

ON SOME ASYMPTOTIC PROPERTIES CONCERNING HOMOGENEOUS DIFFERENTIAL PROCESSES

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1. Introduction. About the behaviour of brownian motion at time point ∞ there are many results by P. Lévy and A. Khintchine etc. The method of W. Feller¹⁾ is applicable to a similar discussion about a homogeneous differential process. In this paper we shall study, applying his method, the properties of a homogeneous differential process.

Let $\{X(t, \omega); 0 \leq t < \infty, \omega \in \Omega\}$ ²⁾ be a homogeneous differential process such that $E(X(t)) = mt$ and $V(X(t)) = \sigma^2 t$.³⁾ After P. Lévy we shall define the concept of upper class and lower class with respect to a homogeneous differential process as follows: if the set of t such that

$$X(t, \omega) > \sigma \sqrt{t} \phi(t)$$

is bounded (unbounded) for almost all ω , then we say that $\phi(t)$ belongs to the upper (lower) class with respect to $\{X(t); 0 \leq t < \infty\}$. Then we may prove the following three theorems. In these theorems, the distribution function of $X(t)$ is denoted by $V_t(x)$.

THEOREM 1. *Let $\{X(t, \omega); 0 \leq t < \infty, \omega \in \Omega\}$ be a right continuous⁴⁾ homogeneous differential process satisfying the following conditions:*

$$(1) \quad \int_{|x-m| < z} |x-m|^3 dV_1(x) = O(z(\log \log z)^{-1/2}) \quad \text{as } z \rightarrow \infty.$$

For any $\varepsilon > 0$,

$$(2) \quad \int_{|x-m| > z} (x-m)^2 dV_1(x) = O((\log \log z)^{-(2+\varepsilon)})$$

or

$$(2)' \quad \int_{|x-m| > z} (x-m)^2 dV_1(x) = o((\log \log z)^{-2}) \quad \text{as } z \rightarrow \infty.$$

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¹⁾ W. Feller: "The law of the iterated logarithm for identically distributed random variables." *Ann. of Math.* vol. **47** (1946).

²⁾ ω is the probability parameter.

³⁾ The symbols E and V denote the expectation and the variance respectively.

⁴⁾ This is not an essential restriction.

There exist two positive numbers α and N such that, for $0 \leq t \leq \alpha$ ⁵⁾ and $0 < N \leq b - a$,

$$(3) \quad \int_{a \leq |x-mt| < b} dV_t(x) = O\left(t \int_{a \leq |x-m| < b} dV_1(x)\right) \quad \text{as } a \rightarrow \infty,$$

uniformly in t .

Then a monotone non-decreasing right continuous function $\phi(t)$ belongs to the upper (lower) class if, and only if,

$$(4) \quad \int_0^\infty \frac{1}{t} \phi(t) e^{-\frac{1}{2} e^{2t}} dt \in \mathfrak{C}(\mathfrak{D}).$$
⁶⁾

Example 1. For a Poisson process, the conditions (1), (2) and (3) are well satisfied.

Example 2. For a process of Pearson type, that is, a differential process $\{X(t, \omega); 0 \leq t < \infty\}$ such that

$$P_r\{X(t, \omega) \leq x\} = \begin{cases} \int_0^x \frac{e^{-y} y^{t-1}}{\Gamma(t)} dy & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

the conditions (1), (2) and (3) are well satisfied.

THEOREM 2. Let $\{X(t, \omega); 0 \leq t < \infty, \omega \in \Omega\}$ be a right continuous homogeneous differential process with symmetric distribution function $V_t(x)$. Then, in Theorem 1, we may remove the assumption (1) and palliate (2) as follows:

$$(2)'' \quad \int_{|x-m| > z} (x-m)^2 dV_1(x) = O((\log \log z)^{-1})$$

Example 3. For a Gaussian process, the conditions (2)'' and (3) are well satisfied.

THEOREM 3. Let $\{X(t, \omega); 0 \leq t < \infty, \omega \in \Omega\}$ be a right continuous homogeneous differential process. If $E((X(t) - mt)^4)$ is finite, the criterion (4) is valid.

