THEORY OF MEROMORPHIC FUNCTIONS ON AN OPEN RIEMANN SURFACE WITH NULL BOUNDARY

MASATSUGU TSUJI

In the former paper,[1] I have developped a theory of meromorphic functions in a neighbourhood of a bounded closed set E of logarithmic capacity zero, by means of Evans' potential fnnction $u(z)$, which tends to ∞, when z tends to any point of E. It is not known, whether such a potential function exists on an open Riemann surface with null boundary, but by a substitute of Evans' function, we shall develop the similar theory of meromorphic functions on an open Riemann surface with null boundary.

§ 1

1. Let F be an open Riemann surface with null boundary, spread over the z-plane. We exhaust F by a sequence of compact Riemann surfaces: $F_0 \subset F_1 \subset \ldots \subset F_n \rightarrow F$, where the boundary Γ_n of F_n consists of a finite number of analytic Jordan curves.

Let $u_n(z)$ be the harmonic measure of Γ_n with respect to $F_n - \overline{F}_0$, such that $u_n(z)$ is harmonic in $F_n - \overline{F}_0$, $u_n(z) = 0$ on Γ_0, $u_n(z) = 1$ on Γ_n. Then as well known, $\lim_{n \to \infty} u_n(z) = 0$ uniformly in any compact domain of F. Let $v_n(z)$ be the conjugate harmonic function of $u_n(z)$ and

$$d_n = \int_{\Gamma_0} dv_n(z), \tag{1}$$

then

$$d_1 \geqq d_2 \geqq \ldots \geqq d_n \rightarrow 0. \tag{2}$$

We put

$$\zeta = e^{\frac{2\pi}{d_n}(u_n(z) + iv_n(z))} = re^{i\theta}, \tag{3}$$

where

$$r = r_n(z) = e^{\frac{2\pi}{d_n} u_n(z)}, \qquad \theta = \theta_n(z) = \frac{2\pi}{d_n} v_n(z), \tag{4}$$

then

$$1 \leqq r \leqq r_n, \qquad r_n = e^{\frac{2\pi}{d_n}}. \tag{5}$$

Received July 31, 1953.

[1] M. Tsuji: On the behaviour of a mernmorphic function in the neighbourhood of a closed set of capacity zero. Proc. Imp. Acad. **18** (1942). M. Tsuji: Theory of meromorphic functions in an neighbourhood of a closed set of capacity zero. Jap. Journ. Math. **19** (1944-48).

By (2),

(6) $$r_1 \leqq r_2 \leqq \ldots \leqq r_n \to \infty.$$

In this paper, r, θ mean always $r_n(z)$, $\theta_n(z)$.

Since $\lim\limits_{n\to\infty} u_n(z)=0$ uniformly in any compact domain of F, the part of F_n, such that $r_n^\delta \leqq r_n(z) \leqq r_n$ $(0<\delta<1)$ tends to the ideal boundary of F, for $n\to\infty$. Hence for a given F_n, we can take m so large that the part $r_m^\delta \leqq r_m(z) \leqq r_m$ of F_m lies outside of F_n.

Let $\varDelta_r$ be the part of $F_n - \overline{F}_0$, such that $1 \leqq r_n(z) \leqq r$ $(\leqq r_n)$ and $C_r : r_n(z)=r$ $(1\leqq r \leqq r_n)$ be the niveau curve of $r_n(z)$, then by (1),

(7) $$\int_{C_r} d\theta = 2\pi.$$

Let $w(z)$ be one-valued and meromorphic on F. We put

(8) $$m(r,a)=\frac{1}{2\pi}\int_{C_r}\log\frac{1}{[w(z),a]}d\theta,$$

where

$$[a,b]=\frac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}}.$$

Let $n(r,a)$ be the number of zero points of $w(z)-a$ in $\overline{F}_0+\varDelta_r$ and put

(9) $$N(r,a)=\int_1^r\frac{n(r,a)}{r}dr - C(a), \qquad C(a)=m(1,a).$$

(10) $$T_n(r,a)=m(r,a)+N(r,a),$$

(11) $$A(r)=A_0+\iint_{\varDelta_r}\left(\frac{|w'|}{1+|w|^2}\right)^2 r\,dr\,d\theta, \qquad S(r)=\frac{A(r)}{\pi},$$

(12) $$T(r)=\int_1^r\frac{S(r)}{r}dr,$$

where $w'=\dfrac{dw}{d\zeta}$, $\zeta=re^{i\theta}$ and A_0 is the area of the image of F_0 by $w=w(z)$ on the w-sphere K.

Then we shall prove an analogue of Nevanlinna's first fundamental theorem.

THEOREM 1. $\quad T_n(r,a)=T_n(r) \quad (1\leqq r\leqq r_n).$

Though $T_n(r)$ is defined for $1\leqq r\leqq r_n$, it is enough for our purpose.

