UNITARY REPRESENTATIONS OF SOME LINEAR GROUPS II

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§ 0. Introudction. In his preceding paper [2], the author determined the types of irreducible unitary representations and cyclic unitary representations of the group of all euclidean motions in 2-space E^2. The purpose of the present paper is to determine the types of irreducible unitary representations and cyclic ones of the group of all euclidean motions in n-space E^n for $n \geqq 3$.[1),2)] In this paper, we shall make use of the results of the preceding paper [2], but notations are independent of those in [2].

§ 1. Preliminaries and main theorems. Let **G** be the group of all euclidean motions in n-space E^n. Then **G** has a compact subgroup $\mathbf{K} \cong SO(n)$ and a normal subgroup **V** isomorphic to the vector group R^n, and

$$(1.1) \qquad \begin{cases} \mathbf{G} = \mathbf{V} \cdot \mathbf{K}, \quad \mathbf{V} \cap \mathbf{K} = \{e\} \quad (e = \text{the identity of } \mathbf{G}) \\ \mathbf{G}/\mathbf{V} \cong \mathbf{K}. \end{cases}$$

Let X be the character group of **V**, and χ_0 be the identity of X; then $X \cong R^n$. Hereafter $g, g', \ldots$ denote elements of **G**, especially $a, b, c, \ldots$ —— of **K**, x, $y, \ldots$ —— of **V**; and $\chi, \chi', \ldots$ —— elements of X. (χ, x) denotes the value of character χ at $x \in \mathbf{V}$. We denote by M_a the orthogonal matrix which realize the element $a \in \mathbf{K}$ and by M_a^* its conjugate matrix, and define that $M_a x$ means to operate M_a to x as a vector in R^n while ax and xa mean the multiplications as elements of the group **G**. We shall denote briefly χa instead of $M_a^* \chi$. Then, if

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \chi = (\chi_1, \ldots, \chi_n) \text{ and } M_a = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix},$$

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1) The author wrote in [2] that it seemed to be difficult to solve such problem for $n \geqq 3$. But he could solve this problem after he finished the proof-reading of the paper [2].

2) Prof. G. W. Mackey kindly informed to the author that the result of [2] was inculuded in the result of his paper [3] which the author had overlooked. Recently more general cases have been treated in [4] and [5]. However, the results of the papers [3], [4] and [5] seem to be not so explicit as the result of our present paper.

we have

$$(\chi,\ M_a x) = (\chi a,\ x) = \exp(\sqrt{-1}\sum_{ij} a_{ij}\chi_i x_j).$$

$\tilde{X} = X - \{\chi_0\}$ is the product space of the unit sphere $S = S^{n-1}$ in R^n and $T = \{0 < t < \infty\}$ as topological spaces; we denote $\chi \in \tilde{X}$ by $\chi = \langle s,\ t\rangle$ $(s \in S,\ t \in T)$. Then $\chi a = \langle sa,\ t\rangle$ by the above definitions.

S may be considered as the factor space $\mathbf{K}/\mathbf{K}'$ of right $\mathbf{K}'$-cosets where $\mathbf{K}' \cong SO(n-1)$. Hereafter $a',\ b',\ c', \ldots$ denote elements of $\mathbf{K}'$. We shall denote by s_b the image of $b \in \mathbf{K}$ under the natural mapping of $\mathbf{K}$ onto S. For every $s \in S$, we fix an inverse image c_s of s under the natural mapping, where we do not demand the B-measurability etc. of the mapping $s \to c_s$. Every $b \in \mathbf{K}$ is uniquely expressible in the form $b = b'c_s,\ b' \in \mathbf{K}'$, as far as the system $\{c_s\}$ is fixed. We shall consider the Haar measures db on $\mathbf{K}$ and db' on $\mathbf{K}'$ and the measure ds on S invariant under $\mathbf{K}$ such that

$$(1.2) \qquad ds \cdot db' = db.^{3)}$$

Let $\{\tilde{U}^\lambda(a') = \|\tilde{u}^\lambda_{pq}(a')\|\ (p,\ q = 1, \ldots, \tilde{n}(\lambda));\ \lambda = 1, 2, \ldots\}$ be a system of irreducible unitary representations of the compact group $\mathbf{K}'$ constructed by selecting a unitary representation from each class of mutually equivalent irreducible representations of $\mathbf{K}'$, and $\{U^\alpha(a) = \|u^\alpha_{ij}(a)\|\ (i,\ j = 1, \ldots, n(\alpha));\ \alpha = 1, 2, \ldots\}$ be a system of irreducible unitary representations of the compact group $\mathbf{K}$ constructed by the same method as above. Then $U^\alpha(a'),\ a' \in \mathbf{K}'$, may be considered as a unitary representation of $\mathbf{K}'$ and hence, by the complete reducibility, we may assume that $U^\alpha(a')$ is of the form:

$$(1.3) \qquad U^\alpha(a') = \begin{pmatrix} \tilde{U}^{\lambda(\alpha,1)}(a') & & 0 \\ & \ddots & \\ 0 & & \tilde{U}^{\lambda(\alpha,m_\alpha)}(a') \end{pmatrix}.$$

We fix such systems $\{U^\alpha(a)\}$ and $\{\tilde{U}^\lambda(a')\}$. We denote the number $\tilde{n}(\lambda(\alpha,\ 1)) + \ldots + \tilde{n}(\lambda(\alpha,\ m-1))$ by $N_m(\alpha)$ or simply by N_m $(m = 1, \ldots, m_\alpha)$. Hereafter $i,\ j,\ k$ run over $\{1, \ldots, n(\alpha)\}$ while $p,\ q,\ r$ —— $\{1, \ldots, \tilde{n}(\lambda(\alpha,\ m))\}$ for α and m being considered. Then, if $\mu = \lambda(\alpha,\ m)$, we have

$$(1.4) \qquad u^\alpha_{N_m+p,\,j}(b'a) = \sum_q \tilde{u}^\mu_{pq}(b') u^\alpha_{N_m+q,\,j}(a) \qquad (by\ (1.3)).$$

We put for any λ and p

$$\mathfrak{S}^\lambda_p = \left\{ u^\alpha_{N_m+p,\,j}(b) \;\middle/\; \begin{array}{l} j = 1, \ldots, n(\alpha), \text{ and } \langle\alpha,\ m\rangle \text{ runs over} \\ \text{all couples such that } \lambda(\alpha,\ m) = \lambda \end{array} \right\}$$

and

$$\mathfrak{H}^\lambda_p = \mathfrak{L}[\mathfrak{S}^\lambda_p]$$

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3) For the precise meaning of this equality, see [6], pp. 42-45.

where $\mathfrak{L}[\mathfrak{S}]$ denotes the closed linear subspace of $L^2(\mathbf{K})$ spanned by $\mathfrak{S}$. Then $\mathfrak{S}_p^\lambda$ is a complete orthogonal basis in $\mathfrak{H}_p^\lambda$, and

$$L^2(\mathbf{K}) = \bigoplus_{\lambda=1}^{\infty} \bigoplus_{p=1}^{\tilde{n}(\lambda)} \mathfrak{H}_p^\lambda. \tag{1.5}$$

Making use of these notions, we state here the main theorems.

THEOREM 1.1. *Fix an arbitrary element* $t \in T$ *and natural numbers* λ *and* p $(1 \leqq p \leqq \tilde{n}(\lambda))$, *and define unitary operators* $U_t(g)$, $g \in \mathbf{G}$, *in the Hilbert space* $\mathfrak{H}_p^\lambda$ *by*

$$U_t(g)f(b) = U_t(xa)f(b) = (\langle s_b, t\rangle, x)f(ba) \quad (f \in \mathfrak{H}_p^\lambda \subset L^2(\mathbf{K})) \tag{1.6}$$

for $g = xa$.[4] *Then* $\{\mathfrak{H}_p^\lambda, U_t(g)\}$ *is an irreducible unitary representation of* $\mathbf{G}$; *and, for any sequence of complex numbers:* $\{\xi_j^{\alpha m} / j = 1, \ldots, n(\alpha); \lambda(\alpha, m) = \lambda\}$ *such that* $\sum_{\lambda(\alpha,m)=\lambda} \sum_j |\xi_j^{\alpha m}|^2 = 1$, *the function*

$$\begin{aligned} \Phi(g) &\equiv \Phi(xa) \\ &= \int_S (\langle s, t\rangle, x) \Big\{ \sum_{\lambda(\alpha,m)=\lambda(\beta,l)=\lambda} \sum_{jk} \xi_j^{\alpha m} \overline{\xi_k^{\beta l}} \times \\ &\quad \times \sum_{ri} u_{N_m+r,i}^{\alpha}(c_s) u_{ij}^{\alpha}(a) \overline{u_{N_l+r,k}^{\beta}(c_s)} \Big\} ds \end{aligned} \tag{1.7}$$[5]

is a normal elementary[6] *p. d.*[7] *function on* $\mathbf{G}$ *corresponding to the above irreducible unitary representation.*

1.2. *For any fixed* t *and* λ, *the unitary representations* $\{\mathfrak{H}_p^\lambda, U_t(g)\}$ *(defined in 1.1),* $p = 1, \ldots, \tilde{n}(\lambda)$, *are mutually unitary equivalent; while* $\{\mathfrak{H}_p^\lambda, U_t(g)\}$ *and* $\{\mathfrak{H}_q^\mu, U_t(g)\}$ *are not mutually unitary equivalent for any* p *and* q *if* $\lambda \neq \mu$.