2. Proofs. Without loss of generality we may assume that $m = 0$ and $\sigma = 1$.

LEMMA 1. Let $\phi(t)$ be a monotone non-decreasing right continuous function. If $\phi(t)$ does not belong to the upper class, then there exists a monotone increasing sequence $\{t_k\}$ such that $\{\phi_k = \phi(t_k)\}$ does not belong to the upper class with respect to $\{X_k; X_k = X(t_k) - X(t_{k-1})\}$.

Proof. Let $\phi(t)$ be a function which does not belong to the upper class. Then there exists a set $\Omega^* \subseteq \Omega$ with positive probability such that, for any $T > 0$,

⁵⁾ We may assume $\alpha \leq 1$ without losing generality.

⁶⁾ $\in \mathfrak{C}(\mathfrak{D})$ denotes the convergence (divergence) of the integrals.

there exists $t(\omega) > T$ such as

$$(5) \quad X(t, \omega) > \sqrt{t} \phi(t) \quad \text{when } \omega \in \mathcal{Q}^*.$$

Since $X(t, \omega)$ is right continuous in t , we have

$$(6) \quad X(r, \omega) > \sqrt{r} \phi(r) \quad \text{when } \omega \in \mathcal{Q}^*,$$

with a rational number $r(\omega) (> T)$. Let us put

$$P_r(\mathcal{Q}^*) = c > 0.$$

Let $\{r_i\}$ be the set of all rational numbers. We shall define $\mathcal{Q}_n^{(1)}$ as follows :

$$\mathcal{Q}_n^{(1)} = \{\omega \in \mathcal{Q}^* ; \exists r_i \leq n, X(r_i, \omega) > \sqrt{r_i} \phi(r_i)\} \quad (n = 1, 2, \dots),$$

where “ $\exists r_i \leq n$ ” means that there exists at least one r_i which does not exceed n . Then we have, by (6) (with exception of the set of zero measure),

$$(7) \quad \bigcup_n \mathcal{Q}_n^{(1)} = \mathcal{Q}^*, \quad \mathcal{Q}_1^{(1)} \subseteq \mathcal{Q}_2^{(1)} \subseteq \dots \subseteq \mathcal{Q}_n^{(1)} \subseteq \dots$$

Hence, for any $\varepsilon > 0$, we may take n_1 such as

$$(8) \quad P_r(\mathcal{Q}_{n_1}^{(1)}) \geq c - \varepsilon/2.$$

Let us put

$$(9) \quad \mathcal{Q}_{r_i} = \{\omega \in \mathcal{Q}^* ; X(r_i, \omega) > \sqrt{r_i} \phi(r_i)\}.$$

Then we have

$$\bigcup_{r_i \leq n} \mathcal{Q}_{r_i} = \mathcal{Q}_{n_1}^{(1)},$$

so that, if i_1 is sufficiently large, we obtain

$$(10) \quad P_r\left(\bigcup_{i \leq i_1} \mathcal{Q}_{r_i}\right) \geq P_r(\mathcal{Q}_{n_1}^{(1)}) - \varepsilon/2 \geq c - \varepsilon.$$

Rearranging $\{r_i ; i \leq i_1\}$ according to the order of magnitude, we obtain the set $\{t_1, \dots, t_{i_1}\}$. Again we shall adopt the following definition :

$$(11) \quad \mathcal{Q}_n^{(2)} = \begin{cases} \{\omega \in \mathcal{Q}^* ; \exists r_i, \max(i_1, t_{i_1}) < r_i \leq n \text{ and} \\ \quad X(r_i, \omega) > \sqrt{r_i} \phi(r_i)\} & \text{if } n > \max(i_1, t_{i_1}), \\ \text{empty set} & \text{otherwise.} \end{cases}$$

Then by (6)

$$\bigcup_n \mathcal{Q}_n^{(2)} = \mathcal{Q}^*, \quad \mathcal{Q}_1^{(2)} \subseteq \mathcal{Q}_2^{(2)} \subseteq \dots \subseteq \mathcal{Q}_n^{(2)} \subseteq \dots$$

Accordingly there exists n_2 such that

$$P_r(\mathcal{Q}_{n_2}^{(2)}) \geq c - \varepsilon^2/2$$

and

$$\bigcup_{\max(t_1, t_{i_1}) < r_i \leq n_2} \mathcal{Q}_{r_i} = \mathcal{Q}_{n_2}^{(2)}.$$