Proof. Considering $w(z)$ as a function of $\zeta=re^{i\theta}$, we have

$$\frac{dm(r,a)}{dr}-\frac{dm(r,b)}{dr}=\frac{1}{2\pi}\int_{C_r}\frac{\partial}{\partial r}\log\left|\frac{w-b}{w-a}\right|d\theta$$
$$=\frac{1}{2\pi r}\int_{C_r} d\arg\left(\frac{w-b}{w-a}\right)=\frac{n(r,b)-n(r,a)}{r},$$

$$\frac{dm(r, a)}{dr} + \frac{n(r, a)}{r} = \frac{dm(r, b)}{dr} + \frac{n(r, b)}{r},$$

hence integrating on [1, r], we have by (9),

$$T_n(r, a) = T_n(r, b).$$

Let $d\omega(b)$ be the surface element on K, then

$$T_n(r, a) = \frac{1}{\pi}\iint_K T_n(r, b)d\omega(b) = \frac{1}{\pi}\iint_K m(r, b)d\omega(b)$$
$$+ \frac{1}{\pi}\iint_K N(r, b)d\omega(b) = \int_1^r \frac{S(r)}{r}dr + \text{const.}.$$

If we put $r=1$, then we see that const. $=0$, so that

$$T_n(r, a) = T_n(r).$$

2. To prove that $0 \leqq C(a) = m(1, a) \leqq K$, where K is a constant independent of a and n, we shall prove a lemma:

LEMMA. *Let $f(z) = u(z) + iv(z)$ ($f(0)=0$) be regular for $|z| \leqq 1$ ($z = x+iy$) and $v(x) = 0$ for $-1 \leqq x \leqq 1$, $v(z) > 0$ for $y > 0$, $|z| \leqq 1$ and $v(z) = -v(\bar{z})$ for $y < 0$, $|z| \leqq 1$.*

Then $f(z)$ is schlicht in $|z| \leqq 1/7$.

Proof. By the hypothesis,

$$(1) \qquad v(z) = v(re^{i\theta}) = a_1 r \sin\theta + \sum_{n=2}^{\infty} a_n r^n \sin n\theta \qquad (a_1 > 0),$$

where

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} v(e^{i\theta}) \sin\theta d\theta = \frac{2}{\pi}\int_0^{\pi} v(e^{i\theta}) \sin\theta d\theta,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(e^{i\theta}) \sin n\theta d\theta = \frac{2}{\pi}\int_0^{\pi} v(e^{i\theta}) \sin n\theta d\theta.$$

Since $|\sin n\theta| \leqq n|\sin\theta|$,

$$(2) \qquad |a_n| \leqq \frac{2}{\pi}\int_0^{\pi} v(e^{i\theta})|\sin n\theta| d\theta \leqq \frac{2n}{\pi}\int_0^{\pi} v(e^{i\theta}) \sin\theta d\theta = na_1.$$

Since $f(z) = a_1 z + \sum_{n=2}^{\infty} a_n z^n$, we have for $|z_1| \leqq r$, $|z_2| \leqq r$ ($r<1$),

$$\left|\frac{f(z_1) - f(z_2)}{z_1 - z_2}\right| = \left|a_1 + \sum_{n=2}^{\infty} a_n (z_1^{n-1} + \dots + z_2^{n-1})\right| \geqq a_1 - \sum_{n=2}^{\infty} n|a_n| r^{n-1}$$
$$\geqq a_1\Big(1 - \sum_{n=2}^{\infty} n^2 r^{n-1}\Big) = a_1 \frac{1 - 7r + 6r^2 - 2r^3}{(1-r)^3} > a_1 \frac{1-7r}{(1-r)^3} \geqq 0, \quad \text{if} \quad r \leqq 1/7.$$

Hence $f(z)$ is schlicht in $|z| \leqq 1/7$. q.e.d.

We shall prove

THEOREM 2. $$0 \leqq C(a) = m(1, a) \leqq K,$$

where K is a constant, which is independent of a and n.

Proof. Let z_0 be a point of Γ_0 and U_0 be its neighbourhood, which consists of regular points of F_1. Then for a suitable U_0, we can map U_0 on $|\tau|<1$ conformally, such that z_0 becomes $\tau = 0$ and the part of U_0, which lies in $F_1 - F_0$ is mapped on the upper half of $|\tau|<1$ and the part of Γ_0, which lies in U_0 becomes the diameter L of $|\tau| = 1$ through $\tau = -1$ and $\tau = 1$.

We put $u_n(z) = u(\tau)$, then $u(\tau) = 0$ on L, $u(\tau) > 0$ on the upper half of $|\tau|<1$, so that $u(\tau)$ can be continued harmonically across L in the lower half of $|\tau|<1$, by putting $u(\tau) = -u(\bar{\tau})$. Hence if we put $f_n(z) = -v_n(z) + iu_n(z) = f(\tau)$, then by the lemma, $f(\tau)$ is schlicht in $|\tau| \leqq 1/7$. Hence there exists a constant R, such that $f_n(z) = -v_n(z) + iu_n(z)$ is regular and schlicht in $|z - z_0| \leqq R$ for any z_0 of Γ_0.