1.3. *If* $t_1 \neq t_2$, *then* $\{\mathfrak{H}_p^\lambda, U_{t_1}(g)\}$ *and* $\{\mathfrak{H}_q^\mu, U_{t_2}(g)\}$ *are not mutually unitary equivalent for any* λ, μ *and* p, q.

1.4. *Put* $\tilde{\mathfrak{H}}_k^\alpha \equiv \mathfrak{L}[\{u_{kj}^\alpha(b) / j = 1, \ldots, n(\alpha)\}]$ *for any fixed* α *and* k $(1 \leqq k \leqq n(\alpha))$, *and define the unitary operator* $U(g)$ *in* $\tilde{\mathfrak{H}}_k^\alpha$ *by*

$$U(g)f(b) = U(xa)f(b) = U(a)f(b) = f(ba) \quad (f \in \tilde{\mathfrak{H}}_k^\alpha \subset L^2(\mathbf{K})) \tag{1.8}$$

for $g = xa$. *Then* $\{\tilde{\mathfrak{H}}_k^\alpha, U(g)\}$ *is an irreducible unitary representation of* $\mathbf{G}$; *and*

$$\Phi(g) = \Phi(xa) = \sum_{ij} \xi_i \bar{\xi}_j u_{ij}^\alpha(a), \quad \sum |\xi_i|^2 = 1, \tag{1.9}$$

is a corresponding normal elementary p. d. function on $\mathbf{G}$.

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4) Any element $g \in \mathbf{G}$ is uniquely expressible in this form by virture of (1.1).

5) The function in { } in the right-hand side is a B-measurable function of s independent of the special choice of the system $\{c_s\}$; —— see Lemma 1 (§ 2).

6) See [1], § 15.

7) p. d. = positive definite.

1.5. $\{\tilde{\mathfrak{H}}_k^\alpha, U(g)\}$, $k=1, \ldots, n(\alpha)$, *are mutually unitary equivalent for any* α; *while, if* $\alpha \neq \beta$, $\{\tilde{\mathfrak{H}}_k^\alpha, U(g)\}$ *and* $\{\tilde{\mathfrak{H}}_j^\beta, U(g)\}$ *are not mutually unitary equivalent for any* k *and* j.

1.6. *Every irreducible unitary representation of* $\mathbf{G}$ *is unitary equivalent to one of the above stated types. Consequently any normal elementary p. d. function on* $\mathbf{G}$ *is expressible in the form* (1.7) *or* (1.9).

THEOREM 2. *Let* σ *be the Haar measure on the compact group* $\mathbf{K}$ *and* ρ *be a measure on* T *such that* $\rho(T) < \infty$, *and define the unitary operator* $U(g)$, $g \in \mathbf{G}$, *in the Hilbert space* $L^2 \equiv L^2(\mathbf{K} \times T, \sigma \otimes \rho)$[8] *by*

$$U(g)f(b, t) = U(xa)f(b, t) = (\langle s_b, t\rangle, x)f(ba, t) \quad (f \in L^2)$$

for $g = xa$.

2.1. *Let* Δ_ν^λ, $\nu = 1, \ldots, N(\lambda)$ $(\leqq \infty)$; $\lambda = 1, 2, \ldots$, *be subsets of* T *such that* $\rho(\Delta_\nu^\lambda) > 0$, *and* $\mathfrak{M}_\nu^\lambda$ *be the totality of functions* $\varphi(b, t)$ *on* $\mathbf{K} \times \Delta_\nu^\lambda$ *of the form:*

$$\varphi(b, t) = \sum_{\lambda(\alpha, m) = \lambda} \sum_j u_{Nm+1, j}^\alpha(b) \varphi_j^{\alpha m}(t) \quad (\textit{convergence in } L^2)$$

where

$$\sum_{\lambda(\alpha, m) = \lambda} \sum_j \int_{\Delta_\nu^\lambda} |\varphi_j^{\alpha m}(t)|^2 \, d\rho(t) < \infty.$$

Then $\mathfrak{M}_\nu^\lambda$ *is a closed linear subspace of* L^2 *invariant under* $U(g)$, $g \in \mathbf{G}$.

2.2. *Let* $\{f_{\nu j}^{\alpha m}(t) / j = 1, \ldots, n(\alpha);\ \lambda(\alpha, m) = \lambda;\ \nu = 1, \ldots, N(\lambda);\ \lambda = 1, 2, \ldots\}$ *be a sequence of functions satisfying:*

1°) $\displaystyle\sum_\lambda \sum_\nu \sum_{\lambda(\alpha, m) = \lambda} \sum_j \int_{\Delta_\nu^\lambda} |f_{\nu j}^{\alpha m}(t)|^2 \, d\rho(t) < \infty$,

2°) $\displaystyle\sum_{\lambda(\alpha, m) = \lambda} \sum_j |f_{\nu j}^{\alpha m}(t)|^2 > 0$ *for* ρ—*a. a.* $t \in \Delta_\nu^\lambda$ (*a. a. = almost all*),

3°) *for any fixed* λ, *there is no function* $\psi_{\nu\nu'}(t)$ *for* $\nu \neq \nu'$ *as follows:* $f_{\nu j}^{\alpha m}(t) = \psi_{\nu\nu'}(t) f_{\nu' j}^{\alpha m}(t)$ *for all* j *and all* $\langle \alpha, m\rangle$ $(\lambda(\alpha, m) = \lambda)$ *for* ρ—*a. a.* $t \in \Delta_\nu^\lambda \cap \Delta_{\nu'}^\lambda$;

and put

$$f_\nu^\lambda(b, t) = \sum_{\lambda(\alpha, m) = \lambda} \sum_j u_{Nm+1, j}^\alpha(b) f_{\nu j}^{\alpha m}(t) \quad (\textit{convergence in } L^2).$$

Put $\mathfrak{N}_\nu^\alpha = \tilde{\mathfrak{H}}_1^\alpha$ (*defined in Theorem* 1.4) *for* $\nu = 1, \ldots, N'(\alpha)$ $(\leqq \infty)$ *and define unitary operators* $U(g)$, $g \in \mathbf{G}$, *by* (1.8) *and let* $\{\xi_{\nu j}^\alpha / j = 1, \ldots, n(\alpha);\ \nu = 1, \ldots, N'(\alpha),\ \alpha = 1, 2, \ldots\}$ *be a sequence as follows:*

4°) $\displaystyle\sum_\alpha \sum_\nu \sum_j |\xi_{\nu j}^\alpha|^2 < \infty$,

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[8] $\sigma \otimes \rho$ denotes the product measure of σ and ρ.

5°) $\sum_j |\xi^\alpha_{\nu j}|^2 > 0$ *for any* α *and* ν,

6°) *for any fixed* α, *there is no constant* $\eta_{\nu\nu'}$ *for* $\nu \neq \nu'$ *such that* $\xi^\alpha_{\nu j} = \eta_{\nu\nu'}\xi^\alpha_{\nu' j}$ *for any* j;

and put

$$h^\alpha_\nu(b) = \sum_j \xi^\alpha_{\nu j} u^\alpha_{1j}(b).$$

Let $\{\lambda\}'$ *and* $\{\alpha\}'$ *be subsequences of the sequence* $\{1, 2, \ldots\}$ *and define the unitary representation* $\{\mathfrak{H}, U(g)\}$ *of* $\mathbf{G}$ *as the direct sum;*

$$\{\mathfrak{H}, U(g)\} = \Big[\bigoplus_{(\lambda)'}\bigoplus_{\nu}\{\mathfrak{M}^\lambda_\nu, U(g)\}\Big] \oplus \Big[\bigoplus_{(\alpha)'}\bigoplus_{\nu}\{\mathfrak{N}^\alpha_\nu, U(g)\}\Big] \tag{1.10}$$

and put

$$f^0 = \sum_{(\lambda)'}\sum_\nu f^\lambda_\nu + \sum_{(\alpha)'}\sum_\nu h^\alpha_\nu. \tag{1.11}$$

Then $\{\mathfrak{H}, U(g), f^0\}$ *is a cyclic unitary representation of* $\mathbf{G}$; *the corresponding p. d. function* $\Psi(g)$ *is expressible as follows:*

$$\begin{aligned}\Psi(g) &\equiv \Psi(xa)\\ &= \sum_{(\lambda)'}\sum_\nu \int_{\Delta^\lambda_\nu} d\rho(t) \int_S \Big\{ \sum_{\lambda(\alpha, m) = \lambda(\beta, l) = \lambda} \sum_{jk} f^{\alpha m}_{\nu j}(t)\overline{f^{\beta l}_{\nu k}(t)} \times \\ &\qquad \times (\langle s, t\rangle, x) \sum_{ri} u^\alpha_{N_m+r, i}(c_s) u^\alpha_{ij}(a) \overline{u^\beta_{N_l+r, k}(c_s)} ds \\ &\qquad + \sum_{(\alpha)'}\sum_\nu \sum_{ij} \xi^\alpha_{\nu j}\overline{\xi^\alpha_{\nu i}} u^\alpha_{ij}(a).\end{aligned} \tag{1.12}$$

2.3. *If we replace* $u^\alpha_{N_m+1, j}(b)$ *in the definition of* $\mathfrak{M}^\lambda_\nu$ *in* 2.1 *by* $u^\alpha_{N_m+p, j}(b)$ *and* $\tilde{\mathfrak{H}}^\alpha_1$ *in* 2.2 *by* $\tilde{\mathfrak{H}}^\alpha_k$ *where* p *may depend on* ν *and* $\lambda = \lambda(\alpha, m)$, *and* k — *on* α *and* ν, *then we obtain a cyclic unitary representation of* $\mathbf{G}$ *which is unitary equivalent to the original one.*

2.4. *Every cyclic unitary representation of* $\mathbf{G}$ *is unitary equivalent to that of above stated type, and any p. d. function on* $\mathbf{G}$ *is expressible in the form* (1.12).

