REMARKS ON THE DIFFERENTIAL FORMS OF THE FIRST KIND ON ALGEBRAIC VARIETIES

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§1. A differential form ω on a complete variety $\mathbf{U}^n$ is said to be of the first kind if it is finite at every simple point of any variety which is birationally equivalent to $\mathbf{U}$. Let k be a common field of definition for $\mathbf{U}$ and ω, and let $\mathbf{P}$ be a generic point of $\mathbf{U}$ over k. If ω is of the first kind, then $\omega(\mathbf{P})$ is of course a differential form of the first kind belonging to the extension $k(\mathbf{P})$ of k. With respect to the converse, we prove the following

THEOREM 1. *Let k be a field of definition for a complete variety $\mathbf{U}^n$ and a differential form ω on $\mathbf{U}$, and let $\mathbf{P}$ be a generic point of $\mathbf{U}$ over k. Let k be a perfect*[1] *field or more generally let k have a perfect*[1] *subfield which is a field of definition for $\mathbf{U}$. If $\omega(\mathbf{P})$ is a differential form of the first kind belonging to the extension $k(\mathbf{P})$ of k, then ω is of the first kind.*[2]

Proof. Let V be a variety which is birationally equivalent to $\mathbf{U}$ and let K be a field of definition of the birational correspondence between $\mathbf{U}$ and V. We may assume without loss of generality that K is algebraically closed and contains k and that $\mathbf{P}$ is a generic point of $\mathbf{U}$ over K. We want to show that ω is finite at every simple point of V. It suffices to show that $\omega(\mathbf{P})$, considered as the differential form belonging to the extension $K(\mathbf{P})$ of K, is of the first kind or that $\omega(\mathbf{P})$ is finite at every prime divisor $\mathfrak{P}$ in the sense of Zariski of $K(\mathbf{P})$ (= valuation of $K(\mathbf{P})$ of dimension $n-1$ over K), namely, that $\omega(\mathbf{P})$ is of the form

$$\omega(\mathbf{P}) = \sum z_{\alpha\beta}\ldots dy_\alpha dy_\beta \ldots ;$$

$z_{\alpha\beta}\ldots$, y_α, y_β, etc. being in the valuation ring of $\mathfrak{P}$.[3]

We first prove

LEMMA. *Let K be a field, k a subfield of K; let (x) be a set of quantities, such that K and $k(x)$ are independent over k. Then if v is a valuation of $K(x)$*

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1) If we omit the condition of perfectness this theorem does not hold in general.

2) If the problem of the reduction of the singularity over perfect field is solved affirmatively, this theorem is an immediate consequence of theorem 1 of S. Koizumi's paper; On the differential forms of the first kind on algebraic varieties, Journal of the Mathematical Society of Japan, Vol. **2**. However it would not be meaningless to give a simple direct proof.

3) See Y. Kawahara, On the differential forms on algebraic varieties, this journal, Vol. **4**, Theorem 1.

of dimension s over K, the induced valuation v' of $k(x)$ is of dimension not smaller than s over k.

Proof. Let R denote the valuation ring of v in $K(x)$, A its valuation ideal, and let R' denote the valuation ring of v' in $k(x)$, A' its valuation ideal. As v is of dimension s, there are s elements $y_1, \ldots, y_s$ in R which are algebraically independent mod A over K. Let K_0 be a finite extension field of k, such that all $y_1, \ldots, y_s$ belong to $K_0(x)$. Then v induces the valuation of $K_0(x)$ of dimension $\geqq s$ over K_0. Therefore we may assume that K is a finite extension field of k.

$$R/A \supseteqq R'/A' \supseteqq k,$$
$$R/A \supseteqq \quad K \quad \supseteqq k.$$

Let the dimension of K over k be t. Then R/A is of dimension $s+t$ over k. On the other hand the dimension of R/A over R'/A' is $\leqq t$. For, if $Z_1, \ldots, Z_{t+1}$ are $t+1$ elements in R, then as the dimension of $K(x)$ over $k(x)$ is t, there is an algebraic relation among them:

$$\sum a_{r_1 \ldots r_{t+1}} Z_1^{r_1} \ldots Z_{t+1}^{r_{t+1}} = 0$$

where all $a_{r_1 \ldots r_{t+1}}$ belong to $k(x)$. We may assume that all $a_{r_1 \ldots r_{t+1}}$ belong to R' and there is an element among them which does not belong to A'. Considering this relation mod A, we see that the dimension of R/A over R'/A' is $\leqq t$. Therefore the dimension of R'/A' over k is $\geqq s$.

From this lemma we see that $\mathfrak{P}$ induces in $k(\mathbf{P})$ the valuation $\mathfrak{p}$ of dimension at least $n-1$. As $\omega(\mathbf{P})$ is the differential form of the first kind belonging to the extension $k(\mathbf{P})$ of k, $\omega(\mathbf{P})$ is finite at $\mathfrak{p}$; hence $\omega(\mathbf{P})$ is finite at $\mathfrak{P}$. This completes the proof of Theorem 1.