Hence by Koebe's distortion theorem, there exists a constant K_0, such that for any two z_1, z_2 on Γ_0,

$$|f'_n(z_1)| \leqq K_0 |f'_n(z_2)|.$$

Since $u_n = 0$ on Γ_0, we have

$$0 \leqq \frac{dv_n(z_1)}{ds} \leqq K_0 \frac{dv_n(z_2)}{ds}, \tag{1}$$

where ds is the arc element of Γ_0 and we choose the sense of Γ_0 positive with respect to F_0. Hence if we put

$$M_n = \operatorname*{Max}_{\Gamma_0} \frac{dv_n(z)}{ds}, \qquad m_n = \operatorname*{Min}_{\Gamma_0} \frac{dv_n(z)}{ds}, \tag{2}$$

then

$$M_n \leqq K_0 m_n. \tag{3}$$

Now

$$m(1, a) = \frac{1}{d_n}\int_{\Gamma_0} \log \frac{1}{[w, a]} \frac{dv_n(z)}{ds} ds \leqq \frac{M_n}{d_n}\int_{\Gamma_0} \log \frac{1}{[w, a]} ds \leqq \frac{K_1 M_n}{d_n}, \tag{4}$$

where K_1 is a constant independent of a and n.

Since

$$d_n = \int_{\Gamma_0} \frac{dv_n}{ds} ds \geqq L m_n, \tag{5}$$

where L is the length of Γ_0, we have

$$m(1, a) \leqq \frac{K_1 M_n}{L m_n} \leqq \frac{K_1 K_0}{L} = K, \tag{6}$$

where K is a constant independent of a and n.

3. By means of Theorems 1 and 2, we shall prove

THEOREM 3.[2] *Let $n(a)$ be the number of zero points of $w(z)-a$ in F and*

$$n_0 = \sup_a . n(a).$$

Let E be the set of a, such that $n(a) < n_0$, then E is of logarithmic capacity zero.

Proof. First suppose that $n_0 < \infty$. Then there exists a_0, such that $n(a_0) = n_0$. We take n so large that $w(z)-a_0$ has n_0 zeros in F_n. Then for any δ $(0<\delta<1)$, we take m so large that the part of F_m, such that $r_m^\delta \leqq r_m(z) \leqq r_m$ lies outside of F_n, then

$$(1) \qquad T_m(r_m) \geqq N(r_m, a_0) \geqq n_0 \int_{r_m^\delta}^{r_m} \frac{dr}{r} - C(a) \geqq n_0(1-\delta)\log r_m - O(1).$$

Let E be the set of a, such that $n(a) \leqq n_0 - 1$ and suppose that cap. $E>0$, then we may assume that E is a bounded closed set. Let $u(w)$ be the equilibrium potential of E:

$$u(w) = \int_E \log \frac{1}{[w, a]} d\mu(a), \qquad \int_E d\mu(a) = 1,$$

such that $u(w)$ is bounded on the w-sphere K. Then from $m(r, a) + N(r, a) = T_m(r)$, we have

$$O(1) + \int_E N(r_m, a) d\mu(a) = T_m(r_m),$$

so that

$$(2) \qquad T_m(r_m) \leqq (n_0 - 1)\log r_m + O(1).$$

Since $r_m \to \infty$, we have from (1), (2),

$$n_0(1-\delta) \leqq n_0 - 1,$$

which is impossible, if $\delta < 1/n_0$. Hence cap. $E = 0$.

If $n_0 = \infty$, then for any $N>0$, there exists a_0, such that $n(a_0) \geqq N$, then the set of a, such that $n(a) \leqq N-1$ is of logarithmic capacity zero. Since N is arbitrary, the set of a, such that $n(a) < \infty$ is of logarithmic capacity zero.

Remark. Let Φ be the Riemann surface of the inverse function $z = z(w)$ of $w(z)$ spread over the w-sphere K. If $n_0 < \infty$, then the set of a, such that $n(a) = n_0$ is an open set, so that Φ consists of n_0 sheets and the projection of singular points of $z(w)$ on K is a closed set of logarithmic capacity zero.

From the above proof, we have easily

2) Y. Nagai: On the behaviour of the boundary of Riemann surfaces, II. Proc. Japan Acad. **26** (1950). Z. Yûjôbô: On the Riemann surfaces, no Green's function of which exists. Mathematica Japonicae. II, No. 2 (1951). M. Tsuji: Some metrical theorems on Fuchsian groups. Kodai Math. Seminar Reports. Nos. 4-5 (1950). A. Mori: On Riemann surfaces on which no bounded harmonic function exists. Journ. Math. Soc. Japan. **3** (1951).