THEOREM 3. (Generalization of Bochner's theorem) *Any p. d. function* $\Psi(g)$ *on* $\mathbf{G}$ *is expressible by means of normal elementary p. d. functions in the following form:*

$$\Psi(g) = \sum_{\lambda=1}^\infty \sum_{\nu=1}^\infty \xi^\lambda_\nu \int_{\Delta^\lambda_\nu} \Phi^\lambda_\nu(g; t) d\rho(t) + \sum_{\alpha=1}^\infty \sum_{\nu=1}^\infty \eta^\alpha_\nu \Phi^\alpha_\nu(g)$$

where $\Phi^\lambda_\nu(g, t)$ *and* $\Phi^\alpha_\nu(g)$ *are normal elementary p. d. functions* (cf. (1.7), (1.9) and (1.12)), $\Delta^\lambda_\nu \subset T$ *and* ξ^λ_ν, $\eta^\alpha_\nu \geqq 0$, $\sum_{\lambda\nu} \xi^\lambda_\nu \rho(\Delta^\lambda_\nu) < \infty$, $\sum_{\alpha\nu} \eta^\alpha_\nu < \infty$.

We shall prove these theorems in § 4 by making use of results of §§ 2 and 3.

Remark. The argument in this paper may be applied to any Lie group **G** of the following type: **G** has a closed normal subgroup **V** isomorphic to a vector group and the factor group **G**/**V** is compact.

§2. **Unitary representations of G in $L^2(\mathbf{K})$.** We fix an element $t_0 \in T$ and denote $(\langle s, t_0\rangle, x)$ by (s, x) briefly, and define unitary operators $U(g)$, $g \in \mathbf{G}$, in the Hilbert space $L^2(\mathbf{K})$ as follows:

$$U(g)f(b) = U(xa)f(b) = (s_b, x)f(ba) \quad (f \in L^2(\mathbf{K})) \quad \text{for} \quad g = xa.$$

We shall use notations defined in §1, but, in this paragraph, $(.,.)$ and $\|.\|$ denote respectively the inner product and the norm in $L^2(\mathbf{K})$.

The following lemma may be verified by making use of (1.4) and the orthogonality-relation of the system $\{\tilde{u}^{\lambda}_{pq}(b')\}$ in $L^2(\mathbf{K}')$.

LEMMA 1. *For any $a \in \mathbf{K}$ and any $s \in S$, it holds that*

$$\int_{\mathbf{K}'} u^{\alpha}_{N_m+p,j}(b'c_s a)\overline{u^{\beta}_{N_l+q,k}(b'c_s)}db'$$

$$= \begin{cases} \sum_{ri} u^{\alpha}_{N_m+r,j}(c_s)u^{\alpha}_{ij}(a)u^{\beta}_{k,N_m+r}(c_s^{-1})/\tilde{n}(\lambda(\alpha, m)) \\ \qquad\qquad\qquad \text{if } \lambda(\alpha, m) = \lambda(\beta, l) \text{ and } p = q, \\ 0 \qquad\qquad\quad \text{if not}; \end{cases}$$

and consequently, for any a, the function of the form in the right-hand side of above equality is a B-measurable function of s independent of the special choice of the system $\{c_s\}$ (see §1).

Next, if we put $\bar{\mathfrak{H}}^{\lambda}_p = \mathfrak{L}[\{U(g)f \,/\, f \in \mathfrak{H}^{\lambda}_p, g \in \mathbf{G}\}]$, then we have

LEMMA 2. *If $\lambda \neq \mu$ or $p \neq q$, then $\bar{\mathfrak{H}}^{\lambda}_p$ and $\bar{\mathfrak{H}}^{\mu}_q$ are mutually orthogonal in $L^2(\mathbf{K})$.*

Proof. It is sufficient to prove that $(U(g)\varphi, \psi) = 0$ for any $\varphi \in \mathfrak{H}^{\lambda}_p$, $\psi \in \mathfrak{H}^{\mu}_q$ and any $g \in \mathbf{G}$. φ, ψ and g are expressible in the form:

$$\varphi = \sum_{\lambda(\alpha, m)=\lambda} \sum_j \xi^{\alpha m}_j u^{\alpha}_{N_m+p,j}, \quad \psi = \sum_{\lambda(\beta, l)=\mu} \sum_k \eta^{\beta l}_k u^{\beta}_{N_l+q,k}, \quad g = xa.$$

Hence we have

$$(U(g)\varphi, \psi) = \int_{\mathbf{K}} (s_b, x)\varphi(ba)\overline{\psi(b)}db$$

$$= \int_S (s, x)ds \int_{\mathbf{K}'} \varphi(b'c_s a)\overline{\psi(b'c_s)}db' = 0$$

by (1.2) and Lemma 1, q.e.d.

COROLLARY. $\bar{\mathfrak{H}}^{\lambda}_p = \mathfrak{H}^{\lambda}_p$; *consequently $\mathfrak{H}^{\lambda}_p$ is a subspace of $L^2(\mathbf{K})$ invariant under $U(g)$, $g \in \mathbf{G}$.*

This fact is proved from (1.5) and Lemma 2.

LEMMA 3. *For any given λ and p, we fix a couple $\langle \alpha, m \rangle$ such that $\lambda(\alpha, m) = \lambda$ and put $k = N_m(\alpha) + p$. If $\varphi \in \mathfrak{H}_p^\lambda$ and if the p. d. function $(U(g)\varphi, \varphi)$*[9] *on* **G** *is a minorant*[10] *of the p. d. function $(U(g)u_{kk}^\alpha, u_{kk}^\alpha)$, then $\varphi = \xi u_{kk}^\alpha$, ξ being a complex number.*

Proof. By the assumption and by Corollary of Lemma 2, there exists an element $\psi \in \mathfrak{H}_p^\lambda$ such that

$$(U(g)\varphi, \varphi) + (U(g)\psi, \psi) = (U(g)u_{kk}^\alpha, u_{kk}^\alpha), \tag{2.1}$$

especially, putting $g = a \in \mathbf{K}$, we have

$$\int_{\mathbf{K}} \varphi(ba)\overline{\varphi(b)}db + \int_{\mathbf{K}} \psi(ba)\overline{\psi(b)}db = u_{kk}^\alpha(a)/n(\alpha).$$

Each term of the left-hand side is p. d. function of $a (\in \mathbf{K})$, while $u_{kk}^\alpha(a)$ is an elementary *p. d.* function on **K**. Hence we have[11]

$$\left.\begin{aligned} &\int_{\mathbf{K}} \varphi(ba)\overline{\varphi(b)}db = \eta u_{kk}^\alpha(a)/n(\alpha) \\ &\int_{\mathbf{K}} \psi(ba)\overline{\psi(b)}db = (1-\eta)u_{kk}^\alpha(a)/n(\alpha) \end{aligned}\right\} \quad 0 < \eta < 1. \tag{2.2}$$

On the other hand, φ is expressible in the form:

$$\varphi = \sum_{\lambda(\beta, l) = \lambda} \sum_j \xi_j^{\beta l} u_{N_l + p, j}^\beta.$$

Hence it follows from the orthogonality-relation of $\{u_{ij}^\beta(b)\}$ that

$$\int_{\mathbf{K}} \varphi(ba)\overline{\varphi(b)}db = \sum_{\lambda(\beta, l) = \lambda} \sum_{ij} \xi_j^{\beta l} \overline{\xi_i^{\beta l}} u_{ij}^\beta(a)/n(\beta).$$

From this equality and (2.2), we get

$$\sum_l^{\lambda(\beta, l) = \lambda} |\xi_j^{\beta l}|^2 = \eta \delta_{\alpha\beta} \delta_{kj} \quad (\delta\text{: Kronecker's delta})$$

where $\sum_l^{\lambda(\beta, l) = \lambda}$ means the summation for all l such that $\lambda(\beta, l) = \lambda$ for fixed β. Hence φ may be expressible as follows:

$$\varphi(b) = \sum_l^{\lambda(\alpha, l) = \lambda} \xi_l u_{N_l + p, k}^\alpha(b), \quad \sum_l^{\lambda(\alpha, l) = \lambda} |\xi_l|^2 = \eta. \tag{2.3}$$

Similarly we get

9) See [1], §7.