Therefore, if i_2 is sufficiently large, we have

$$(12) \quad P_r\left(\bigcup_{i_1 < i \leq i_2} \Omega_{r_i}\right) \geq P_r(\Omega_{n_2}^{(2)}) - \varepsilon^2/2 \geq c - \varepsilon^2.$$

By the same method as in the previous discussion we have a monotone sequence $\{t_{i_1+1}, \dots, t_{i_2}\}$. Repeating this process, we have a monotone sequence such that

$$(13) \quad t_1 < t_2 < \dots < t_{i_1} < \dots < t_{i_j} < t_{i_{j+1}} \dots < t_{i_{j+1}} < \dots, \\ (t_i \rightarrow \infty \text{ as } i \rightarrow \infty)$$

and

$$P_r\left(\bigcup_{i_{j-1} < i \leq i_j} \Omega_{t_i}\right) \geq c - \varepsilon^j.$$

Hence, if $\varepsilon < c/2$, we obtain

$$(14) \quad P_r\left(\bigcap_j \bigcup_{i_{j-1} < i \leq i_j} \Omega_{t_i}\right) \geq c - (\varepsilon + \varepsilon^2 + \dots + \varepsilon^n + \dots) = c - \frac{\varepsilon}{1-\varepsilon} > 0.$$

(9) and (14) show that $\{\phi_i\}$ does not belong to the upper class with respect to $\{X_i\}$.

According to the following lemma which will be proved after the method of W. Feller, we can exchange in Lemma 1 the condition “ $\{\phi_k\}$ does not belong to the upper class” by the condition “ $\{\phi_k\}$ belongs to the lower class.”

LEMMA 2. *Let the conditions in Theorem 1 be satisfied. Let $\{t_k\}$ be a monotone increasing sequence such that $t_k \rightarrow \infty$ (as $k \rightarrow \infty$) and $t_k - t_{k-1} \leq \alpha \leq 1$. Then the monotone increasing sequence $\{\phi_k = \phi(t_k)\}$ belongs to the upper (lower) class with respect to $\{X_k; X_k = X(t_k) - X(t_{k-1})\}$ if, and only if,*

$$(15) \quad \sum_k \frac{t_k - t_{k-1}}{t_k} \phi_k e^{-\frac{1}{2}\sigma_k^2} \in \mathfrak{G}(\mathfrak{D}).$$

Theorem 1 is a simple corollary to Lemma 1 and Lemma 2.

Proof of Theorem 1.

a) The case of convergence. Let us suppose that $\phi(t)$ does not belong to the upper class. Then, according to Lemma 1 and Lemma 2, there exists a monotone increasing sequence $\{t_k\}$ such that $t_k - t_{k-1} \leq \alpha$ and $\{\phi_k = \phi(t_k)\}$ belongs to the lower class with respect to $\{X_k; X_k = X(t_k) - X(t_{k-1})\}$. Hence by Lemma 2

$$\sum_k \frac{t_k - t_{k-1}}{t_k} \phi_k e^{-\frac{1}{2}\sigma_k^2} \in \mathfrak{D}.$$

On the other hand, by the monotony of $\phi(t)$ and the assumption of convergence,

$$\sum_k \frac{t_k - t_{k-1}}{t_k} \phi_k e^{-\frac{1}{2}\sigma_k^2} \leq \sum_k \int_{t_{k-1}}^{t_k} \frac{1}{t} \phi(t) e^{-\frac{1}{2}\sigma^2(t)} dt$$

$$= \int^{\infty} \frac{1}{t} \phi(t) e^{-\frac{1}{2} \beta^2(t)} dt \in \mathfrak{C}.$$

This is a contradiction. So $\phi(t)$ must belong to the upper class.

b) The case of divergence. Let us consider the monotone increasing sequences $\{t_k = k\alpha\}$ and $\{\phi_k = \phi(t_k)\}$. Then we have

$$\begin{aligned} \int^{\infty} \frac{1}{t} \phi(t) e^{-\frac{1}{2} \beta^2(t)} dt &= \sum_k \int_{t_{k-1}}^{t_k} \frac{1}{t} \phi(t) e^{-\frac{1}{2} \beta^2(t)} dt \\ &\cong \sum_k \frac{t_k - t_{k-1}}{t_{k-1}} \phi_{k-1} e^{-\frac{1}{2} \beta_{k-1}^2} \\ &= \sum_k \frac{t_k - t_{k-1}}{t_k} \phi_k e^{-\frac{1}{2} \beta_k^2}. \end{aligned}$$

Thus the divergence of the integrals yields that of the series (15). Therefore, by Lemma 2, $\{\phi_k\}$ must belong to the lower class and accordingly $\phi(t)$ belongs to the lower class with respect to the process $\{X(t, \omega)\}$.