THEOREM 4. *If* $n_0 = \sup_a n(a) = \infty$, *then* $w(z)$ *takes any value infinitely often, except a set of logarithmic capacity zero and*

$$\lim_{n\to\infty} \frac{T_n(r_n)}{\log r_n} = \infty.$$

Conversely, if this condition is satisfied, then $w(z)$ *takes any value infinitely often, except a set of logarithmic capacity zero.*

§ 2

1. Let Φ be the Riemann surface of the inverse function $z = z(w)$ of $w(z)$ spread over the w-plane and w_0 be its regular point. We continue $z(w)$ along a half-line $L(\varphi)$: $\arg (w - w_0) = \varphi$ till we meet a singular point of $z(w)$. Then we obtain the Mittag-Leffler's principal star region $H(w_0)$. Let E be the set of φ, such that $L(\varphi)$ meets a singular point of $z(w)$ at a finite distance. Then

THEOREM 5.[3] *E is of measure zero.*

This is an extension of Gross' theorem.[4]

Proof. Let $H_R(w_0)$ be the part of $H(w_0)$, which lies in $|w - w_0| < R$ and E_R be the set of φ, such that $L(\varphi)$ meets a singular point of $z(w)$ in $|w - w_0| < R$. Let F_R be the image of $H_R(w_0)$ on F and $C_r(R)$ be the part of C_r contained in F_R and $s(r)$ be the length of its image in $H_R(w_0)$, then writing $w' = \dfrac{dw}{d\zeta}$, $\zeta = re^{i\theta}$, we have

$$s(r)^2 = \left(\int_{C_r(R)} |w'|\, r d\theta\right)^2 \leqq 2\pi r \int_{C_r(R)} |w'|^2 r d\theta = 2\pi r \frac{dA(r)}{dr},$$

where $A(r)$ is the area of the image of $\Delta_r \cdot F_R$ in $H_R(w_0)$. Hence

$$\int_{\sqrt{r_n}}^{r_n} \frac{s(r)^2}{r} dr \leqq 2\pi A(r_n) \leqq 2\pi^2 R^2.$$

Hence if we put $\underset{\sqrt{r_n} \leqq r \leqq r_n}{\operatorname{Min}} s(r) = s_n$, then

$$s_n^2 \log r_n \leqq 4\pi^2 R^2.$$

Since $r_n \to \infty$, we have $s_n \to 0$, so that $mE_R = 0$. Since R is arbitrary, we have $mE = 0$.

2. Let a Riemann surface F be spread over the z-shpere K. If there exists a sequence of compact Riemann surfaces $F_n \to F$, such that $L_n/|F_n| \to 0$ $(n \to \infty)$,

3) K. Noshiro: Open Riemann surface with null boundary. Nagoya Math. Journ. **3** (1951). Z. Yûjôbô. l. c. 2).

4) W. Gross: Über die Singularitäten analytischer Funktionen. Monatshefte f. Math. u. Phys. **29** (1918).

then F is called regularly exhaustible in Ahlfors' sense, where L_n is the length of the boundary of F_n and $|F_n|$ is its area measured on K.

THEOREM 6.[5] *The Riemann surface Φ of the inverse function $z(w)$ of $w(z)$ is regularly exhaustible in Ahlfors' sense.*

Proof. Writing $w' = \dfrac{dw}{d\zeta}$, $\zeta = re^{i\theta}$, we put as in § 1,

$$A(r) = A_0 + \iint_{\Delta r} \left(\frac{|w'|}{1+|w|^2} \right)^2 r dr d\theta, \tag{1}$$

$$L(r) = \int_{C_r} \frac{|w'|}{1+|w|^2} r d\theta. \tag{2}$$

Then

$$L(r)^2 \leqq 2\pi r \frac{dA(r)}{dr}. \tag{3}$$

(i) First suppose that $A(r_n) \to \infty$ $(n \to \infty)$ and suppose that

$$L(r) > (A(r))^{3/4} \tag{4}$$

for any r, such that $\sqrt{r_n} \leqq r \leqq r_n$, then

$$\int_{\sqrt{r_n}}^{r_n} \frac{dr}{r} \leqq 2\pi \int_1^{r_n} \frac{dA(r)}{(A(r))^{3/2}} \leqq \frac{4\pi}{A_0}.$$

Since $\displaystyle\int_{\sqrt{r_n}}^{r_n} \frac{dr}{r} = \frac{1}{2} \log r_n \to \infty$, this is absurd, hence there exists τ_n $(\sqrt{r_n} \leqq \tau_n \leqq r_n)$, such that

$$L(\tau_n) \leqq (A(\tau_n))^{3/4}. \tag{5}$$

Since $A(\tau_n) \to \infty$ with $A(r_n) \to \infty$, we have

$$\frac{L(\tau_n)}{A(\tau_n)} \leqq \frac{1}{(A(\tau_n))^{1/4}} \to 0 \qquad (n \to \infty). \tag{6}$$

(ii) If $A(r_n) \leqq K$ $(n \to \infty)$, then

$$\int_{\sqrt{r_n}}^{r_n} \frac{L(r)^2}{r} dr \leqq 2\pi A(r_n) \leqq 2\pi K,$$

so that there exists τ_n $(\sqrt{r_n} \leqq \tau_n \leqq r_n)$, such that $L(\tau_n) \to 0$, hence

$$\frac{L(\tau_n)}{A(\tau_n)} \to 0 \qquad (n \to \infty). \tag{7}$$

Hence our theorem is proved.