10) See [1], §11; —— of couse, we do not mean the trivial one: the function identically equal to zero.

11) See Theorem 7 in [1].

(2.3′) $$\psi(b)=\sum_{l}^{\lambda(\alpha,l)=\lambda}\eta_l u^{\alpha}_{N_l+p,k}(b).\qquad \sum_{l}^{\lambda(\alpha,l)=\lambda}|\eta_l|^2=1-\eta.$$

Consequently

(2.4) $$\sum_{l}^{\lambda(\alpha,l)=\lambda}\{|\xi_l|^2+|\eta_l|^2\}=1.$$

If we put $g=x\in\mathbf{V}$ in (2.1), we have (by (1.2))

$$\int_S(s,\ x)ds\int_{\mathbf{K}'}|\varphi(b'c_s)|^2db'+\int_S(s,\ x)ds\int_{\mathbf{K}'}|\psi(b'c_s)|^2db'$$

$$=\int_S(s,\ x)ds\int_{\mathbf{K}'}|u^{\alpha}_{kk}(b'c_s)|^2db'.$$

Since $\varphi(b)$ and $\psi(b)$ are continuous by (2.3) and (2.3′), and since $x(\in\mathbf{V})$ is arbitrary in the above equality, we obtain for any $s\in S$

$$\int_{\mathbf{K}'}|\varphi(b'c_s)|^2db'+\int_{\mathbf{K}'}|\psi(b'c_s)|^2db'=\int_{\mathbf{K}'}|u^{\alpha}_{kk}(b'c)|^2db'.$$

Putting $s=s_e$ (whence we may put $c_s=e$) in this equality, we have

(2.5) $$\int_{\mathbf{K}'}|\varphi(b')|^2db'+\int_{\mathbf{K}'}|\psi(b')|^2db'=\int_{\mathbf{K}'}|u^{\alpha}_{kk}(b')|^2db'$$

$$=\tilde{u}^{\lambda}_{pp}(e)/\tilde{n}(\lambda)\neq 0.$$

By (1.3) and by the assumption: $k=N_m(\alpha)+p$,

$$u^{\alpha}_{N_l+p,k}(b')\equiv 0\quad\text{on}\quad\mathbf{K}'\quad\text{for}\quad l\neq m.$$

Hence, from (2.3), (2,3′) and (2.5), we get

$$|\xi_m|^2+|\eta_m|^2=1.$$

From this and (2.4), we obtain $\xi_l=\eta_l=0$ for $l\neq m$, and hence $\varphi=\xi_m u^{\alpha}_{N_m+p,k}$ by (2.3), q.e.d.

LEMMA 4. *Let α, m and k be as in Lemma 3 for any given λ and p. Then $\{\mathfrak{H}^{\lambda}_p,\ U(g),\ u^{\alpha}_{kk}\}$ is a cyclic unitary representation of* $\mathbf{G}$.

Proof. For any β, l and any i, j $(1\leqq i,\ j\leqq n(\beta))$ it holds that

$$u^{\beta}_{N_l+p,i}\in\mathfrak{L}[\{U(a)u^{\beta}_{N_l+p,j}\ /\ a\in\mathbf{K}(\subset\mathbf{G})\}]$$

by the irreducibility of $U^{\beta}(a)$ as a representation of $\mathbf{K}$. By virtue of this fact and Corollary of Lemma 2, it suffices to prove that $\lambda(\beta,\ l)=\lambda$ implies

(2.6) $$u^{\beta}_{N_l+p,1}\in\mathfrak{L}[\{U(g)u^{\alpha}_{kj}\ /\ j=1,\ \dots,\ n(\alpha)\ ;\ g\in\mathbf{G}\}].$$

Now, if $\lambda(\beta,\ l)=\lambda=\lambda(\alpha,\ m)$, then, by Lemma 1, the functions $\varphi_j(s)$ $(j=1,\ \dots,\ n(\alpha))$ defined by

$$\varphi_j(s) \equiv \tilde{n}(\lambda)\int_{K'} u^{\beta}_{Nl+p,1}(b'c_s)\cdot\overline{u^{\alpha}_{Nm+p,j}(b'c_s)}db'$$
$$= \sum_q u^{\beta}_{Nl+q,1}(c_s)u^{\alpha}_{j,Nm+q}(c_s^{-1})$$

are bounded B-measurable functions on S and it holds for any r $(1 \leqq r \leqq \tilde{n}(\lambda))$ and any $s \in S$ that

$$\sum_j u^{\alpha}_{Nm+r,j}(c_s)\varphi_j(s) = \sum_q\sum_j u^{\alpha}_{Nm+r,j}(c_s)u^{\alpha}_{j,Nm+q}(c_s^{-1})u^{\beta}_{Nl+q,1}(c_s)$$
$$= \sum_q u^{\alpha}_{Nm+r,Nm+q}(e)u^{\beta}_{Nl+q,1}(c_s) = u^{\beta}_{Nl+r,1}(c_s).$$

Hence, by means of the relation: $u^{\beta}_{Nl+p,Nl+q}(b') = \tilde{u}^{\lambda}_{pq}(b') = u^{\alpha}_{Nm+p,Nm+q}(b')$, we get (for $b = b'c_s$)

$$u^{\beta}_{Nl+p,1}(b) = \sum_r u^{\beta}_{Nl+p,Nl+r}(b')u^{\beta}_{Nl+r,1}(c_s)$$
$$= \sum_{ri} u^{\alpha}_{Nm+p,Nm+r}(b')u^{\alpha}_{Nm+r,j}(c_s)\varphi_j(s) = \sum_j u^{\alpha}_{Nm+p,j}(b)\varphi_j(s).$$

On the other hand, there exist complex numbers $\xi_{j\nu}$ and elements $x_{j\nu}$ of $\mathbf{V}$ $(\nu = 1, \ldots, N(j))$ for any $\varepsilon > 0$ and every j such that

$$\int_S |\varphi_j(s) - \sum_\nu \xi_{j\nu}\cdot(s, x_{j\nu})|^2 ds < \varepsilon^2/n(\alpha)^2,$$

since $\varphi_j(s)$, $j = 1, \ldots, n(\alpha)$, are bounded and B-measurable on S. Therefore, by simple calculation, we get

$$\|u^{\beta}_{Nl+p,1} - \sum_{j\nu} \xi_{j\nu}U(x_{j\nu})\cdot u^{\alpha}_{Nm+p,j}\| < \varepsilon.$$

This result shows (2.6), q.e.d.

PROPOSITION 1. *$\{\mathfrak{H}^{\lambda}_p, U(g)\}$ is an irreducible unitary representation of* $\mathbf{G}$ *for any λ and p $(1 \leqq p \leqq \tilde{n}(\lambda))$.*

This proposition is clear by Corollary of Lemma 2, Lemmas 3 and 4, and Theorem 7 in [1].

COROLLARY. i) *If a unitary operator U in $\mathfrak{H}^{\lambda}_p$ is permutable with any $U(g)$, $g \in \mathbf{G}$, then $U = \xi I$, $|\xi| = 1$; consequently* ii) *If $\varphi, \psi \in \mathfrak{H}^{\lambda}_p$ and $(U(g)\varphi, \varphi) = (U(g)\psi, \psi)$ for any $g \in \mathbf{G}$, then $\psi = \xi\varphi$, $|\xi| = 1$.*

These are immediate results of Proposition 1.

PROPOSITION 2. *For any fixed λ, the unitary representations $\{\mathfrak{H}^{\lambda}_p, U(g)\}$, $p = 1, \ldots, \tilde{n}(\lambda)$, are mutually unitary equivalent.*

Proof. We fix a couple $\langle\alpha, m\rangle$ such that $\lambda(\alpha, m) = \lambda$. Then $\{\mathfrak{H}^{\lambda}_p, U(g), u^{\alpha}_{Nm+p,1}\}$, $p = 1, \ldots, \tilde{n}(\lambda)$, are cyclic unitary representations of $\mathbf{G}$ (by Lemma

4). Hence it is sufficient to prove that p. d. functions $(U(g)u^{\alpha}_{N_m+p,1},\ u^{\alpha}_{N_m+p,1})$, $p=1,\ldots,\tilde{n}(\lambda)$, are mutually identical. For any $g=xa\in\mathbf{G}$, we have by (1.2) and Lemma 1

$$(U(g)u^{\alpha}_{N_m+p,1},\ u^{\alpha}_{N_m+p,1})=\int_S(s,\ x)ds\int_{\mathbf{K}'}u^{\alpha}_{N_m+p,1}(b'c_sa)\overline{u^{\alpha}_{N_m+p,1}(b'c_s)}db'$$

$$=\int_S(s,\ x)\Big\{\sum_{qi}u^{\alpha}_{N_m+q,i}(c_s)u^{\alpha}_{i1}(a)u^{\alpha}_{1,N_m+q}(c_s^{-1})/\tilde{n}(\lambda)\Big\}ds\,;$$

this is independent of p, q.e.d.