Now our purpose is to prove Lemma 2. We put

$$(16) \quad \eta_k^2 = \begin{cases} t_k (\log \log t_k)^{-3} & \text{for } t_k > 80, \\ \text{arbitrary in such a way that } \{\eta_k\} \text{ becomes a} \\ \text{monotone increasing sequence} & \text{for } t_k \leq 80. \end{cases}$$

Furthermore we put

$$(17) \quad P_r\{X_k \leq x\} = F_k(x),$$

$$(18) \quad b_k = \int_{|x| < \eta_k} x^2 dF_k(x),$$

$$(19) \quad B_n = \sum_{k=1}^n b_k,$$

$$(20) \quad \mu_k' = - \int_{|x| < \eta_k} x dF_k(x), \quad \mu_k'' = - \int_{\eta_k \leq |x| < t_k^{1/2}} x dF_k(x),$$

$$\mu_k''' = - \int_{t_k^{1/2} \leq |x|} x dF_k(x)$$

and

$$(21) \quad \sigma_k^2 = b_k - \mu_k'^2, \quad s_n^2 = \sum_{k=1}^n \sigma_k^2.$$

We shall introduce three new sequences of random variables as follows;

$$(22) \quad X_k' = \begin{cases} X_k + \mu_k' & \text{if } |X_k| < \eta_k \\ \mu_k' & \text{otherwise,} \end{cases}$$

$$(23) \quad X_k'' = \begin{cases} X_k + \mu_k'' & \text{if } \eta_k \leq |X_k| < t_k^{1/2} \\ \mu_k'' & \text{otherwise,} \end{cases}$$

$$(24) \quad X_k''' = \begin{cases} X_k + \mu_k''' & \text{if } |X_k| \geq t_k^{1/2} \\ \mu_k''' & \text{otherwise.} \end{cases}$$

Then we have

$$(25) \quad X_k = X_k' + X_k'' + X_k''',$$

and the variables of each of the three sequences are mutually independent. Moreover

$$(26) \quad E(X_k) = E(X_k') = E(X_k'') = E(X_k''') = 0$$

and

$$(27) \quad V(X_k') = \sigma_k^2.$$

If we define S_n' as follows

$$(28) \quad S_n' = X_1' + X_2' + \dots + X_n',$$

(the sums S_n'' and S_n''' are defined similarly), then we have

$$(29) \quad V(S_n') = s_n^2.$$

LEMMA 3. *With probability one*

$$(30) \quad S_n''' = O(t_n^{1/2} (\log \log t_n)^{-1/2}).$$

Proof. From the assumption (3) we have

$$\begin{aligned} \sum_k P_T \{X_k''' \neq \mu_k'''\} &= \sum_k \int_{|x| \geq t_k^{1/2}} dF_k(x) = O(1) \sum_k (t_k - t_{k-1}) \int_{|x| \geq t_k^{1/2}} dV_1(x) \\ &= O(1) \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} (t_k - t_{k-1}) \int_{t_j^{1/2} \leq |x| < t_{j+1}^{1/2}} dV_1(x) \\ &= O(1) \sum_{j=1}^{\infty} \sum_{k=1}^j (t_k - t_{k-1}) \int_{t_j^{1/2} \leq |x| < t_{j+1}^{1/2}} dV_1(x) \\ &= O(1) \sum_{j=1}^{\infty} t_j \int_{t_j^{1/2} \leq |x| < t_{j+1}^{1/2}} dV_1(x) = O(1) \int_{-\infty}^{\infty} x^2 dV_1(x) < \infty. \end{aligned}$$