§ 3

1. As an application of Theorem 6, we shall prove an extension of Myrberg's

[5] K. Noshiro: l. c. 3).

theorem. Let F be a closed Riemann surface of genus $p \geqq 2$, spread over the z-sphere K. We make F become a surface of planar character by cutting along p disjoint ring cuts C_i $(i=1, 2, \ldots, p)$, and let F_0 be the resulting surface. We take infinitely many same samples as F_0 and connect them along the opposite shores of C_i as in the well known way, then we obtain a covering surface $F^{(\infty)}$ of F, which is of planar character. Hence by Koebe's theorem, we can map $F^{(\infty)}$ on a schlicht domain D on the ζ-plane. The boundary E of D is a bounded perfect set, which is the singular set of a certain linear group of Schottky type. Myrberg[6] proved that E is of positive logarithmic capacity, hence $F^{(\infty)}$ is of positive boundary.

We shall generalize this Myrberg's theorem as follows.

Instead of cutting F along p ring cuts, we cut F along q $(1 \leqq q \leqq p)$ ring cuts C_i $(i=1, 2, \ldots, q)$ and let F_0 be the resulting surface. We take infinitely many same samples as F_0 and connect them along the opposite shores of C_i $(i=1, 2, \ldots, q)$, then we obtain a covering surface $F_{(q)}^{(\infty)}$ of F, which is of infinite genus, if $q<p$.

2. We shall prove

THEOREM 7. *$F_{(1)}^{(\infty)}$ is of null boundary, while if $q \geqq 2$, $F_{(q)}^{(\infty)}$ is of positive boundary and there exists a non-constant bounded harmonic function $u(z)$ on $F_{(q)}^{(\infty)}$, whose Dirichlet integral $D[u]$ on $F_{(q)}^{(\infty)}$ is finite.*

Proof. (i). First we shall prove that $F_{(1)}^{(\infty)}$ is of null boundary. We cut F along $C_1 = C$ and let F_0 be the resulting surface. We take infinitely many same samples as F_0:

$$(1) \qquad \begin{array}{l} F_1', F_2', \ldots, F_n', \ldots \\ F_1'', F_2'', \ldots, F_n'', \ldots \end{array}$$

Let C^+, C^- be the both shores of C. When we consider them belong to the boundary of F_n', we denote them by $(C^+)_n'$, $(C^-)_n'$.

Similarly we define $(C^+)_n''$, $(C^-)_n''$ for F_n''.

We connect $\{F_n'\}$, $\{F_n''\}$ as follows.

We identify C^+ of F_0 with $(C^-)_1'$ of F_1', $(C^+)_1'$ of F_1' with $(C^-)_2'$ of F_2' and so on. We identify C^- of F_0 with $(C^+)_1''$ of F_1'', $(C^-)_1''$ of F_1'' with $(C^+)_2''$ of F_2'' and so on and put

$$(2) \qquad F_n = F_0 + \sum_{\nu=1}^{n} F_\nu' + \sum_{\nu=1}^{n} F_\nu'', \qquad F_n - F_{n-1} = F_n' + F_n''.$$

We take a circular disc Δ_0 in F_0 and let Γ_0 be its boundary.

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6) P. J. Myrberg: Die Kapazität der singulären Menge der linearen Gruppen. Ann. Acad. Fenn. Ser. A. Math.-Phys. **10** (1941). M. Tsuji: On the uniformization of an algebraic function of genus $p \geqq 2$. Tohoku Math. Journ. **3** (1951).

Then

(3) $$\Delta_0 \subset F_0 \subset F_1 \subset \ldots \subset F_n \to F^{(\infty)}_{(1)}.$$

The boundary Γ_n of F_n is

(4) $$\Gamma_n = (C^+)'_n + (C^-)''_n.$$

Let $u^{(0)}_n(z)$ be the harmonic measure of Γ_n with respect to $F_n - \bar{\Delta}_0$ and let $v^{(0)}_n(z)$ be its conjugate harmonic function and put

(5) $$d^{(0)}_n = \int_{\Gamma_0} dv^{(0)}_n(z), \qquad \mu^{(0)}_n = 2\pi / d^{(0)}_n.$$

Let $u'_n(z)$ be the harmonic measure of $(C^+)'_n$ with respect to F'_n, such that $u'_n(z) = 0$ on $(C^-)'_n$, $u'_n(z) = 1$ on $(C^+)'_n$ and let $v'_n(z)$ be its conjugate harmonic function.

