CORRECTIONS TO MY PAPER "ON THE STRUCTURE OF COMPLETE LOCAL RINGS"[1]

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The proof of Proposition 2 and that of Corollary to Proposition 3 in my previous paper "On the structure of complete local rings"[1] are not correct.[2] Here we want to correct them.

Proof of Proposition 2.

Since the previous proof of Proposition 2 is valid when $R/\mathfrak{m}$ is perfect, we treat only the case when $R/\mathfrak{m}$ is not perfect.

Starting from $K_0 = R/\mathfrak{m}$, we obtain K_n $(n = 1, 2, \ldots)$ from K_{n-1} by adjoining all p-th roots of elements of K_{n-1}.

Definition. Let a local ring R_1 with maximal ideal $\mathfrak{m}_1$ be a subring of another local ring R_2 with maximal ideal $\mathfrak{m}_2$. We say that R_2 is unramified with respect to R_1 if $\mathfrak{m}_2 = \mathfrak{m}_1 R_2$ and $\mathfrak{m}_2^k \cap R_1 = \mathfrak{m}_1^k$ for every positive integer k.

(1) Equal characteristic case.

We construct a sequence of local rings $R = R^{(0)} \subset R^{(1)} \subset \ldots$ such that (1) $R^{(n)}$ is unramified with respect to R, (2) $R^{(n)}/\mathfrak{m}R^{(n)} = K_n$ and (3) $(R^{(n)})^p \subseteqq R^{(n-1)}$.

The existence of such a sequence obviously follows from Zorn's Lemma if we observe that a monic polynomial $f(x)$ over a local ring, say R^*, is irreducible modulo its maximal ideal, then $R^*[x]/(f(x))$ is unramified with respect to R^*. (We may use the p-basis).

Let S be the union of all $R^{(n)}$. Then S is a local ring unramified with respect to R. For every element a^* of $R/\mathfrak{m}$, we construct a sequence (a_n) as follows: Let b_n be a representative of $a^{*p^{-n}}$ in R_n and set $a_n = b_n^{p^n}$. Then $a_n \in R$ and the limit a, which is the multiplicative representative of a^*, is in R. Thus we have Proposition 2 in this case.

(2) Unequal characteristic case.

As in above, we construct a sequence of local rings $R = R^{(0)} \subset R^{(1)} \subset \ldots$ satisfying the above conditions (1) and (2) as follows: Let $\mathfrak{M} = \mathfrak{M}^{(0)}$ be a sys-

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[1] Nagoya Math. Journ. 1 (1950), pp. 63-70.

[2] Prof. I. S. Cohen (Massachusetts Institute of Technology, U.S.A.) pointed out the error of the proof of Proposition 2. I am grateful to him for his kind communication.

tem of representatives of a p-basis of M $R/\mathfrak{m}$. Let $\mathfrak{M}^{(n)}$ be, when $\mathfrak{M}^{(n-1)}$ is already given, a set such that (1) for every element of $\mathfrak{M}^{(n-1)}$, $\mathfrak{M}^{(n)}$ contains one and only one p-th root of it and (2) $\mathfrak{M}^{(n)}$ consists merely of p-th roots of elements of $\mathfrak{M}^{(n-1)}$. Set $R^{(n)}=R[\mathfrak{M}^{(n)}]$.

Let S be the union of all $R^{(n)}$ and let $\bar{S}$ be its completion. Then we see easily that the multiplicative representative of an arbitrary element of $\mathfrak{M}$ is itself. Let R_0 be the absolutely unramified local ring which is generated by multiplicative representatives for $R/\mathfrak{m}$. Now, for our purpose, it is sufficient to prove the following.

Lemma. For every element a of R_0, there exists an element a_n of R such that $a\equiv a_n$ (mod. $\mathfrak{m}^n\bar{S}$).

Proof. For $n=1$, our assertion is evident. We assume that this is true for $n=r$ and we prove the case $n=r+1$. Since $R_0/(p)=R/\mathfrak{m}=(R_0/(p))^{p^r}(M)$, we can find an element $c_1=\sum_i b_i^{p^r} m_i$ (where $b_i\in R_0$ and m_i is a monomial on elements of $\mathfrak{M}$) such that $a=c_1+pc_2$ ($c_2\in R_0$). Let b_i' be an element of R such that $b_i\equiv b_i'$ (mod. $\mathfrak{m}\bar{S}$) and let c' be an element of R such that $c\equiv c'$ (mod. $\mathfrak{m}^r\bar{S}$). Then $a_n=\sum_i b_i'^{p^r} m_i+pc'$ is a required element.

Proof of the Corollary to Proposition 3.

As is obvious, we have only to treat the case when $\bar{R}_0$ is of characteristic 0 and $p\neq 0$. Let B be a complete valuation ring (of characteristic 0) such that $B/(p)=\bar{R}_0/(p)$.

(1) When $\bar{R}_0/(p)$ is perfect:

Let $\{\bar{y}_\lambda\}$ be a transcendental basis for $\bar{R}_0/(p)$ over the prime field. Then we can find its multiplicative representative systems $\{y_\nu\}$, $\{z_\lambda\}$ in $\bar{R}_0$ and B. Then we can identify z_λ with y_λ. The same holds for $\{\bar{y}_\lambda^{p^{-n}}\}$ and the similar identification allows the above identification of y_λ and z_λ. Therefore we may consider that $\bar{R}_0$ and B contains the same complete valuation ring B_1 such that its residue field is the least perfect field containing $\{\bar{y}_\lambda\}$. Since $\bar{R}_0/(p)$ is separably algebraic over $B_1/(p)$ and since B is complete, we see that B and $\bar{R}_0$ are isomorphic over B_1.

(2) General case:

Considering $\bar{R}_0$ as R in the above proof of Proposition 2, we construct the valuation ring $\bar{S}$. Let K be the largest perfect subfield of $\bar{R}_0/(p)$. Then using multiplicative representatives for K in $\bar{R}_0$ and B, we see that $\bar{R}_0$ and B contain, respectively, complete valuation rings B_1 and B_1' with the same residue field K. Then by (1), we may identify B_1' with B_1. Further, we may assume without loss of generality that $\mathfrak{M}$ (= a system of representatives of p-basis in $\bar{R}_0$) is also contained in B. Then our assertion follows immediately by our above proof of Proposition 2.

Errata:

p. 63, *l.* 21 and p. 64, *l.* 27; For "form" read "forms", p. 66, Proposition 2; For "with maximal ideal" read "with maximal ideal $\mathfrak{m}$", p. 69, Proposition 7; For "With these conditions" read "If these conditions".

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