REMARKS ON THE ELLIPTIC CASE OF THE MAPPING THEOREM FOR SIMPLY-CONNECTED RIEMANN SURFACES

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1. It is well-known that the conformal equivalence of a compact simply-connected Riemann surface to the extended plane is readily established once it is shown that given a local uniformizer $t(p)$ which carries a given point p_0 of the surface into 0, there exists a function u harmonic on the surface save at p_0 which admits near p_0 a representation of the form

$$R\left[\frac{\alpha}{t(p)}\right] + h(p) \tag{1.1}$$

(α complex $\neq 0$; h harmonic at p_0). For the monodromy theorem then implies the existence of a meromorphic function on the surface whose real part is u. Such a meromorphic function has a simple pole at p_0 and elsewhere is analytic. It defines a univalent conformal map of the surface onto the extended plane.

In the author's paper "The conformal mapping of simply-connected Riemann surface" (Annals of Mathematics, vol. 50 (1949), pp. 686–690), it was observed that the existence of such a generating function u could be established on the basis of the Perron method. The object of the present note is to give the details of the proof to which allusion was made.

2. We consider a compact Riemann surface F and a given point $p_0 \in F$. Let φ denote a univalent conformal map of $|t| < 2$ into F which takes $t = 0$ into p_0. Let R denote a fixed number satisfying $1 < R < 2$ and let ρ denote a generic number satisfying $0 < \rho < 1$. Let Δ_r denote the complement with respect to F of the φ-image of $|t| \leqq r$ and let Γ_r denote the φ-image of $|t| = r$ $(0 < r < 2)$. Let U_ρ denote the solution of the Dirichlet problem for Δ_ρ which satisfies the boundary condition

$$U_\rho[\varphi(\rho e^{i\theta})] = A(\rho)\cos\theta + B(\rho) \tag{2.1}$$

where $A(\rho)$ and $B(\rho)$ are so chosen that

Received April 14, 1955.

$$\max_{\Gamma_1} U_\rho = 1, \quad \min_{\Gamma_1} U_\rho = -1. \tag{2.2}$$

We now consider the classical expansion for $u_\rho(re^{i\theta}) = U_\rho[\varphi(re^{i\theta})]$, $(\rho \leqq r < 2)$:

$$\frac{a_0(\rho)\log r + b_0(\rho)}{2} + \sum_{k=1}^{\infty}[\{a_k(\rho)r^k + a_{-k}(\rho)r^{-k}\}\cos k\theta + \{b_k(\rho)r^k - b_{-k}(\rho)r^{-k}\}\sin k\theta] \tag{2.3}$$

and observe that

$$a_0(\rho)\log r + b_0(\rho) = \frac{1}{\pi}\int_0^{2\pi} u_\rho(re^{i\theta})d\theta \tag{2.4}$$

and

$$\begin{cases} a_k(\rho)r^k + a_{-k}(\rho)r^{-k} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} u_\rho(re^{i\theta})\cos k\theta d\theta, \\ b_k(\rho)r^k - b_{-k}(\rho)r^{-k} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} u_\rho(re^{i\theta})\sin k\theta d\theta, \qquad k = 1, 2, \ldots . \end{cases} \tag{2.5}$$

From (2.2) and (2.4) we infer that $|b_0(\rho)| \leqq 2$ and $|a_0(\rho)| \leqq 4(\log 2)^{-1}$. Furthermore we conclude on setting $r = 1$ and $r = R$ in the equalities (2.5) that there exists a positive number C such that

$$|a_{-1}(\rho)| \leqq C \tag{2.6}$$

and

$$|a_k(\rho)|, \ |b_k(\rho)| \leqq CR^{-k}, \qquad k = 1, 2, \ldots . \tag{2.7}$$

On the other hand, on setting $r = \rho$ in (2.5) we conclude

$$|a_{-k}(\rho)| \leqq C\frac{\rho^{2k}}{R^k}, \qquad k = 2, 3, \ldots, \tag{2.8}$$

and

$$|b_{-k}(\rho)| \leqq C\frac{\rho^{2k}}{R^k}, \qquad k = 1, 2, \ldots . \tag{2.9}$$

For each r, $0 < r \leqq 1$, there exists a positive number $M(r)$ such that for $0 < \rho \leqq r$,

$$\max_{\Gamma_r} |U_\rho| \leqq M(r).$$

This is readily concluded from the above estimates of the coefficients. It follows that there exists a decreasing sequence of ρ, say $\{\rho_n\}_1^\infty$, with $\lim \rho_n = 0$ such that $\{U_{\rho_n}\}$ converges to a harmonic function U on $F-\{p_0\}$, uniformly on each compact subset. Clearly U is not constant by virtue of (2.2). Further in the expansion

$$\frac{a_0 \log r + b_0}{2} + \sum_{k=1}^{\infty} [(a_k r^k + a_{-k} r^{-k}) \cos k\theta + (b_k r^k - b_{-k} r^{-k}) \sin k\theta]$$

of $U[\varphi(re^{i\theta})]$, $0 < r < 2$, $a_{-k} = 0$ for $k = 2, 3, \ldots$ and $b_{-k} = 0$ for $k = 1, 2, \ldots$ as is readily concluded from (2.8) and (2.9).

However $a_{-1} \neq 0$. Otherwise U would attain either its maximum or minimum in $F-\{p_0\}$ and would be constant. We conclude that there exists a function V_1 harmonic on $F-\{p_0\}$ and such that $V_1[\varphi(t)]$ admits a representation of the form

$$R\left[\frac{1}{t}\right] + A \log |t| + h_1(t)$$

where h_1 is harmonic in $|t| < 2$. If $A = 0$, our construction is achieved. Actually $A = 0$. Since we are avoiding appeal to Green's theorem, we do not assume that this need be the case. Instead we note that on replacing $\cos\theta$ by $\sin\theta$ in (2.1) and paraphrasing the above argument we are led to the existence of a function V_2 harmonic on $F-\{p_0\}$ and such that $V_2[\varphi(t)]$ admits a representation of the form

$$R\left[\frac{i}{t}\right] + B \log |t| + h_2(t)$$

where h_2 is harmonic in $|t| < 2$. It follows that there exists a linear combination of V_1 and V_2 say, V, such that $V[\varphi(t)]$ admits a representation of the form

$$R\left[\frac{\alpha}{t}\right] + h(t)$$

where α is a complex number $\neq 0$ and h is harmonic in $|t| < 2$.

The required existence theorem of § 1 follows.

It is to be noted that we have not assumed that F is simply-connected in this section.

It is also worth noting that on replacing $\cos\theta$ in (2.1) by $\cos n\theta$ and $\sin n\theta$ ($n = 2, 3, \ldots$) and repeating the above argument we are led to the existence of a function V harmonic on $F-\{p_0\}$ and such that $V[\varphi(t)]$ admits a representation of the form

$$R\left[\frac{\alpha}{t^n}\right] + h(t)$$

where again α is complex $\neq 0$ and h is harmonic in $|t| < 2$. We conclude that the present method yields the existence of elementary differentials of the second kind having a pole of assigned order $\geqq 2$ at an assigned point of F.

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