Let $u''_n(z)$ be the harmonic measure of $(C^-)''_n$ with respect to F''_n, such that $u''_n(z) = 0$ on $(C^+)''_n$, $u''_n(z) = 1$ on $(C^-)''_n$. We put

(6) $$d_n = \int_{(C^-)'_n} dv'_n(z) + \int_{(C^+)''_n} dv''_n(z), \qquad \mu_n = 2\pi / d_n.$$

Then as Noshiro[7] proved,

(7) $$\mu^{(0)}_n \geqq \mu^{(0)}_0 + \mu_1 + \ldots + \mu_n.$$

Since $\mu_n \geqq \text{const.} = a > 0$, we have $\lim_{n\to\infty} \mu^{(0)}_n = \infty$, so that $\lim_{n\to\infty} d^{(0)}_n = 0$, hence $F^{(\infty)}_{(1)}$ is of null boundary.

(ii) Next we shall prove that $F^{(\infty)}_{(q)}$ $(q \geqq 2)$ is of positive boundary. Suppose that $F^{(\infty)}_{(q)}$ is of null boundary, then by Theorem 6, $F^{(\infty)}_{(q)}$ is regularly exhaustible in Ahlfor's sense, so that there exists a sequence of compact Riemann surfaces: $F_1 \subset F_2 \subset \ldots \subset F_n \to F^{(\infty)}_{(q)}$, such that

(1) $$\frac{L_n}{S_n} \to 0, \qquad (n \to \infty),$$

where L_n is the length of the boundary Γ_n of F_n, measured on the z-sphere K and

(2) $$S_n = \frac{|F_n|}{|F|},$$

where $|F_n|$, $|F|$ are the spherical areas of F_n and F respectively.

As seen from the proof of Theorem 6, Γ_n is the niveau curve of a harmonic measure, so that Γ_n consists of a finite number ν_n of disjoint closed curves, which are not homotop null, hence the length of each curve is $\geqq a > 0$, where a is a constant, which depends on F only. Hence

(3) $$L_n \geqq a\nu_n.$$

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[7] K. Noshiro: l, c. 3).

We denote the Euler's characteristic of F_n by ρ_n.

Let C_i $(i=q+1, \ldots, p)$ be covered $\mu_i^{(n)}$-times by F_n, then we see easily that

$$\rho_n \leqq 2(\mu_{q+1}^{(n)}+ \ldots + \mu_p^{(n)}) + \nu_n \leqq 2(\mu_{q+1}^{(n)}+ \ldots + \mu_p^{(n)}) + L_n/a. \tag{4}$$

Now by Ahlfors' second covering theorem,[8]

$$\mu_i^{(n)} \leqq S_n + hL_n, \tag{5}$$

where h is a constant, which depends on F only, so that

$$\rho_n \leqq 2(p-q)S_n + hL_n \tag{6}$$

with a suitable h.

Since $\rho_0 = 2(p-1)$ is the Euler's characteristic of F, we have by Ahlfors' fundamental theorem on covering surfaces,[9]

$$\rho_n \geqq 2(p-1)S_n - hL_n, \tag{7}$$

so that by (6),

$$2(q-1)S_n \leqq hL_n, \tag{8}$$

which contradicts (1), if $q \geqq 2$. Hence $F_{(q)}^{(\infty)}$ $(q \geqq 2)$ is of positive boundary.

Next we shall prove that there exists a non-constant bounded harmonic function $u(z)$ on $F_{(q)}^{(\infty)}$, whose Dirichlet integral $D[u]$ is finite.

We take off F_0 from $F_{(q)}^{(\infty)}$, then there remains $2q$ connected surfaces Φ_i^+, Φ_i^- $(i=1, 2, \ldots, q)$, where Φ_i^+ abutts on F_0 along C_i^+ and Φ_i^- abutts on F_0 along C_i^-.

We exhaust $\Phi_1^+ = \Phi$ by a sequence of compact Riemann surfaces: $\Phi_1 \subset \Phi_2 \subset \ldots \subset \Phi_n \rightarrow \Phi$, where $C_1^+ + \Gamma_n$ is the boundary of Φ_n. Let $u_n(z)$ be the harmonic measure of Γ_n with respect to Φ_n and let $v_n(z)$ be its conjugate harmonic function and put

$$d_n = \int_{C_1^+} dv_n(z). \tag{9}$$

Since $u_n(z)$ decreases with n, d_n decreases with n.

From the above proof, we see that

$$\lim_{n\rightarrow\infty} d_n > 0. \tag{10}$$

Let

$$\lim_{n\rightarrow\infty} u_n(z) = u(z) \tag{11}$$

and $v(z)$ be its conjugate harmonic function and put

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8) L. Ahlfors: Zur Theorie der Überlagerungsflächen. Acta Math. **65** (1935).

9) L. Ahlfors: l. c. 8).

$$d = \int_{C_1^+} dv(z), \tag{12}$$

then $\lim_{n\to\infty} d_n = d > 0$, so that $u(z) \not\equiv$ const., hence $u(z) = 0$ on C_1^+, $0 < u(z) < 1$ in Φ.