PROPOSITION 3. *If $\lambda\neq\mu$, then the unitary representations $\{\mathfrak{H}^{\lambda}_p,\ U(g)\}$ and $\{\mathfrak{H}^{\mu}_q,\ U(g)\}$ are not mutually unitary equivalent for any p and q.*

Proof. By Proposition 2, it suffices to prove this for $p=q=1$. We denote the operator $U(g)$ considered in $\mathfrak{H}^{\lambda}_1$ and $\mathfrak{H}^{\mu}_1$ by $U_1(g)$ and $U_2(g)$ respectively. If $\{\mathfrak{H}^{\lambda}_1,\ U_1(g)\}$ is unitary equivalent to $\{\mathfrak{H}^{\mu}_1,\ U_2(g)\}$, then there exists a unitary transformation U of $\mathfrak{H}^{\lambda}_1$ onto $\mathfrak{H}^{\mu}_1$ such that $U_2(g)=U\cdot U_1(g)\cdot U^{-1}$. We fix a couple $\langle\alpha,\ m\rangle$ such that $\lambda(\alpha,\ m)=\lambda$, and put $k=N_m(\alpha)+1$. Then $u^{\alpha}_{kk}\in\mathfrak{H}^{\lambda}_1$ and $f=U\cdot u^{\alpha}_{kk}\in\mathfrak{H}^{\mu}_1$. The element f is expressible in the form: $f=\sum_{\lambda(\beta,\,l)=\mu}\sum_j\xi^{\beta l}_j u^{\beta}_{N_l+1,j}$, and hence for any $a'\in\mathbf{K}'$

$$(U_2(a')f,\ f)=\sum_{\lambda(\beta,\,l)=\mu}\sum_{ij}\xi^{\beta l}_j\overline{\xi^{\beta l}_i}u^{\beta}_{ij}(a')/n(\beta)$$

$$=\sum_{pq}\tilde{u}^{\mu}_{pq}(a')\sum_{\lambda(\beta,\,l)=\mu}\xi^{\beta l}_{N_l+q}\overline{\xi^{\beta l}_{N_l+p}}/n(\beta)\quad\text{(by (1.3))}.$$

On the other hand

$$(U_2(a')f,\ f)=(U\cdot U_1(a')\cdot U^{-1}f,\ f)$$

$$=(U_1(a')u^{\alpha}_{kk},\ u^{\alpha}_{kk})=\tilde{u}^{\lambda}_{11}(a')/n(\alpha).$$

This is a contradiction, because $\lambda\neq\mu$ implies that $\tilde{u}^{\mu}_{pq}(a')$ and $\tilde{u}^{\lambda}_{11}(a')$ are mutually orthogonal in $L^2(\mathbf{K}')$ for any p and q, q.e.d.

§3. Unitary representations of G in $L^2(\mathbf{K}\times T,\ \sigma\otimes\rho)$. Let Δ be a subset of T and $\mathfrak{M}^{\lambda}_p(\Delta)$ be the totality of functions $\varphi(b,\ t)\in L^2\equiv L^2(\mathbf{K}\times\Delta,\ \sigma\otimes\rho)$ of the form

$$\varphi(b,\ t)=\sum_{\lambda(\alpha,\,m)=\lambda}\sum_j u^{\alpha}_{N_m+p,j}(b)\varphi^{\alpha m}_{pj}(t),\quad \sum\sum\int_{\Delta}|\varphi^{\alpha m}_{pj}(t)|^2d\rho(t)<\infty.$$

We may prove easily the following

LEMMA 5. *Any function $\varphi(b,\ t)\in L^2(\mathbf{K}\times T,\ \sigma\otimes\rho)$ is uniquely expressible in the form:*

$$\varphi(b,\ t)=\sum_{\mu}\sum_{p}\sum_{\lambda(\alpha,\,m)=\mu}\sum_j u^{\alpha}_{N_m+p,j}(b)\varphi^{\alpha m}_{pj}(t)\quad(\textit{convergence in } L^2)\tag{3.1}$$

where

$$\varphi_{pj}^{\alpha m}(t) = \int_{\mathbf{K}} \varphi(b,\ t)\overline{u^{\alpha}_{N_m+p,j}(b)}db; \tag{3.2}$$

and consequently

$$\sum_{\mu}\sum_{p}\sum_{\lambda(\alpha,m)=\mu}\sum_{j}\int_{T}|\varphi_{pj}^{\alpha m}(t)|^2 d\rho(t) = \int_{\mathbf{K}\times T}|\varphi(b,\ t)|^2 db d\rho(t). \tag{3.3}$$

PROPOSITION 4. *$\mathfrak{M}_p^{\lambda}(\varDelta)$ is a closed linear subspace of $L^2(\mathbf{K}\times T,\ \sigma\otimes\rho)$ invariant under $U(g)$, $g\in\mathbf{G}$, defined in Theorem 2.1.*

It is clear from the definition of $U(g)$ and by Lemma 2 that $\mathfrak{M}_p^{\lambda}(\varDelta)$ is a linear subspace of $L^2(\mathbf{K}\times T,\ \sigma\otimes\rho)$ invariant under $U(g)$, $g\in\mathbf{G}$. The closedness of $\mathfrak{M}_p^{\lambda}(\varDelta)$ may be proved by virtue of Lemma 4.

Thus $\{\mathfrak{M}_p^{\lambda}(\varDelta),\ U(g)\}$, $p=1, \ldots, \tilde{n}(\lambda)$; $\lambda=1, 2, \ldots,$ may be considered as unitary representations of $\mathbf{G}$.

LEMMA 6. *If $f_1\in\mathfrak{M}_p^{\lambda}(\varDelta_1)$, $f_2\in\mathfrak{M}_p^{\mu}(\varDelta_2)$ and if p. d. functions $(U(g)f_1,\ f_1)$ and $(U(g)f_2,\ f_2)$ have a common minorant,*[12] *then there exist a Borel set $\varDelta_0\subset\varDelta_1\cap\varDelta_2$ such that $\rho(\varDelta_0)>0$ and a B-measurable function $\omega(t)$ defined on $\varDelta_0$ such that $0<|\omega(t)|<\infty$ and $f_1(b,\ t)=\omega(t)f_2(b,\ t)$ for σ—a. a. $b\in\mathbf{K}$ for ρ—a. a. $t\in\varDelta_0$; consequently $\lambda=\mu$.*

Proof. Let $\Psi(g)$ be a common minorant of $(U(g)f_1,\ f_1)$ *and* $(U(g)f_2,\ f_2)$. Then, by Theorem 5 in [1], $\Psi(g)$ is expressible as follows:

$$\Psi(g) = (U(g)\psi_1,\ \psi_1) = (U(g)\psi_2,\ \psi_2), \quad \psi_1\in\mathfrak{M}_p^{\lambda}(\varDelta_1), \quad \psi_2\in\mathfrak{M}_p^{\mu}(\varDelta_2); \tag{3.4}$$

furthermore there exist $\varphi_1\in\mathfrak{M}_p^{\lambda}(\varDelta_1)$ and $\varphi_2\in\mathfrak{M}_p^{\mu}(\varDelta_2)$ such that

$$\begin{aligned}&\int_{\mathbf{K}\times T}(\langle s_b,\ t\rangle,\ y)(\langle s_b,\ t\rangle,\ x)f_\nu(ba,\ t)\overline{f_\nu(b,\ t)}dbd\rho(t)\\ &= \int_{\mathbf{K}\times T}(\langle s_b,\ t\rangle,\ y)\{(\langle s_b,\ t\rangle,\ x)\psi_\nu(ba,\ t)\overline{\psi_\nu(b,\ t)}+\\ &\qquad+(\langle s_b,\ t\rangle,\ x)\varphi_\nu(ba,\ t)\overline{\varphi_\nu(b,\ t)}\}dbd\rho(t), \quad \nu=1,\ 2,\end{aligned} \tag{3.5}$$

for any $y,\ x\in\mathbf{V}$ and $a\in\mathbf{K}$ (we put $f(b,\ t)\equiv 0$ on $\mathbf{K}\times(T-\varDelta_\nu)$ for any function $\in\mathfrak{M}_p^{\lambda}(\varDelta_\nu)$). For any Borel set $\varDelta\subset T$, the characteristic function of the set $\mathbf{K}\times\varDelta$ may be approximated in $L^2(\mathbf{K}\times T,\ \sigma\otimes\rho)$ by means of linear combinations of "characters" $(\langle s_b,\ t\rangle,\ y)$. Hence (3.5) implies that

$$\begin{aligned}&\int_{\mathbf{K}}(\langle s_b,\ t\rangle,\ x)f_\nu(ba,\ t)\overline{f_\nu(b,\ t)}db\\ &= \int_{\mathbf{K}}(\langle s_b,\ t\rangle,\ x)\psi_\nu(ba,\ t)\overline{\psi_\nu(b,\ t)}db+\end{aligned} \tag{3.6}$$

[12]) See the foot-note 10).