§ 2. We prove the following

THEOREM 2. *Let U^n be a projective model without singular point and let ω be a differetial form on U^n, defined over k. Let U'^{n-1} be the generic hyperplane section of U^n (over k) on which ω induces the differential form ω' of the first kind. Then ω is of the first kind.*[4]

Proof. Let $\mathbf{U}'$ be the intersection of $\mathbf{U}^n$ and a hyperplane $\mathbf{H}$ defined by a homogeneous equation

$$\sum_{i=0}^{N} u_i X_i = 0$$

in $\mathbf{P}^N$. where $u_0, u_1, \ldots, u_N$ are algebraically independent over k. Let $\mathbf{W}^{n-1}$ be a subvariety of $\mathbf{U}^n$ which is algebraic over k and let $\mathbf{W}'^{n-2}$ be a component of $\mathbf{W} \cap \mathbf{H}$, which is contained in $\mathbf{U}'$.

[4] This theorem has been proved also by S. Koizumi.

Without loss of generality we may assume that $\mathbf{W}^{n-1}$ has a representative $\mathbf{W}_0^{n-1}$. Put $K=k(u_0, \ldots, u_N)$ and let $P=(x)$ be a generic point of U_0' over $\overline{K}$ and Q a generic point of $\mathbf{W}_0'$ over $\overline{K}$. Then P is also a generic point of U_0 over k and Q is a generic point of W_0 over $\bar{k}$. Further we may assume that $x_1, \ldots, x_n$ is a set of uniformizing parameters for U_0 at Q. Then it is easily seen that $x_2, \ldots, x_n$ is a set of uniformizing parameters for U_0' at Q. For simplicity we assume that ω is a simple differential form. We may treat the case of the differential form of the higher degrees analogously.

Let ω be defined by $\omega(P)=\sum_{i=1}^{n} a_i dx_i$, $a_i \in k(P)$. Then ω' is defined over $\overline{K}$ by $\omega'(P)=\sum_{i=1}^{n} a_i dx_i$, where $\sum_{i=1}^{n} a_i dx_i$ is considered as the differential form belonging to the extension $\overline{K}(P)$ of $\overline{K}$. Now since

$$u_0+u_1x_1+\ldots+u_Nx_N=0,$$
$$\sum_{j=1}^{N} u_j dx_j=0, \text{ i.e.}$$
$$-u_1dx_1=\sum_{k=2}^{N} u_k dx_k .$$

If we put $dx_k=\sum_{i=1}^{n} b_{ki}dx_i$, $b_{ki}\in k(P)$, $k=n+1, \ldots . N$, we get

$$-u_1dx_1=\sum_{i=2}^{n} u_i dx_i+\sum_{k=n+1}^{N} u_k dx_k$$
$$=\sum_{i=2}^{n}\Big(u_i+\sum_{k=n+1}^{N} u_k b_{ki}\Big)dx_i+\sum_{k=n+1}^{N} u_k b_{k1} dx_1 .$$
$$-dx_1=\sum_{i=2}^{n}\frac{\left(u_i+\sum_{k=n+1}^{N} u_k b_{ki}\right)}{\left(u_1+\sum_{k=n+1}^{N} u_k b_{k1}\right)}dx_i,$$
$$\omega'(P)=\sum_{i=2}^{n}\left(a_i-a_1\frac{\left(u_i+\sum_{k=n+1}^{N} u_k b_{ki}\right)}{\left(u_1+\sum_{k=n+1}^{N} u_k b_{k1}\right)}\right)dx_i .$$

By the assumptions that ω' is of the first kind and $x_2, \ldots, x_n$ form a set of uniformizing parameters,

$$A_i=a_i-a_1\frac{u_i+\sum_k u_k b_{ki}}{u_1+\sum_k u_k b_{k1}}$$

is in the specialization ring of Q in $\overline{K}(P)$, therefore A_i has a finite specialization over $P\to Q$ with respect to $\overline{K}$, and hence it has a finite specialization over $P\to Q$ with respect to $k(u_1, \ldots, u_N)$. Now as P is a generic point of U_0^n over $k(u_1, \ldots, u_N)$ and Q is a generic point of W_0^{n-1} over $\bar{k}(u_1, \ldots, u_N)$, either a or $1/a$ is in the specialization ring $\mathfrak{O}$ of Q in $k(u_1, \ldots, u_N)(P)$, where a is an arbitrary element in $k(u_1, \ldots, u_N)(P)$. Therefore A_i must be in the specialization ring $\mathfrak{O}$; moreover since $1/(u_1 + \sum_{k=n+1}^{N} u_k b_{k1})$ is in $\mathfrak{O}$,

$$A_i/(u_1 + \sum_k u_k b_{k1}) = a_i u_1 + a_1 u_i + \sum_{k=n+1}^{N} u_k \ (a_i b_{k1} + a_1 b_{ki})$$

is in $\mathfrak{O}$ for $i = 2, \ldots, n$, where a_i and b_{kj} are in $k(P)$. As $u_1, \ldots, u_N$ are algebraically independent over $k(P)$, a_i and a_1 must belong to the specialization ring of Q in $k(P)$.[5] This shows that $\omega(P)$ is finite at Q. Since ω is finite at the generic point of every $(n-1)$-dimensional subvariety of U, ω is of the first kind.[6]

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5) See A. Weil's book, Foundations of Algebraic Geometry, Prop. 8 in Chapter IV.
6) See Prop. 4 of Koizumi's paper loc. cit. 2).