ON THE VECTOR IN HOMOGENEOUS SPACES

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The main purpose of this paper is to investigate the parallelism of vectors in homogeneous spaces. The definition of a vector and the condition for spaces under which a covariant differential of a vector is also a vector were given by E. Cartan [4] in a very intuitive way. Here I formulate this in a stricter way by his method of moving frame. Even if a homogeneous space has the property that the covariant differential of a vector is of the same kind, another definition of covariant differential may also have the required property. I will give a necessary and sufficient condition under which the definition of covariant differential is unique. Once the covariant differential has been defined it is easy to introduce a parallelism of vectors in the space. But the parallelism depends in general on the path along which we translate a vector. The condition for the spaces with an absolute parallelism can be obtained. A symmetric space in E. Cartan's sense with an absolute parallelism is an affine space with points as its elements, rotational part being a full linear group or its subgroup. Next we define a geodesic in a space admitting a parallelism of a vector and prove that under a certain condition the geodesic thus defined is an extremal for any invariant integral of our homogeneous space. When our space admits a Riemann metric which is invariant under the transformation of our group we have two sorts of parallelism of a vector, namely the one defined above and the one derived from the Riemann connection attached to our Riemann metric, and two sorts of geodesics. We give a necessary and sufficient condition in order that the two sorts of parallelism and geodesics coincide. We get in addition a sufficient condition for a homogeneous Riemann space under which all the geodesics in the sense of Riemann metric are generated by a certain one-parametric subgroup. In the last we define a space with a connection associated with a homogeneous space from our point of view and investigate some properties of the space. If such a space has not a torsion and admits an absolute parallelism of a vector the space itself is a homogeneous space which is different from the fundamental homogeneous space. Under a certain condition the geodesic defined from the viewpoint of vector-translation coincides with the extremal of the integral on our space corresponding to an invariant integral of the fundamental space. This is the generalization of the

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well known fact that the geodesic of a Riemann space gives rise to a straight line when we develop the geodesic in the euclidean space.

Throughout the whole we treat local problems only, and the word "*group*" is often used in the sense of "*group germ.*"

1. Preliminaries

1.1 We quot from E. Cartan [2] [3] the matters necessary for our purpose. Throughout the whole discussion let the indices run as follows,

$$i,\ j,\ k,\ l,\ h=1,\ 2,\ \ldots,\ n$$
$$\alpha,\ \beta,\ \gamma=n+1,\ n+2,\ \ldots,\ r$$
$$p,\ q,\ s,\ t,\ u,\ v=1,\ 2,\ \ldots,\ r$$

and let the summation $\sum_{ij}$ range over all the permutations of i and j while $\sum_{(ij)}$ ranges over all the combinations of i and j. Let the element of the r-parametric Lie group $\mathfrak{G}$ which operates transitively on a point $x=(x_1,\ \ldots,\ x_n)$ of an n-dimensional space be $x'_i=f_i(x_1,\ \ldots,\ x_n;\ a_1,\ \ldots,\ a_r)$ which we write in short $S_a: x'=f(x;\ a)$. When $S_c=S_bS_a$ we have $c=\varphi(a;\ b)$ namely $c_p=\varphi_p(a_1,\ \ldots,\ a_r;\ b_1,\ \ldots,\ b_r)$. Let S_0 be an identity transformation. We put $S_a^{-1}S_{a+da}=S_\varepsilon$ and expand $\varepsilon=(\varepsilon_1,\ \ldots,\ \varepsilon_r)$ with respect to $da=(da_1,\ \ldots,\ da_r)$. Then the parts linear in da are relative components $\omega_p(a,\ da)$ of our group $\mathfrak{G}$. These are independent Pfaffians. Let