$$+\int_{\mathbf{K}}(\langle s_b,\ t\rangle,\ x)\varphi_\nu(ba,\ t)\overline{\varphi_\nu(b,\ t)}db, \quad \nu=1,\ 2,$$

for any $x\in\mathbf{V}_0$ and $a\in\mathbf{K}_0$ for ρ—a. a. $t\in T$ where $\mathbf{V}_0$ and $\mathbf{K}_0$ are dense subsets of $\mathbf{V}$ and $\mathbf{K}$ respectively such that $\overline{\overline{\mathbf{V}}}_0=\overline{\overline{\mathbf{K}}}_0=\aleph_0$; and hence, by Lebesgue's convergence theorem, (3.6) is true for any $x\in\mathbf{V}$ and $a\in\mathbf{K}$ for ρ—a. a. $t\in T$. Similar argument shows that (3.4) implies

$$\int_{\mathbf{K}}(\langle s_b,\ t\rangle,\ x)\psi_1(ba,\ t)\overline{\psi_1(b,\ t)}db$$
$$(3.7)\qquad =\int_{\mathbf{K}}(\langle s_b,\ t\rangle,\ x)\psi_2(ba,\ t)\overline{\psi_2(b,\ t)}db$$

for ρ—a. a. $t\in T$. Each term in (3.6) and (3.7) expresses a p. d. function of $g=xa$; especially the left-hand side of (3.6) expresses an elementary p. d. function corresponding to the irreducible unitary representation $\{\mathfrak{H}_p^\lambda,\ U_t(g)\}$ or $\{\mathfrak{H}_p^\mu,\ U_t(g)\}$ stated in §2 if $\nu=1$ or $\nu=2$ respectively. Hence, by Theorem 7 in [1], there exists a function $\omega_0(t)\geqq 0$ such that

$$\int_{\mathbf{K}}(\langle s_b,\ t\rangle,\ x)f_1(ba,\ t)\overline{f_1(b,\ t)}db$$
$$=\omega_0(t)\int_{\mathbf{K}}(\langle s_b,\ t\rangle,\ x)f_2(ba,\ t)\overline{f_2(b,\ t)}db$$

for any $x\in V$ and $a\in K$ for a. a. $t\in T$, and hence, by Proposition 3 and Corollary of Proposition 1, we obtain that $\lambda=\mu$ and that

$$f_1(b,\ t)=\omega(t)f_2(b,\ t)\quad\text{for}\quad \sigma\text{—a. a. } b$$

for ρ—a. a. t for a certain $\omega(t)$ ($|\omega(t)|^2=\omega_0(t)$), which is B-measurable in t by Fubini's theorem. If we put

$$\Delta_0=\left\{t\Big/\int_{\mathbf{K}}|\psi_1(b,\ t)|^2db=\int_{\mathbf{K}}|\psi_2(b,\ t)|^2db\neq 0\right\}\quad(\text{see } (3.7)),$$

then we may easily show that the set Δ_0 and the function $\omega(t)$, considered on Δ_0, have the properties stated in Lemma 6, q.e.d.

PROPOSITION 5. *The unitary representations* $\{\mathfrak{M}_p^\lambda(\Delta),\ U(g)\}$ *and* $\{\mathfrak{M}_q^\lambda(\Delta),\ U(g)\}$ *are mutually unitary equivalent for any p and q* ($1\leqq p,\ q\leqq \tilde{n}(\lambda)$).

This fact is easily verified from the definition of $\mathfrak{M}_p^\lambda(\Delta)$ and by Proposition 2.

PROPOSITION 6. *If* $\lambda\neq\mu$, *then, for any p, q, any* Δ_1, Δ_2, *and any* $f_1\in\mathfrak{M}_p^\lambda(\Delta_1)$ *and* $f_2\in\mathfrak{M}_q^\mu(\Delta_2)$, *the p. d. functions* $(U(g)f_1,\ f_1)$ *and* $(U(g)f_2,\ f_2)$ *are mutually disjoint.*[13]

[13] See [1], §12.

This proposition is evident by Lemma 6, Proposition 5 and the definition of $\mathfrak{M}_p^\lambda(\varDelta)$.

PROPOSITION 7. *Assume that* $f_1\in\mathfrak{M}_p^\lambda(\varDelta_1)$ *and* $f_2\in\mathfrak{M}_p^\lambda(\varDelta_2)$. *In order that the p. d. functions* $(U(g)f_1, f_1)$ *and* $(U(g)f_2, f_2)$ *are not mutually disjoint, it is necessary and sufficient that there exist a Borel set* $\varDelta\subset\varDelta_1\cap\varDelta_2$ *such that* $\rho(\varDelta)>0$ *and a B-measurable function* $\omega(t)$ *defined on* $\varDelta$ *such that* $0<|\omega(t)|<\infty$ *and that* $f_1(b, t)=\omega(t)f_2(b, t)$ *for* σ*—a. a.* $b\in\mathbf{K}$ *for* ρ*—a. a.* $t\in\varDelta$.

Proof. The necessity is clear by Lemma 6.

To prove the sufficiency, we put $\omega_1(t)=\min\{1, |\omega(t)|\}$ on $\varDelta$ and define

$$f(b, t)=\begin{cases}\omega_1(t)f_1(b, t) & \text{on} \quad \mathbf{K}\times\varDelta,\\ 0 & \text{on} \quad \mathbf{K}\times(T-\varDelta).\end{cases}$$

Then we may prove that $f\in\mathfrak{M}_p^\lambda(\varDelta)\subset\mathfrak{M}_p^\lambda(\varDelta_1)\cap\mathfrak{M}_p^\lambda(\varDelta_2)$ and that p. d. function $(U(g)f, f)$ is a common minorant of $(U(g)f_1, f_1)$ and $(U(g)f_2, f_2)$, q.e.d.

PROPOSITION 8. *In order for* $\{\mathfrak{M}_p^\lambda(\varDelta), U(g), f\}$ $(f\equiv f(b, t)\in\mathfrak{M}_p^\lambda(\varDelta))$ *to be a cyclic unitary representation of* $\mathbf{G}$, *it is necessary and sufficient that* $f(b, t)\neq 0$ *as an element of* $\mathfrak{H}_p^\lambda(\subset L^2(\mathbf{K}))$ *for* ρ*—a. a.* $t\in\varDelta$.

Proof. The necessity is clear by the definition of $U(g)$.

We shall prove the sufficiency. Put

$$\mathfrak{M}'=\mathfrak{L}[\{U(g)f \,/\, g\in\mathbf{G}\}]$$

and let φ be any element of $\mathfrak{M}_p^\lambda(\varDelta)\ominus\mathfrak{M}'$. Then

$$\int_{\mathbf{K}\times\Delta}(\langle s_b, t\rangle, x)f(ba, t)\overline{\varphi(b, t)}dbd\rho(t)=0 \quad \text{for any } x \text{ and } a.$$

By the similar argument as in the proof of Lemma 6, it follows from the above equality that

$$\int_{\mathbf{K}}(\langle s_b, t\rangle, x)f(ba, t)\overline{\varphi(b, t)}db=0 \quad \text{for any } x \text{ and } a$$

for ρ—a. a. $t\in\varDelta$. Since the unitary representation $\{\mathfrak{H}_p^\lambda, U_t(g)\}$ is irreducible for any t (Proposition 1) and since $f\neq 0$ in $\mathfrak{H}_p^\lambda$ for ρ—a. a. $t\in\varDelta$ by the assumption, we get $\varphi(b, t)\equiv 0$ in $\mathfrak{H}_p^\lambda$ for ρ—a. a. $t\in\varDelta$, and hence $\varphi(b, t)\equiv 0$ in $\mathfrak{M}_p^\lambda(\varDelta)$. Thus we obtain $\mathfrak{M}'=\mathfrak{M}_p^\lambda(\varDelta)$, q.e.d.

§4. **Proof of Theorems.** Throughout this paragraph, we notice that the spaec $\mathfrak{M}_\nu^\lambda$ defined in Theorem 2 is identical with the space $\mathfrak{M}_1^\lambda(\varDelta_\nu^\lambda)$ in the notation stated in §3 for any λ and ν.

Theorems 1.1 and 1.2 have been proved in §2—the formula (1.7) may be shown by calculating $\Phi(g) \equiv (U(g)f, f)$, $f = \sum_{\lambda(\alpha, m)=\lambda} \sum_j \xi^{\alpha}_{jm} u^{\alpha}_{\lambda m+p,j}$. Theorems 1.4 and 1.5 are evident from the fact $\mathbf{G}/\mathbf{V} \cong \mathbf{K}$ and by Peter-Weyl's theory. (Theorem 1.3 shall be proved after the proof of Theorems 2.1——2.3.)