Thus, by Borel-Cantelli's lemma, it follows that with probability one there will be only finitely many k such that $X_k''' \neq \mu_k'''$. So, by the assumptions (1) and (2), we have

$$\begin{aligned} |S_n''| &= O(1) \left\{ 1 + \left| \sum_{k=1}^n \int_{|x| \geq t_k^{1/2}} x dF_k(x) \right| \right\} = O(1) \left\{ 1 + \sum_{k=1}^n (t_k - t_{k-1}) \int_{|x| \geq t_k^{1/2}} |x| dV_1(x) \right\} \\ &= O(1) \left\{ 1 + \sum_{k=1}^n \sum_{j=k}^{\infty} (t_k - t_{k-1}) \int_{t_j^{1/2} \leq |x| < t_{j+1}^{1/2}} |x| dV_1(x) \right\} \\ &= O(1) \left\{ 1 + \sum_{j=1}^{n-1} \sum_{k=1}^j (t_k - t_{k-1}) \int_{t_j^{1/2} \leq |x| < t_{j+1}^{1/2}} |x| dV_1(x) \right. \\ &\quad \left. + \sum_{j=n}^{\infty} \sum_{k=1}^n (t_k - t_{k-1}) \int_{t_j^{1/2} \leq |x| < t_{j+1}^{1/2}} |x| dV_1(x) \right\} \end{aligned}$$

$$\begin{aligned} &= O(1)\left\{1 + \sum_{j=1}^{n-1} t_j \int_{t_j^{1/2} \leq |x| < t_{j+1}^{1/2}} |x| dV_1(x) + \sum_{j=n}^{\infty} t_n \int_{t_j^{1/2} \leq |x| < t_{j+1}^{1/2}} |x| dV_1(x)\right\} \\ &= O(1)\left\{1 + \int_{|x| < t_n^{1/2}} |x|^3 dV_1(x) + t_n \int_{t_n^{1/2} \leq |x|} |x| dV_1(x)\right\} \\ &= O(1)\left(\frac{t_n}{\log \log t_n}\right)^{1/2} \end{aligned}$$

This proves the lemma.

LEMMA 4. *With probability one*

$$(31) \quad S''_n = O((t_n \log \log \log t_n)^{1/2}).$$

Proof. According to a theorem of L. Kronecker,⁷⁾ it will be sufficient to prove that the series

$$(32) \quad \sum_n \frac{1}{(t_n \log \log \log t_n)^{1/2}} X''_n$$

converges with probability one. By a theorem of Khintchine and Kolmogoroff,⁸⁾ it is sufficient to show that

$$(33) \quad \sum_n \frac{1}{t_n \log \log \log t_n} E(X''_n{}^2) \in \mathfrak{C}.$$

To prove (33) we shall consider the following function

$$(34) \quad S(n) = \min_{k \in T_n} k, \quad T_n = \left\{k; \frac{t_k^{1/2}}{(\log \log t_k)^{3/2}} > t_n^{1/2}\right\}.$$

Obviously $S(n)$ is monotone non-decreasing. Hence we can define the inverse function of $S(n)$ as follows

$$(35) \quad S^{-1}(n) = \min_{S(l) \leq n} l.$$

By the definition (23), we obtain

$$\begin{aligned} \sum_n \frac{1}{t_n \log_{(3)} t_n} E(X''_n{}^2) &\leq \sum_n \frac{1}{t_n \log_{(3)} t_n} \int_{\eta_n \leq |x| < t_n^{1/2}} x^2 dF_n(x) \\ &= O(1) \sum_n \frac{t_n - t_{n-1}}{t_n \log_{(3)} t_n} \int_{\eta_n \leq |x| < t_n^{1/2}} x^2 dV_1(x) \\ &= O(1) \sum_n \frac{t_n - t_{n-1}}{t_n \log_{(3)} t_n} \sum_{k=n}^{S(n)} \int_{\eta_k \leq |x| < \eta_{k+1}} x^2 dV_1(x) \\ &= O(1) \sum_k \int_{\eta_k \leq |x| < \eta_{k+1}} x^2 dV_1(x) \sum_{n=S^{-1}(k)}^k \frac{t_n - t_{n-1}}{t_n \log_{(3)} t_n} \end{aligned}$$

⁷⁾ K. Knopp: *Theorie und Anwendung der Unendlichen Reihen*, 2 ed., Berlin, 1924, p. 127.

⁸⁾ A. Kolmogoroff: *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin, 1933, p. 59.

$$= O(1) \sum_k \int_{\eta_k \leq |x| < \eta_{k+1}} x^2 dV_1(x) < \infty,$$

where $\log_{(k)}$ denotes the k -times iterated logarithm. This proves the lemma.