$$(S_\varepsilon x)_i=x_i+\sum_p X_{ip}(x)\varepsilon_p+\ldots$$

be expansion with respect to ε_p, then

$$(S_a^{-1}S_{a+da}x)_i=x_i+\sum_p X_{ip}(x)\omega_p(a,\ da)+\ldots.$$

1.2 The parameter of the product of two infinitesimal transformations

$$S_\xi:\ x'=f(x;\ \xi)\qquad S_\eta:\ x''=f(x';\ \eta)$$

is $\xi+\eta$ if we neglect the terms of higher order. In fact

$$\begin{aligned} S_\eta S_\xi:\ x''_i=f_i(x';\ \eta) &= x'_i+\sum_p X_{ip}(x')\eta_p+\ldots \\ &= x_i+\sum_p X_{ip}(x)\xi_p+\ldots+\sum_p X_{ip}(x_i+\ldots)\eta_p+\ldots \\ &= x_i+\sum_p X_{ip}(x)\ (\xi_p+\eta_p)+\ldots. \end{aligned}$$

1.3 When we put $S_{a'}=S_tS_a$, where t is independent of a, we have $S_{a'}^{-1}S_{a'+da'}=S_a^{-1}S_{a+da}$. In other words $\omega(a',\ da')=\omega(a,\ da)$ when $a'=\varphi(a,\ t)$. On the other hand if we put $S_{a'}=S_aS_t$ where t is independent of a, we get on account of the relation $S_{a'}^{-1}S_{a'+da'}=S_t^{-1}(S_a^{-1}S_{a+da})S_t$

$$\omega_p(a',\ da')=\sum_q \tau_{pq}(t)\omega_q(a,\ da) \tag{1.1}$$

where $(\tau_{pq}(t))$ is a transformation of a linear adjoint group corresponding to S_t. For an exterior differential of ω_s we have

$$d\omega_s = \sum_{(pq)} c_{pqs}[\omega_p\omega_q] \qquad (c_{pqs} = -c_{qps} \text{ const.}). \tag{1.2}$$

Putting $\sum_i X_{ip}\frac{\partial}{\partial x_i} = X_p$ we get

$$(X_p,\ X_q) = \sum_s c_{pqs}X_s, \tag{1.3}$$

$$\sum_u (c_{pqu}c_{ust} + c_{qsu}c_{upt} + c_{spu}c_{uqt}) = 0. \tag{1.4}$$

Let the variation from a to a' determined by $S_{a'} = S_aS_\varepsilon$, where ε is independent of a, be δ, and an arbitrary variation be d. Then by δ $S_a^{-1}S_{a+da}$ is transformed into $(S_aS_\varepsilon)^{-1}S_{a+da}S_\varepsilon = S_\varepsilon^{-1}(S_a^{-1}S_{a+da})S_\varepsilon$, while by $d\,S_a^{-1}S_{a+\delta a} = S_\varepsilon$ is invariant. Hence $d\omega_p(a,\ \delta a) = 0$, and by the definition of an exterior derivative $d\omega_p = d\omega_p(a,\ \delta a) - \delta\omega_p(a,\ da)$ we have $\delta\omega_s(a,\ da) = -\sum_{pq} c_{pqs}e_q\omega_p(a,\ da)$ where e_q is a parameter of S_ε. If $S_\varepsilon = S_t^{-1}S_{t+dt}$ this is an infinitesimal transformation of a linear adjoint group, so we have

$$\delta\tau_{pq} = \sum_{us} c_{usp}\omega_u^{(0)}\tau_{sq} \qquad (\omega_u^{(0)} = \omega_u(t,\ dt)). \tag{1.5}$$

1.4 Between the coefficients of a transformation $\tau = (\tau_{pq})$ of a linear adjoint group there exist the following relations

$$\sum_q c_{stq}\tau_{pq} = \sum_{qr} c_{qrp}\tau_{qs}\tau_{rt}. \tag{1.6}$$

In fact if we put $P_{pst} = \sum_q c_{stq}\tau_{pq} - \sum_{qr} c_{qrp}\tau_{qs}\tau_{rt}$ we have $\delta P_{pst} = \sum_{uv} c_{uvp}P_{stv}\omega_u^{(0)}$. On the other hand for $S_t = S_0$ (τ_{pq}) is a unit matrix and so $P_{pst} = c_{stp} - c_{stp} = 0$. Hence $P_{pst} = 0$.