Next, let $\mathfrak{M}^{\lambda}_{\nu}$ and f^{λ}_{ν} ($\nu = 1, \ldots, N(\lambda)$; $\lambda = 1, 2, \ldots$) be as stated in Theorem 2. Theorem 2.1 have been proved in §3 (Proposition 4). By the conditions 1°) and 2°), we have $f^{\lambda}_{\nu} \in \mathfrak{M}^{\lambda}_{\nu}$ and $f^{\lambda}_{\nu}(b, t) \neq 0$ in $\mathfrak{H}^{\lambda}_{t}(\subset L^2(\mathbf{K}))$ for ρ—a. a. $t \in \Delta^{\lambda}_{\nu}$. Hence the unitary representation $\{\mathfrak{M}^{\lambda}_{\nu}, U(g), f^{\lambda}_{\nu}\}$ is cyclic by Proposition 8 for every λ and ν. The p. d. functions $(U(g)f^{\lambda}_{\nu}, f^{\lambda}_{\nu})$, $\nu = 1, 2, \ldots$, are mutually disjoint from the condition 3°) and by Proposition 7. Hence, by Theorem 8 in [1], the direct sum $\{\bigoplus_{\nu} \mathfrak{M}^{\lambda}_{\nu}, U(g), f^{\lambda}\}$, $f^{\lambda} = \sum_{\nu} f^{\lambda}_{\nu}$, is a cyclic unitary representation of $\mathbf{G}$. We may further show by Proposition 6 that the p. d. functions $(U(g)f^{\lambda}, f^{\lambda})$ and $(U(g)f^{\mu}, f^{\mu})$ are mutually disjoint for $\lambda \neq \mu$. Similar argument is possible for $\{\mathfrak{N}^{\alpha}_{\nu}, U(g)\}$, $\nu = 1, \ldots, N'(\alpha)$; $\alpha = 1, 2, \ldots$. Therefore, by the same argument as in the proof of Theorem 2 in [2], we may prove that the unitary representation $\{\mathfrak{H}, U(g), f^0\}$ stated in Theorem 2.2 is cyclic. The formula (1.12) may be verified by calculating $\Psi(g) = (U(g)f^0, f^0)$. Theorem 2.2 is thus proved. Theorem 2.3 may be seen by Proposition 5.

We now prove Theorem 1.3. If $\{\mathfrak{H}^{\lambda}_{p}, U_{t_1}(g)\}$ and $\{\mathfrak{H}^{\mu}_{q}, U_{t_2}(g)\}$ $(t_1 \neq t_2)$ are mutually unitary equivalent, there exist $f_1 \in \mathfrak{H}^{\lambda}_{p}$ and $f_2 \in \mathfrak{H}^{\mu}_{q}$ such that $(U_{t_1}(g)f_1, f_1) = (U_{t_2}(g)f_2, f_2)$ for any $g \in \mathbf{G}$, and hence the direct sum $\{\mathfrak{H}^{\lambda}_{p} \oplus \mathfrak{H}^{\mu}_{q}, U(g), f_1 + f_2\}$ $(U(g) = U_{t_1}(g) \oplus U_{t_2}(g))$ is not cyclic by Theorem 8 in [1]. But we may prove by means of Theorems 2.2 and 2.3 verified above that $\{\mathfrak{H}^{\lambda}_{p} \oplus \mathfrak{H}^{\mu}_{q}, U(g), f_1 + f_2\}$ is a cyclic unitary representation of $\mathbf{G}$. That is a contradiction.

In order to prove Theorems 1.6 and 2.4, we first modify Lemma 2 in [2] to the following form:

LEMMA 7. *Let $\tilde{X}$, S, T and $\mathbf{K}$ be as stated in §1 and $F(\Lambda)$ $(\Lambda \subset \tilde{X} \equiv S \times T)$ be a measure on $\tilde{X}$ such that $F(\tilde{X}) < \infty$, and assume that there exists a non-negative function $u(a; \chi)$ on $\mathbf{K} \times \tilde{X}$, measurable in $\langle a, \chi \rangle$ and summable on $\tilde{X}$ with respect to F for every $a \in \mathbf{K}$, such that*

$$F(\Lambda a) = \int_{\Lambda} u(a; \chi) dF(\chi) \quad (\Lambda a = \{\chi a \,/\, \chi \in \Lambda\}) \tag{4.1}$$

for any $\Lambda \subset \tilde{X}$ and any $a \in \mathbf{K}$. Then there exist a non-negative B-measurable function $\omega(s, t)$ on $\tilde{X} \equiv S \times T$ and a measure $\rho(\Delta)$ on T, $\rho(T) < \infty$, such that

$$F(\Lambda) = \int_{\Lambda} \omega(s, t) ds d\rho(t) \quad \textit{for any} \quad \Lambda \subset \tilde{X}, \tag{4.2}$$

ds being the invariant measure on S defined in §1.

Proof. We put $B_{\Lambda} = \{\langle b, t \rangle / \langle s_b, t \rangle \in \Lambda\} \subset \mathbf{K} \times T$ (see §1) for any $\Lambda \subset \tilde{X} \equiv S \times T$, and define a measure $F^*(B)$ on $\mathbf{K} \times T$ by the formula:

(4.3) $$\int_{\mathbf{K}\times T}\varphi(b,\ t)dF^*(b,\ t)=\int_{S\times T}dF(s,\ t)\int_{\mathbf{K}'}\varphi(b'c_s,t)db' \quad (\text{see } \S 1)$$

for any continuous function $\varphi(b,\ t)$ on $\mathbf{K}\times T$ with compact carrier. Then we have

(4.4) $$F^*(B_\Lambda)=F(\Lambda) \quad \text{for any} \quad \Lambda\subset\tilde{X},$$

and (4.1) implies

$$F^*(Ba)=\int_B u^*(a\,;\ b,\ t)dF^*(b,\ t) \quad (Ba=\{\langle ba,\ t\rangle\ /\ \langle b,\ t\rangle\in B\})$$

where $u^*(a\,;\ b,\ t)=u(a\,;\ \langle s_b,\ t\rangle)$ is non-negative, B-measurable in $\langle a,\ b,\ t\rangle$ and summable (in $\langle b,\ t\rangle$) on $\mathbf{K}\times T$ with respect to F^* for any $a\in\mathbf{K}$. Therefore, by the same argument as the proof of Lemma 2 in [2], we may show that there exist a non-negative B-measurable function $\omega^*(s,\ t)$ on $\mathbf{K}\times T$ and a measure ρ on T, $\rho(T)<\infty$, such that

$$F^*(B)=\int_B \omega^*(b,\ t)dbd\rho(t) \quad \text{for any} \quad B\subset\mathbf{K}\times T.$$

Hence we obtain from (4.4), (1.2) and by simple calculation that

$$F(\Lambda)=\int_\Lambda dsd\rho(t)\int_{\mathbf{K}'}\omega^*(b'c_s,\ t)db' \quad \text{for any} \quad \Lambda\subset\tilde{X},$$

and hence we get (4.2) by putting $\omega(s,\ t)=\int_{\mathbf{K}'}\omega^*(b'c_s,\ t)db'$, q.e.d.

Hereafter the indices j and k may run over all natural numbers, not following after the rule defined in §1.

Now let $\{\mathfrak{H},\ U(g),\ f^0\}$ be a cyclic unitary representation of $\mathbf{G}$. Then, making use of Lemma 7, we can achieve the same argument as in [2]——from the beginning of §3 (p. 6) to L. 14 in p. 10——, and obtain the following result:

$\{\mathfrak{H},\ U(g)\}=\{\mathfrak{N},\ U(g)\}\oplus\{\mathfrak{M},\ U(g)\}$; $\{\mathfrak{N},\ U(g)\}$ is equivalent to a cyclic unitary representation of the group $\mathbf{K}(\cong\mathbf{G}/\mathbf{V})$, and $\{\mathfrak{M},\ U(g)\}$ is given as follows: there exists a unitary space $\mathfrak{H}_0$ of all sequences of complex numbers: $\{\xi_1,\ \ldots,\ \xi_n\}$, $n\leqq\infty$, such that $\|\xi\|^2=\sum_{j=1}^{\infty}|\xi_j|^2<\infty$ (*if* $n=\infty$), and exists a matrix of functions $M(a\,;\ s,\ t)=\|u_{jk}(a\,;\ s,\ t)\|$ whose elements $u_{jk}(a\,;\ s,\ t)$ $(j,\ k=1,\ \ldots,\ n)$ are B-measurable in $\langle a,\ s,\ t\rangle$; and every $f\in\mathfrak{M}$ is realized as a $\mathfrak{H}_0$-valued function $\mathbf{f}(s,\ t)\equiv\{f_1(s,\ t),\ \ldots,\ f_n(s,\ t)\}$ defined on $\tilde{X}\equiv S\times T$, and $f\sim\mathbf{f}(s,\ t)$[14] implies that

$$\begin{cases}\|f\|^2=\int_{S\times T}\|\mathbf{f}(s,\ t)\|^2dsd\rho(t) \quad (\|\mathbf{f}(s,\ t)\|^2=\sum_j|f_j(s,\ t)|^2),\\ U(x)f\sim(\langle s,\ t\rangle,\ x)\mathbf{f}(s,\ t) \quad \text{for any} \quad x\in\mathbf{V},\\ U(a)f\sim M(a\,;\ s,\ t)\mathbf{f}(sa,\ t) \quad \text{for any} \quad a\in\mathbf{K}\,;\end{cases}$$

[14] $f\sim\mathbf{f}(s,\ t)$ means that f is realized as $\mathbf{f}(s,\ t)$.