LEMMA 5. For any $\delta > 0$ the probability is zero that there exist infinite many n for which the inequalities

$$(36) \quad S''_n > \delta t_n^{1/2} / (\log \log t_n)^{1/2}$$

and

$$(37) \quad S'_n > \delta (t_n \log \log t_n)^{1/2}$$

hold simultaneously.

Proof. Let us denote by A_n the event that there exists at least one t_k such that

$$(38) \quad 10 t_n \log_{(2)}^{-3} t_n < t_k \leq t_n \quad \text{and} \quad X''_k \neq \mu''_k,$$

and by \bar{A}_n its complementary event. Choosing m for which $[t_m] = [10 t_n \log_{(2)}^{-3} t_n]^{91}$ holds, we have

$$\begin{aligned} \left| \sum_{k=m}^n \mu''_k \right| &= O(1) \sum_{k=m}^n (t_k - t_{k-1}) \int_{\eta_k \leq |x| < t_k^{1/2}} |x| dV_1(x) \\ &= O(1) \sum_{k=m}^n (t_k - t_{k-1}) \frac{1}{\eta_k (\log_{(2)} \eta_k)^{2+\varepsilon}} \\ &= O(1) \sum_{k=m}^n \frac{t_k - t_{k-1}}{t_k^{1/2} (\log_{(2)} t_k)^{1/2+\varepsilon}} \\ &= O(1) \frac{t_n^{1/2}}{(\log \log t_n)^{1/2+\varepsilon}}. \end{aligned}$$

Accordingly, if \bar{A}_n occurs, then we have by Lemma 4

$$\begin{aligned} S''_n &= S''_m + (S''_n - S''_m) = O((t_m \log_{(3)} t_m)^{1/2}) + O\left(\frac{t_n^{1/2}}{(\log \log t_n)^{1/2+\varepsilon}}\right) \\ &= o(t_n^{1/2} / (\log \log t_n)^{1/2}). \end{aligned}$$

This excludes (36). Therefore, for sufficiently large n , the event (36) will occur only in conjunction with the event A_n with probability one. Let B_n denote the event of a simultaneous realization of (37) and A_n . It suffices to prove that the probability that B_n occurs for infinitely many n is zero. To this purpose, we consider the event

$$(39) \quad C_\nu = \sum_{e^{\nu-1} < t_m \leq e^\nu} B_n$$

which implies the realization of at least one B_n with

⁹¹ $[x]$ denotes the largest integer which does not exceed x .

$$(40) \quad e^{\nu-1} < t_n \leq e^\nu.$$

Our lemma will be proved if we show that

$$(41) \quad \sum_\nu P_r(C_\nu) < \infty.$$

Put

$$(42) \quad P_\nu = \sum_{e^\nu \log^{-3} \nu < t_k \leq e^\nu} P_r\{X_k'' \neq \mu_k''\}.$$

Then we obtain

$$(43) \quad \sum_\nu \frac{P_\nu}{(\log \nu)^{100}} < \infty^{10)}$$

and

$$(44) \quad P_r(C_\nu) = O(1)P_\nu/(\log \nu)^{100}^{11)}$$

Accordingly the series (41) converges.

LEMMA 6. For any monotone increasing sequence $\{\phi_n\}$ the divergence (convergence) of the series (15) is a necessary and sufficient condition that with probability one the inequality

$$(45) \quad \sum_{k=1}^n X_k > B_n^{1/2} \phi_n$$

be satisfied for infinitely (only finitely) many n .

Proof. Without loss of generality, we may assume that

$$(46) \quad \log \log t_n \leq \phi_n^2 \leq 4 \log \log t_n.^{12)}$$

If a and b are sufficiently large and $b - a \geq N$, then we have by the assumption (3)

$$\int_a^b x^2 dF_n(x) = O(1)(t_n - t_{n-1}) \int_a^b x^2 dV_1(x).$$

So we have

$$\begin{aligned} \sigma_n^2 &= b_n - \mu_n^2 = \int_{|x| < \eta_n} x^2 dF_n(x) - \left(\int_{|x| < \eta_n} x dF_n(x) \right)^2 \\ &= t_n - t_{n-1} - \int_{|x| \geq \eta_n} x^2 dF_n(x) - \left(\int_{|\tau| < \eta_n} x dF_n(x) \right)^2 \\ &= t_n - t_{n-1} - O(1)(t_n - t_{n-1})(\log \log t_n)^{-(2+\epsilon)}. \end{aligned}$$