1.5 All the transformations of our group which fix a certain fixed point of our space form a subgroup. We denote it by $\mathfrak{H}$. Then we can attach to each point of our homogeneous space a set of frames $S_a\mathfrak{H}R$, where R is a fundamental frame. For the transformation S_a and S_{a+da} belonging to $\mathfrak{H}$ there exist n linearly independent combinations of relative components ω_p with constant coefficients. Let these be $\omega_1, \ldots, \omega_n$ anew. These are *principal relative components* of our homogeneous space $\mathfrak{G}/\mathfrak{H}$. Then we have the relations

$$d\omega_i = \sum_{(pj)} c_{pji}[\omega_p\omega_j], \qquad c_{\alpha\beta i} = 0. \tag{1.7}$$

If we take $(x_1, \ldots, x_n;\ u_{n+1}, \ldots, u_r)$ as parameters of our space we get

$$\omega_i = \sum_j p_{ij}(x_1, \ldots, x_n;\ u_{n+1}, \ldots, u_r)dx_j.$$

For a transformation of a linear adjoint group which corresponds to S_t belonging to $\mathfrak{H}$, which we assume connected hereafter, we have

(1.8) $$\tau_{i\alpha}=0.$$

Hence if $S_{a'}=S_aS_t$ principal relative components are transformed in such a way that

(1.9) $$\omega_i(a',\ da')=\sum_j \tau_{ij}\omega_j(a,\ da)$$

and the infinitesimal transformation of $\omega_i=\omega_i(a,\ da)$ is given by

(1.10) $$\delta\omega_i=\sum_{\alpha k} c_{\alpha k i}\omega_\alpha(t,\ dt)\omega_k.$$

Hence

(1.11) $$\delta\tau_{ij}=\sum_{\alpha k} c_{\alpha k i}\omega_\alpha(t,\ dt)\tau_{kj}.$$

The matrix group with $\tau=(\tau_{ij})$ as its elements is called a *linear group of isotropy.*

1.6 A transformation S_a of our group $\mathfrak{G}$ operates transitively on our homogeneous space $\mathfrak{G}/\mathfrak{H}$. We assume throughout the whole discussion that the only transformation S_a which fixes each point of our homogeneous space $\mathfrak{G}/\mathfrak{H}$ is an identity transformation S_0, namely $\mathfrak{G}$ is *effective* on $\mathfrak{G}/\mathfrak{H}$.

1.7 Let σ be an involutive automorphism of a Lie group $\mathfrak{G}$. All the elements of $\mathfrak{G}$ which are invariant under σ form a subgroup $\mathfrak{H}$. If this subgroup is closed and the homogeneous space $\mathfrak{G}/\mathfrak{H}$ has the property stated in **1.6** we call this space a symmetric space. For the principal relative components ω_i and the secondary relative components ω_α which we choose suitably we have the relations

(1.12) $$d\omega_i=\sum_{\alpha j} c_{\alpha j i}[\omega_\alpha\omega_j],\quad d\omega_\alpha=\sum_{(ij)} c_{ij\alpha}[\omega_i\omega_j]+\sum_{(\beta\gamma)} c_{\beta\gamma\alpha}[\omega_\beta\omega_\gamma],$$

namely we have in addition to $c_{\alpha\beta i}=0$

(1.13) $$c_{ijk}=0$$

(1.14) $$c_{\alpha i\beta}=0.$$

2. Covariant differential of a vector

2.1 To any point of our homogeneous space we attach a frame S_aR, and let $S_a\mathfrak{H}R$ be a set of frames attached to the point x. We transform a frame from S_aR to $S_{a'}R=S_aS_tR$ where S_t belongs to $\mathfrak{H}$. Then we have