ρ being a measure on T such that $\rho(T)<\infty$ (obtaind from Lemma 7).

Next, for any B-measurable function $f(s, t)$ on $S\times T$, we define a function $f^*(b, t)$ on $\mathbf{K}\times T$ by

$$f^*(b, t)\equiv f(s_b, t)$$

and put $M^*(a; b, t)\equiv\|u^*_{jk}(a; b, t)\|$. Then, as is easily seen, the above result concerning $\{\mathfrak{M}, U(g)\}$ is translated into the following form: every $f\in\mathfrak{M}$ is realized as a $\mathfrak{H}_0$-valued function $\mathbf{f}(b, t)$ defined on $\mathbf{K}\times T$ and $f\sim\mathbf{f}(b, t)$ implies that

$$\begin{cases}\|f\|^2=\int_{\mathbf{K}\times T}\|\mathbf{f}(b, t)\|^2 db d\rho(t),\\ U(x)f\sim(\langle s_b, t\rangle, x)\mathbf{f}(b, t) \quad \text{for any} \quad x\in\mathbf{V},\\ U(a)f\sim M^*(a; \cdot b, t)\mathbf{f}(ba, t) \quad \text{for any} \quad a\in\mathbf{K};\end{cases}$$

moreover, if $M_1^*(a; b, t)=M_2^*(a; b, t)$ as operators in $\mathfrak{M}$, then $M_1^*(a; bc, t)=M_2^*(a; bc, t)$ in the same sense for any $c\in\mathbf{K}$——see p. 10 in [2].

Starting from this result, we can achieve the similar argument to that in [2]——from p. 10, L. 15 to p. 11, L. 15.[15] Thus $\mathfrak{M}$ may be realized as a subspace of the direct sum of at most countable number of $L^2(\mathbf{K}\times T, \sigma\otimes\rho)$, and $f\sim\{\varphi_\nu(b, t)\}\equiv\{\varphi_1(b, t), \varphi_2(b, t), \ldots\}$ implies

$$\begin{cases}\|f\|^2=\sum_{\nu=1}^{n}\int_{\mathbf{K}\times T}|\varphi_\nu(b, t)|^2 db d\rho(t), \quad n\leqq\infty,\\ U(x)f\sim\{(\langle s_b, t\rangle, x)\varphi_\nu(b, t)\} \quad \text{for any} \quad x\in\mathbf{V},\\ U(a)f\sim\{\varphi_\nu(ba, t)\} \quad \text{for any} \quad a\in\mathbf{K}.\end{cases}$$

Since $L^2(\mathbf{K}\times T, \sigma\otimes\rho)=\bigoplus_{\lambda=1}^{\infty}\bigoplus_{p=1}^{\tilde{n}(\lambda)}\mathfrak{M}_p^\lambda(T)$ by Lemma 5 and Proposition 4 (§3), it follows that $\mathfrak{M}$ may be expressible in the form:

$$\mathfrak{M}=\bigoplus_{\lambda=1}^{\infty}\bigoplus_{p=1}^{\tilde{n}(\lambda)}\bigoplus_{\nu=1}^{n(\lambda, p)}\mathfrak{M}_{\nu p}^\lambda \quad (n(\lambda, p)\leqq\infty), \quad \mathfrak{M}_{\nu p}^\lambda\subset\mathfrak{M}_p^\lambda(T) \quad \text{for any} \quad \nu,$$

and every $\mathfrak{M}_{\nu p}^\lambda$ is a closed linear subspace of $\mathfrak{M}$ invariant under $U(g)$, $g\in\mathbf{G}$.

Put

$$f^0=f+h, \quad f\in\mathfrak{M} \quad \text{and} \quad h\in\mathfrak{N},$$

and

$$f=\sum_\lambda\sum_p\sum_\nu f_{\nu p}^\lambda, \quad f_{\nu p}^\lambda\in\mathfrak{M}_{\nu p}^\lambda \quad (\subset\mathfrak{M}_p^\lambda(T)).$$

Then $\{\mathfrak{M}, U(g), f\}$ is——and consequently every $\{\mathfrak{M}_{\nu p}^\lambda, U(g), f_{\nu p}^\lambda\}$ is a cyclic unitary representation of $\mathbf{G}$. We put

[15] Such argument is impossible without extending functions on $S\times T$ to those on $\mathbf{K}\times T$ as stated above. The author owes to Mr. S. Murakami's suggestion for this improvement.

$$\Delta_{\nu p}^{\lambda}=\left\{t \Big/ \int_{\mathrm{K}}|f_{\nu p}^{\lambda}(b,\ t)|^{2}db \neq 0\right\}\ (\subset T).$$

Then $\{\mathfrak{M}_{\nu p}^{\lambda},\ U(g),\ f_{\nu p}^{\lambda}\}$ is cyclic if and only if $\mathfrak{M}_{\nu p}^{\lambda}=\mathfrak{M}_{p}^{\lambda}(\Delta_{\nu p}^{\lambda})$ by Proposition 8. We may consider by Proposition 5 that $\mathfrak{M}_{\nu p}^{\lambda}=\mathfrak{M}_{1}^{\lambda}(\Delta_{\nu p}^{\lambda})$ and $f\in\mathfrak{M}_{1}^{\lambda}(\Delta_{\nu p}^{\lambda})$. Exchanging indices, we denote for any λ

$$\Delta_{\nu}^{\lambda} \text{ and } f_{\nu}^{\lambda},\quad \nu=1,\ldots,N(\lambda)\ (\leqq\infty)$$

instead of

$$\Delta_{\nu p}^{\lambda} \text{ and } f_{\nu p}^{\lambda},$$
$$\nu=1,\ldots,n(\lambda,\ p)\ (\leqq\infty);\quad p=1,\ldots,\tilde{n}(\lambda)\ (<\infty);$$

and put $\mathfrak{M}_{\nu}^{\lambda}=\mathfrak{M}_{1}^{\lambda}(\Delta_{\nu}^{\lambda})$. Then we may consider that

$$\{\mathfrak{M},\ U(g)\}=\bigoplus_{\lambda=1}^{\infty}\bigoplus_{\nu=1}^{N(\lambda)}\{\mathfrak{M}_{\nu}^{\lambda},\ U(g)\},\quad f=\sum_{\lambda}\sum_{\nu}f_{\nu}^{\lambda}, \tag{4.5}$$

and

$$f_{\nu}^{\lambda}\in\mathfrak{M}_{1}^{\lambda}(\Delta_{\nu}^{\lambda}),\quad f_{\nu}^{\lambda}(b,\ t)\not\equiv 0 \text{ in } \mathfrak{H}_{1}^{\lambda} \text{ for } \rho\text{—a. a. } t\in\Delta_{\nu}^{\lambda}.$$

Hence

$$f_{\nu}^{\lambda}(b,\ t)=\sum_{\lambda(\alpha,\ m)=\lambda}\sum_{j}u_{\nu m+1,\ j}^{\alpha}(b)f_{\nu j}^{\alpha m}(t)\quad(\text{convergence in } L^{2}(\mathrm{K}\times T,\ \sigma\otimes\rho))$$

for any λ and ν where the series of functions

$$\left\{f_{\nu j}^{\alpha m}\Big/\begin{array}{l} j=1,\ldots,n(\lambda);\ \lambda(\alpha,\ m)=\lambda;\\ \nu=1,\ldots,N(\lambda);\quad \lambda=1,\ 2,\ldots\end{array}\right\}$$

satisfies the conditions 1°) and 2°) in Theorem 2.2. Since $\{\mathfrak{M},\ U(g),\ f\}$ is cyclic, it follows from (4.5) and by Theorem 8 in [1] that p. d. functions $(U(g)f_{\nu}^{\lambda},\ f_{\nu}^{\lambda})$, $\nu=1,\ldots,N(\lambda)$, $\lambda=1,\ 2,\ldots$, are mutually disjoint. Hence the series $\{f_{\nu j}^{\alpha m}\}$ satisfies the condition 3°) by Propositions 6 and 7. Therefore $\{\mathfrak{M},\ U(g),\ f\}$ must be of form stated in Theorem 2.2. Similar argument may be achieved for $\{\mathfrak{N},\ U(g),\ h\}$. Consequently we obtain (1.10), (1.11) and (1.12) by simple calculations. Theorem 2.4 is thus proved.

Next, assume that the cyclic unitary representation $\{\mathfrak{H},\ U(g),\ f^{0}\}$ is irreducible. (Notice that any irreducible representation is cyclic.) Then only one couple $\langle\lambda,\ \nu\rangle$ or $\langle\alpha,\ \nu\rangle$ may be appear in (1.10). In the case $\{\mathfrak{H},\ U(g)\}=\{\mathfrak{M}_{\nu}^{\lambda},\ U(g)\}$, by the irreducibility, there exists a point $t_{0}\in T$ such that $\rho(T-\{t_{0}\})=0$. Hence $\{\mathfrak{H},\ U(g)\}$ must be of the form stated in Theorem 1.1 or 1.4. Thus we obtain Theorem 1.6.

Finally, Theorem 3 is easily seen from Theorems 1 and 2.

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