Let $D[u]$ be the Dirichlet integral of $u(z)$ on Φ, then

$$0 < D[u] \leqq d < \infty. \tag{13}$$

Hence there exists a non-constant bounded harmonic function on Φ, which vanishes on C_1^+ and whose Dirichlet integral is finite. Similarly there exists a similar harmonic function on Φ_i^+, Φ_i^-.

Hence as proved by R. Nevanlinna[10] and Bader and Pareau,[11] there exists a non-constant bounded harmonic function on $F_{(q)}^{(\infty)}$, whose Dirichlet integral is finite.

Remark. By Sario's theorem,[12] there exists no non-constant one-valued regular function on $F_{(q)}^{(\infty)}$, whose Dirichlet integral is finite.

§ 4

Let $w(z)$ be one valued and meromorphic on F and Φ be the Riemann surface of the inverse function $z(w)$ of $w(z)$ spread over the w-plane and $\Phi^{(\rho)}$ be a connected piece of Φ, which lies above $|w - w_0| < \rho$ and $F^{(\rho)}$ be its image on F. We assume that $F^{(\rho)}$ is non-compact.

With the same notations as § 1 we put

$$\Delta_r^{(\rho)} = \Delta_r \cdot F^{(\rho)}, \quad F_n^{(\rho)} = F_n \cdot F^{(\rho)}, \quad C_r^{(\rho)} = C_r \cdot F^{(\rho)}. \tag{1}$$

For the sake of brevity we assume that $w_0 = 0$.

To define $m(r, a)$, we introduce a metric (a, b) in $|w| < \rho$ as follows.[13]

For $|a| < \rho$, we put

$$(a, 0) = \frac{2\rho|a|}{\rho^2 + |a|^2}. \tag{2}$$

Let $U_a(w) = \dfrac{\rho^2(w-a)}{\rho^2 - \bar{a}w}$, then for $|a| < \rho$, $|b| < \rho$, we define (a, b) by

$$(a, b) = (U_a(b), 0) = \frac{2\rho|b-a|/|\rho^2 - \bar{a}b|}{1 + \rho^2|b-a|^2/|\rho^2 - \bar{a}b|^2}. \tag{3}$$

By this metric, we put

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10) R. Nevanlinna: Über der Existenz von beschränkten Potentialfunktionen auf Flächen von unendlichem Geschlecht. Math. Zeits. **52** (1950).

11) R. Bader and M. Pareau: Domaines non-compacts et classification des surfaces de Riemann. C.R. **232** (1951). A. Mori: On the existence of harmonic functions on a Riemann surface. Journ. Fac. Sci. Tokyo Univ. Section I, Vol. VI, Part 4 (1951).

12) L. Sario: Über Riemannsche Fläche mit hebbarem Rand. Ann. Acad. Fenn. A. I. **50** (1948).

13) M. Tsuji: On a regular function which is of constant absolute value on the boundary of an infinite domain. Tohoku Math. Journ. **3** (1951).

$$(4)\qquad m(r, a) = \frac{1}{2\pi}\int_{C_r^{(\rho)}} \log \frac{1}{(w(z), a)} d\theta.$$

We use the same notations as § 1, since no confusion occurs.

Let $n(r, a)$ be the number of zero points of $w(z) - a$ in $\overline{F}_0^{(\rho)} + \Delta_r^{(\rho)}$ and put

$$(5)\qquad N(r, a) = \int_1^r \frac{n(r, a)}{r} dr - C(a), \qquad C(a) = m(1, a).$$

Then as Theorem 2,

$$(6)\qquad 0 \leqq C(a) \leqq K,$$

where K is a constant independent of a and n. We put

$$(7)\qquad T_n(r, a) = m(r, a) + N(r, a),$$

$$(8)\qquad A(r) = A_0 + \iint_{\Delta_\rho^{(r)}} \left(\frac{|w'|}{1+|w|^2}\right)^2 r dr d\theta, \quad S(r) = \frac{A(r)}{\sigma(\rho)},$$

where A_0 is the area of the image of $F_0^{(\rho)}$ on the w-sphere K by $w = w(z)$, $w' = \dfrac{dw}{d\zeta}$, $\zeta = re^{i\theta}$ and $\sigma(\rho) = \dfrac{\pi\rho^2}{1+\rho^2}$ is the area of the projection of $|w| \leqq \rho$ on K.

$$(9)\qquad T_n(r) = \int_1^r \frac{S(r)}{r} dr,$$

$$(10)\qquad L(r) = \int_{C_r^{(\rho)}} \frac{|w'|}{1+|w|^2} r d\theta.$$

Then similarly as my former paper,[14] we have

THEOREM 8. $T_n(r, a) = T_n(r) + O(\Phi(r)), \qquad (1 \leqq r \leqq r_n),$

where

$$\Phi(r) = \int_1^r \frac{L(r)}{r} dr.$$

For $|a| \leqq \rho_1 < \rho$,

$$|O(\Phi(r))| \leqq K\Phi(r),$$

where K is a constant, which depends ρ_1 on only.