$$S_{a'}^{-1}S_{a'+da'}=(S_aS_t)^{-1}S_{a+da}S_{t+dt}=S_t^{-1}(S_a^{-1}S_{a+da})S_t\cdot S_t^{-1}S_{t+dt},$$

hence by (1.1) and **1.2** we get

(2.1) $$\omega_p(a',\ da')=\sum_q \tau_{pq}\omega_q(a,\ da)+\omega_p(t,\ dt).$$

So if we put

$$\omega_p = \omega_p(a, da), \quad \bar{\omega}_p = \omega_p(a', da'), \quad \omega_p^{(0)} = \omega_p(t, dt)$$

then by virtue of the relations $\omega_i^{(0)} = 0$ and $\tau_{i\alpha} = 0$ we have

$$\bar{\omega}_i = \sum_j \tau_{ij}\omega_j, \quad \bar{\omega}_\alpha = \sum_p \tau_{\alpha p}\omega_p + \omega_\alpha^{(0)}. \tag{2.2}$$

Here we state lemmas.

LEMMA 1. *For $\bar{\omega}_j$, $\bar{\omega}_\alpha$ given by* (2.2)

$$d\bar{\omega}_i = \sum_{(pj)} c_{pji}[\bar{\omega}_p\bar{\omega}_j], \quad d\bar{\omega}_\alpha = \sum_{(pq)} c_{pq\alpha}[\bar{\omega}_p\bar{\omega}_q],$$

namley c_{pqr} are the same for ω_p and $\bar{\omega}_p$.

We can state in a more general form which will be used later.

LEMMA 2. *Let π_p and $\bar{\pi}_p$ be Pfaffians such that*

$$\bar{\pi}_p = \sum_q \tau_{pq}\pi_q + \omega_p(t, dt)$$

where (τ_{pq}) is an element of a linear adjoint group and $\omega_p(t, dt)$ are relative components corresponding to S_t (not necessarily an element of $\mathfrak{H}$). Then putting

$$\Omega_p = d\pi_p - \sum_{(qs)} c_{qsp}[\pi_q\pi_s], \quad \bar{\Omega}_p = d\bar{\pi}_p - \sum_{(qs)} c_{qsp}[\bar{\pi}_q, \bar{\pi}_s] \tag{2.3}$$

we get

$$\bar{\Omega}_p = \sum_q \tau_{pq}\Omega_q. \tag{2.4}$$

Proof. From the assumed equality we get by putting $\omega_p^{(0)} = \omega_p(t, dt)$

$$d\bar{\pi}_p = \sum_q [d\tau_{pq}\pi_q] + \sum_q \tau_{pq}d\pi_q + d\omega_p^{(0)}$$

$$[\bar{\pi}_q\bar{\pi}_s] = [\sum_u \tau_{qu}\pi_u + \omega_q^{(0)}, \ \sum_v \tau_{sv}\pi_v + \omega_s^{(0)}].$$

Hence

$$\begin{aligned}\bar{\Omega}_p &= d\bar{\pi}_p - \sum_{(qs)} c_{qsp}[\bar{\pi}_q\bar{\pi}_s] \\ &= \sum_q \tau_{pq}(d\pi_q - \sum_{(st)} c_{stq}[\pi_s\pi_t]) + \sum_{(uv)}(\sum_q c_{uvq}\tau_{pq} - \sum_{qs} c_{qsp}\tau_{qu}\tau_{sv})[\pi_u\pi_v] \\ &\quad + \sum_q [d\tau_{pq} - \sum_{st} c_{stp}\omega_s^{(0)}\tau_{tq}, \ \pi_q] + d\omega_p^{(0)} - \sum_{(qs)} c_{qsp}[\omega_q^{(0)}\omega_s^{(0)}].\end{aligned}$$

By virtue of (1.6) (1.5) (1.2)

$$\bar{\Omega}_p = \sum_q \tau_{pq}\Omega_q.$$

If we notice that $\Omega_p = 0$ leads to $\bar{\Omega}_p = 0$ we get lemma 1.