Thus $t_n - t_{n-1}/\sigma_n^2 \rightarrow 1$ and therefore $t_n/s_n^2 \rightarrow 1$ as $n \rightarrow \infty$. So the divergence (convergence) of (15) is equivalent to

^{10) 11) 12)} loc. cit. 1).

$$(47) \quad \sum_n \frac{\phi_n^2}{s_n^2} \phi_n e^{-\frac{1}{2} \phi_n^2} \in \mathfrak{D}(\mathfrak{L}).$$

According to a theorem of W. Feller,¹³⁾ (47) implies that with probability one there are infinitely (only finitely) many n such that

$$(48) \quad S'_n > s_n(\phi_n + c/\phi_n),$$

where c is an arbitrary constant. From the definition (19) and (21), we have

$$B_n - s_n^2 = \sum_{k=1}^n \mu_k^2$$

and

$$|\mu_k^2| = \left| \int_{|x| \cong \gamma_k} x dF_k(x) \right| \leq \frac{1}{\gamma_k} \int_{|x| \cong \gamma_k} x^2 dF_k(x) \leq \frac{t_k - t_{k-1}}{\gamma_k}.$$

Hence $B_n - s_n^2 = O((\log t_n)^2)$ and we may take $B_n^{1/2}$ for s_n in (48), so we have

$$S'_n > B_n^{1/2}(\phi_n + c/\phi_n).$$

Hence, using Lemma 3 and Lemma 5, the divergence of (15) yields that with probability one there exist infinitely many n for which the inequalities

$$(49) \quad S'_n > B_n^{1/2}(\phi_n + c/\phi_n)$$

and

$$(50) \quad |S''_n + S'''_n| < M(t_n/\log \log t_n)^{1/2},$$

where c is an arbitrary constant and M is a sufficiently large number, hold simultaneously. Let us put $c = 2M$ in (49). Then we see that with probability one there exist infinitely many n such that

$$(51) \quad \sum_{k=1}^n X_k = S'_n + S''_n + S'''_n > B_n^{1/2} \phi_n.$$

Conversely if (51) holds for infinitely many n with probability one, it follows, by (30) and (31), that with probability one

$$S'_n > \frac{1}{2} B_n^{1/2} \phi_n$$

for infinitely many n appearing in (51). From Lemma 5, it follows that with probability one there exist infinitely many n for which (50) and (51) hold simultaneously, so that we have

$$S'_n > B_n^{1/2}(\phi_n - 2M/\phi_n) > s_n(\phi_n - 2M/\phi_n).$$

This means that $\{\phi_n - 2M/\phi_n\}$ belongs to the lower class with respect to $\{X'_n\}$.

¹³⁾ W. Feller: "The general form of the so-called law of the iterated logarithm." Trans. Amer. Math. Soc. vol. 54 (1943), pp. 373-402.

Then, by a theorem of W. Feller,¹⁴⁾ we have

$$\frac{\sigma_n^2}{S_n} (\phi_n - 2M/\phi_n) e^{-\frac{1}{2}(\phi_n - 2M/\phi_n)^2} \in \mathfrak{D},$$

and accordingly

$$\sum_n \frac{\sigma_n^2}{S_n} \phi_n e^{-\frac{1}{2}\phi_n^2} \in \mathfrak{D}.$$

This is equivalent to the divergence of (15).

Now Lemma 2 will be proved easily.

Proof of Lemma 2. If the series (15) diverges, then it is clear, from Lemma 6, that for any constant c

$$\sum_{k=1}^n X_k > B_n^{1/2}(\phi_n + c/\phi_n)$$

will be satisfied for infinitely many n with probability one. Therefore it is sufficient to show that

$$(52) \quad t_n^{1/2} - B_n^{1/2} = O(t_n^{1/2}/\phi_n^2)$$

or, by (46),

$$(53) \quad t_n - B_n = O(t_n/\log \log t_n).$$

But we have

$$\begin{aligned} t_n - B_n &= \sum_{k=1}^n \int_{|x| \geq \gamma_k} x^2 dF_k(x) = O(1) \sum_{k=1}^n (t_k - t_{k-1}) \int_{|x| \geq \gamma_k} x^2 dV_1(x) \\ &= O(1)(t_n/\log \log t_n + t_n \int_{|x| \geq t_n^{1/2}/\log_{(2)} t_n} x^2 dV_1(x)) \end{aligned}$$

(the first term on the right is the contribution of the terms in the sum with $t_k < t_n/\log \log t_n$, and the integral is an upper bound for the contribution of the remaining terms). Hence, by the assumption (2), we have

$$t_n - B_n = O(t_n/\log \log t_n).$$

The converse is trivial.

