &= f^*(\alpha'_{f(\Gamma_1^{i})+}) \wedge f^*(\alpha'_{f(\Gamma_1^{j})+}) \\ &= f^*(\alpha'_{f(\Gamma_1^{i})+} \wedge \alpha'_{f(\Gamma_1^{j})+}) = f^*(\alpha'_{f(\Gamma_1^{i})+f(\Gamma_1^{i})+}). \end{aligned}$$

3), 4) W. V. D. Hodge, The theory and applications of harmonic integrals, **IV**; 51, 2 (London), 1940.

5) See J. Igusa, On Picard varieties § II, 11, American Journal, **74**, 1-22 (1952).

Therefore

$$\sum_{i,j} s_{ij}\,\alpha'_{f(\Gamma_1^i)^+ f(\Gamma_1^j)^+} = 0$$

if and only if

$$\sum_{i,j} s_{ij}\,\alpha_{\Gamma_1^{i+}\Gamma_1^{j+}} = 0.$$

This shows that

$$\sum_{i,j} s_{ij} f(\Gamma_1^i)^+ f(\Gamma_1^j)^+ \approx 0$$

if and only if

$$\sum_{i,j} s_{ij}\,\Gamma_1^{i+}\,\Gamma_1^{j+} \approx 0.$$

Let $\alpha'_1, \ldots, \alpha'_{B_1/2}$ be differentials of the first kind on $\mathbf{V}'$ such that $\Omega^{(1,0)}$ is the period matrix of $f^*(\alpha'_1), \ldots, f^*(\alpha'_{B_1/2})$ with period cycles $\Gamma_1^1, \ldots, \Gamma_1^{B_1}$. Then the period matrix of $\alpha'_1, \ldots, \alpha'_{B_1/2}$ with period cycles $f(\Gamma_1^1), \ldots, f(\Gamma_1^{B_1})$ is also $\Omega^{(1,0)}$. Therefore, by virtue of Theorem 3, we get

$$\mathfrak{H}_{n-1}(\mathbf{V}, Q)/K^*_{2n-2}(\mathbf{V}, Q) \wedge \mathfrak{H}_{n-1}(\mathbf{V}, Q)$$
$$\cong \mathfrak{H}_{n-1}(\mathbf{V}', Q)/K^*_{2n-2}(\mathbf{V}', Q) \wedge \mathfrak{H}_{n-1}(\mathbf{V}', Q).$$

This proves our assertion.

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