2.2 To any point x of our n-dimensional homogeneous space we attach an n-dimensional vector space which we call a *tangent space* at x. Let $S_a\mathfrak{H}R$ be a set of frames corresponding to the point x. Then by a frame transformation from S_aR to S_aS_tR, where S_t belongs to $\mathfrak{H}$, the principal relative components undergo the transformation given by (1.9). Now we take in a tangent space a frame $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ corresponding to S_aR and let a frame $(\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n)$ corresponding to S_aS_tR be given by $\mathbf{e}_j = \sum \tau_{ij}\bar{\mathbf{e}}_i$. Then the components of a vector v in our tangent space undergo the transformation

$$(2.5) \qquad \bar{v}_i = \sum_j \tau_{ij} v_j .$$

The relation (1.9) shows that the principal relative components ω_i's are components of an infinitesimal vector. Let the components of a vector v corresponding to S_aS_tR be $v_i(t)$. Then $v_i(t) = \sum_j \tau_{ij} v_i(0)$, hence by (1.11) we get for the components $v_i(t)$ of the same vector the relation

$$(2.6) \qquad dv_i - \sum_{\alpha j} c_{\alpha ji}\omega_\alpha(t,\, dt) v_j = 0.$$

Now we take a vector field $v(x)$ on our space $\mathfrak{G}/\mathfrak{H}$. Components of a vector $v(x)$ with respect to a frame S_aR are functions of a. By a frame transformation from S_aR to $S_{a'}R = S_aS_tR$, where S_t is an element of $\mathfrak{H}$ with variable t, we have by (2.2)

$$(2.7) \qquad \bar{\omega}_i = \sum_j \tau_{ij}\omega_j$$

$$(2.8) \qquad \bar{\omega}_\alpha = \sum_p \tau_{\alpha p}\omega_p + \omega_\alpha^{(0)} .$$

We get on account of the relation (2.5)

$$(2.9) \qquad d\bar{v}_i = \sum_j \tau_{ij} dv_j + \sum_j d\tau_{ij} v_j .$$

By (1.11) and (2.8)

$$d\bar{v}_i = \sum_j \tau_{ij} dv_j + \sum_{\alpha kj} c_{\alpha ki}\omega_\alpha^{(0)}\tau_{kj} v_j = \sum_j \tau_{ij} dv_j + \sum_{\alpha kj} c_{\alpha ki}(\bar{\omega}_\alpha - \sum_p \tau_{\alpha p}\omega_p)\tau_{kj} v_j .$$

Hence taking (2.5) into consideration we get

$$(2.10) \qquad d\bar{v}_i - \sum_{\alpha k} c_{\alpha ki}\bar{\omega}_\alpha \bar{v}_k = \sum_j \tau_{ij} dv_j - \sum_{\alpha kpj} c_{\alpha ki}\tau_{\alpha p}\tau_{kj}\omega_p v_j .$$

It is quite natural to define a covariant differential of a vector v_i by

$$(2.11) \qquad Dv_i = dv_i - \sum_{\alpha k} c_{\alpha ki}\omega_\alpha v_k ,$$

because by (2.6) it vanishes for a vector at a fixed point. But it is not always a vector. Here it is important to remark that by the discussion of **2.1** $c_{\alpha ki}$ is the same for ω_p and $\bar{\omega}_p$. In order that Dv is a vector (2.10) must be transformed in the form

$$d\bar{v}_i - \sum_{\alpha k} c_{\alpha ki}\bar{\omega}_\alpha \bar{v}_k = \sum_j \tau_{ij}(dv_j - \sum_{\alpha k} c_{\alpha kj}\omega_\alpha v_k).$$