Proof of Theorem 2. In the proof of Theorem 1, the condition (1) was used to evaluate

$$\sum_{k=1}^n \int_{|x| \geq t_k^{1/2}} x dF_k(x) = O(t_n/\log \log t_n)^{1/2}.$$

But this is equal to zero in our case. Also the condition (2) was used to evaluate

¹⁴⁾ loc. cit. 13).

$$\left| \sum_{k=1}^n \mu_k'' \right| = O(t_n / (\log \log t_n)^{1+2\varepsilon})^{1/2}$$

and

$$t_n - B_n = O(t_n / \log \log t_n).$$

In our case the former is equal to zero and for the latter the condition (2)'' is sufficient. These prove Theorem 2.

Proof of Theorem 3. Let $t = q/p$ be a rational number. Then we have

$$\begin{aligned} E((X(t))^4) &= \int_{-\infty}^{\infty} x^4 dV_t(x) = \int (X(q/p))^4 P(d\omega) \\ &= \int \{ (X(1/p) - X(0)) + (X(2/p) - X(1/p)) \\ &\quad + \dots + (X(q/p) - X(q-1/p)) \}^4 P(d\omega) \\ &= q \int_{-\infty}^{\infty} x^4 dV_{1/p}(x) + 3q(q-1)/p^2. \end{aligned}$$

Put $p = q$ and $E((X(1))^4) = a$. Then we obtain

$$a = E((X(1))^4) = p \int_{-\infty}^{\infty} x^4 dV_{1/p}(x) + 3(p-1)/p,$$

and accordingly

$$\int_{-\infty}^{\infty} x^4 dV_{1/p}(x) = (a - 3(p-1)/p)/p.$$

Hence

$$\begin{aligned} (54) \quad E((X(t))^4) &= E((X(q/p))^4) = a q/p - 3(1 - q/p) q/p \\ &= at - 3t(1-t). \end{aligned}$$

Since $E((X(t))^4)$ is a monotone increasing function of t , (54) holds for any real number t .

Let us consider the sequence $\{X_k\}$ in Lemma 2. Using the notations in the previous proofs we have

$$\begin{aligned} P_r\{|X_k| \geq \eta_k\} &= \int_{|x| \geq \eta_k} dF_k(x) \\ &\leq \frac{1}{\eta_k^4} \int_{-\infty}^{\infty} x^4 dF_k(x) < a(t_k - t_{k-1})/\eta_k^4, \end{aligned}$$

and so

$$\sum_k P_r\{|X_k| \geq \eta_k\} \leq a \sum_k (t_k - t_{k-1})(\log \log t_k)^6 / t_k^2 \in \mathbb{C}.$$

According to Borel-Cantelli's lemma, it follows that with probability one there will be only finitely many k such that $|X_k| \geq \eta_k$.

Put

$$X'_k = \begin{cases} X_k + \int_{|x| \cong \eta_k} x dF_k(x) & \text{if } |X_k| < \eta_k \\ \int_{|x| \cong \eta_k} x dF_k(x) & \text{otherwise.} \end{cases}$$

Then we have

$$E(X'_k) = 0, \quad V(X'_k) = t_k - t_{k-1} + O(1)(t_k - t_{k-1})/\eta_k^2$$

and so

$$t_n - \sum_{k=1}^n V(X'_k) = O((\log t_n)^2).$$

On the other hand, we have

$$\sum_k \left| \int_{|x| \cong \eta_k} x dF_k(x) \right| \leq a \sum_k (t_k - t_{k-1})/\eta_k^3 \in \mathcal{O}.$$

Hence

$$\sum_{k=1}^n X_k = \sum_{k=1}^n X'_k + O(1).$$

By a theorem of W. Feller¹⁵⁾ the criterion (4) is valid for $\{X'_k\}$ and so for $\{X_k\}$. Thus we may apply Lemma 2, and Theorem 3 will be proved similarly as in the previous proof of Theorem 1.

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¹⁵⁾ loc. cit. 13).