THEOREM 9. *For any δ $(0<\delta<1)$, there exists τ_n $(r_n^{1-\delta} \leqq \tau_n \leqq r_n)$ $(n \geqq n_0)$, such that*

$$\Phi(\tau_n) \leqq \sqrt{T_n(\tau_n)} \log T_n(\tau_n).$$

Hence for any $\varepsilon > 0$,

$$(1-\varepsilon)T_n(\tau_n) \leqq T_n(\tau_n, a) \leqq (1+\varepsilon)T_n(\tau_n) \qquad (n \geqq n_1).$$

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14) M. Tsuji: l. c. 13).

Proof. We follow Dinghas.[15] By (10), (8),

$$L(r)^2 \leqq 2\pi r \int_{C_r^{(\rho)}} \left(\frac{|w'|}{1+|w|^2}\right)^2 r d\theta = 2\pi r \frac{dA(r)}{dr},$$

$$\frac{L(r)}{r} \leqq \sqrt{2\pi}\sqrt{\frac{A'(r)}{r}},$$

so that

$$\Phi(r) \leqq \sqrt{2\pi}\int_1^r \sqrt{\frac{A'(r)}{r}}\,dr.$$

Hence

$$(1) \qquad (\Phi(r))^2 \leqq 2\pi \int_1^r \frac{dr}{r}\int_1^r A'(r)dr = 2\pi\sigma(\rho) r \log r \cdot T_n'(r).$$

Suppose that

$$(2) \qquad \Phi(r) > \sqrt{T_n(r)}\,\log T_n(r)$$

for any r, such that $r_n^{1-\delta} \leqq r \leqq r_n$, then

$$\int_{r_n^{1-\delta}}^{r_n} \frac{dr}{r\log r} \leqq 2\pi\sigma(\rho)\int_{r_n^{1-\delta}}^{r_n} \frac{dT_n(r)}{T_n(r)\log^2 T_n(r)} \leqq 2\pi\sigma(\rho)\frac{1}{\log T_n(r_n^{1-\delta})} \to 0$$
$$(n \to \infty).$$

Since $\int_{r_n^{1-\delta}}^{r_n} \frac{dr}{r\log r} = \log\frac{1}{1-\delta}$, this is absurd, hence there exists τ_n ($r_n^{1-\delta} \leqq \tau_n \leqq r_n$) ($n \geqq n_0$), such that

$$\Phi(\tau_n) \leqq \sqrt{T_n(\tau_n)}\,\log T_n(\tau_n).$$

THEOREM 10.[16] *Under the same condition as Theorem 8, let $n(a)$ be the number of zero points of $w(z)-a$ ($|a|<\rho$) in $F^{(\rho)}$ and*

$$n_0 = \sup_a .\, n(a).$$

Let E be the set of a, such that $n(a) < n_0$. Then E is of logarithmic capacity zero.

Proof. (i) First suppose that $n_0 < \infty$. Then there exists a_0, such that $n(a_0) = n_0$ and let $w(z) - a_0$ has n_0 zeros in $F_n^{(\rho)}$.

We take m so large that the part: $r_m^\delta \leqq r_m(z) \leqq r_m$ of $F_m^{(\rho)}$ lies outside of $F_n^{(\rho)}$. By Theorem 9, there exists τ_m ($r_m^{1-\delta} \leqq \tau_m \leqq r_m$), such that

$$(1) \qquad (1+\delta)T_m(\tau_m) \geqq T_m(\tau_m,\ a_0) \geqq N(\tau_m,\ a_0) \geqq n_0\int_{r_m^\delta}^{r_m^{1-\delta}} \frac{dr}{r} - C(a)$$
$$= n_0(1-2\delta)\log r_m - O(1).$$

Let E be the set of a, such that $n(a) \leqq n_0 - 1$ and suppose that cap. $E > 0$.

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[15] A. Dinghas: Eine Bemerkung zur Ahlforsschen Theorie der Überlagerungsflächen. Math. Zeits. **44** (1936).

[16] Y. Nagai: l. c. 2). Z. Yûjòbô: l. c. 2). M. Tsuji: l. c. 2). A. Mori: l. c. 2).

Then we may assume that E is a closed set contained in $|w| \leqq \rho_1 < \rho$. Then similarly as Theorem 3, we have

(2) $$(1-\delta)T_m(r_m) \leqq (n_0 - 1)\log r_m + O(1).$$

Hence from (1), (2),

$$\frac{n_0(1-2\delta)}{1+\delta} \leqq \frac{n_0-1}{1-\delta},$$

which is impossible, if δ is sufficiently small. Hence cap. $E=0$. If $n_0 = \infty$, then we can prove similarly as Theorem 3, that the set E of a, such that $n(a) < \infty$ is of logarithmic capacity zero.

Mathematical Institute,
Tokyo University