So it is necessary and sufficient that the following relations hold, (2.12) $$\sum_{\beta j} c_{\beta ji}\tau_{\beta\alpha}\tau_{jk} = \sum_{j} c_{\alpha kj}\tau_{ij}$$

(2.13) $$\sum_{\alpha j} c_{\alpha ji}\tau_{\alpha h}\tau_{jk} = 0.$$

Since $c_{\alpha\beta i}=0$, $\tau_{i\alpha}=0$ (2.12) holds identically by virtue of (1. 6). As for (2. 13) we get on account of the non-singularity of (τ_{jk}) $\sum_{\alpha} c_{\alpha ji}\tau_{\alpha h}=0$. Now putting $B_{jih}=\sum_{\alpha} c_{\alpha ji}\tau_{\alpha h}$ we get by virtue of the relations $\delta\tau_{\alpha h}=\sum_{\gamma p} c_{\gamma p\alpha}\omega_{\gamma}^{(0)}\tau_{ph}$

$$\delta B_{jih} = \sum_{\alpha\gamma} c_{\alpha ji}\Big(\sum_{\beta} c_{\gamma\beta\alpha}\tau_{\beta h} + \sum_{k} c_{\gamma k\alpha}\tau_{kh}\Big)\omega_{\gamma}^{(0)}.$$

Here

$$\sum_{\alpha\beta} c_{\gamma\beta\alpha}c_{\alpha ji}\tau_{\beta h} = -\sum_{k\beta}(c_{\beta jk}c_{k\gamma i} + c_{j\gamma k}c_{k\beta i})\tau_{\beta h} = -\sum_{k} c_{k\gamma i}B_{jkh} + \sum_{k} c_{j\gamma k}B_{kih}.$$

And so

$$\delta B_{jih} = \sum_{\gamma}\Big(-\sum_{k} c_{k\gamma i}B_{jkh} + \sum_{k} c_{j\gamma k}B_{kih} + \sum_{\alpha k} c_{\alpha ji}c_{\gamma k\alpha}\tau_{kh}\Big)\omega_{\gamma}^{(0)}.$$

For $S_t=S_0$ we have $B_{ijh}=0$. Hence the equality $B_{jih}=0$ is equivalent to

(2.14) $$\sum_{\alpha} c_{\alpha ji}c_{\gamma k\alpha} = 0.$$

Now we assert that *under the condition* (2.14)

(2.15) *matrices* $C_\alpha=(c_{\alpha ij})$ $(\alpha=n+1, \dots, r)$ *are linearly independent.*

In fact for the set of constants $(\lambda_{n+1}, \dots, \lambda_r)$ satisfying the relations $\sum_{\alpha}\lambda_\alpha c_{\alpha ji}=0$ for all i and j we consider $\sum_{\alpha}\lambda_\alpha X_\alpha$. All of such infinitesimal operators generate an invariant subgroup $\mathfrak{K}$ of $\mathfrak{G}$. The verification runs as follows,

$$\Big(\sum_{\alpha}\lambda_\alpha X_\alpha,\ X_\beta\Big) = \sum_{\alpha\gamma}\lambda_\alpha c_{\alpha\beta\gamma}X_\gamma$$

$$\sum_{\gamma}\Big(\sum_{\alpha}\lambda_\alpha c_{\alpha\beta\gamma}\Big)c_{\gamma ij} = -\sum_{\alpha h}\lambda_\alpha c_{\beta ih}c_{h\alpha j} - \sum_{\alpha h}\lambda_\alpha c_{i\alpha h}c_{h\beta j} = 0$$

$$\Big(\sum_{\alpha}\lambda_\alpha X_\alpha,\ X_i\Big) = \sum_{\alpha p}\lambda_\alpha c_{\alpha ip}X_p = \sum_{\alpha\beta}\lambda_\alpha c_{\alpha i\beta}X_\beta$$

$$\sum_{\beta}\Big(\sum_{\alpha}\lambda_\alpha c_{\alpha i\beta}\Big)c_{\beta jk} = \sum_{\alpha}\lambda_\alpha\Big(\sum_{\beta} c_{\beta jk}c_{\alpha i\beta}\Big) = 0.$$

Each element of $\mathfrak{K}$ leaves all the points of our homogeneous space $\mathfrak{G}/\mathfrak{H}$ invariant. Hence by the assumption of **1.6** (2.15) must be satisfied.

By (2.14) and (2.15) we get

(2.16) $$c_{\alpha i\beta} = 0.$$

This is equivalent to $\tau_{\beta i}=0$. Thus

THEOREM 2.1 *In order that* $Dv_i = dv_i - \sum_{\alpha j} c_{\alpha ji}\omega_\alpha v_j$ *is a component of a vector it is necessary and sufficient that* $c_{\alpha i\beta}=0$ *holds in our space.*

This result was obtained by E. Cartan in his paper [4]. But the discussion there is too intuitive to be understood rigorously. Symmetric spaces satisfy our condition (2.16).

2.3[1] In the spaces where $c_{\alpha i\beta}=0$ holds we can define a covariant differential which is also a vector. But in some cases it is not unique. The reason is that for a given space relative components are not uniquely determined. They admits the following three sorts of transformations

$$\text{(i)}\qquad \bar{\omega}_i=\sum_j A_{ij}\omega_j,\quad \bar{\omega}_\alpha=\omega_\alpha\quad (|A_{ij}|\neq 0)$$

$$\text{(ii)}\qquad \bar{\omega}_i=\omega_i,\quad \bar{\omega}_\alpha=\sum_\beta A_{\alpha\beta}\omega_\beta\quad (|A_{\alpha\beta}|\neq 0)$$

$$\text{(iii)}\qquad \bar{\omega}_i=\omega_i,\quad \bar{\omega}_\alpha=\omega_\alpha+\sum_i A_{\alpha i}\omega_i$$

where A_{ij}, $A_{\alpha\beta}$, $A_{\alpha i}$ are constants. Putting

$$d\bar{\omega}_i=\sum_{(pk)}\bar{c}_{pki}[\bar{\omega}_p\bar{\omega}_k],\quad d\bar{\omega}_\alpha=\sum_{(pq)}\bar{c}_{pq\alpha}[\bar{\omega}_p\bar{\omega}_q]$$

we can easily see that the condition $c_{\alpha i\beta}=0$ is equivalent to $\bar{c}_{\alpha i\beta}=0$ for the transformations (i) and (ii). For the transformation (i) let a_{ij} be numbers such that $\omega_i=\sum_j a_{ij}\bar{\omega}_j$ then

$$d\bar{\omega}_i=\sum_j A_{ij}d\omega_j=\sum_{j(pk)}A_{ij}c_{pkj}[\omega_p,\ \sum_l a_{kl}\bar{\omega}_l].$$

Hence

$$\bar{c}_{\alpha ki}=\sum_{jl}A_{ij}c_{\alpha lj}a_{lk}.$$

Let v_i be components of a vector v with respect to ω_i. Then it is natural to define components $\bar{v}_i$ of the same vector v with respect to $\bar{\omega}_i$ by $\bar{v}_i=\sum_j A_{ij}v_j$, and we have

$$d\bar{v}_i-\sum_{\alpha k}\bar{c}_{\alpha ki}\bar{\omega}_\alpha\bar{v}_k=\sum_j A_{ij}dv_j-\sum_{\alpha jkl}A_{ij}c_{\alpha lj}a_{lk}\omega_\alpha\bar{v}_k=\sum_j A_{ij}(dv_j-\sum_{\alpha l}c_{\alpha lj}\omega_\alpha v_l).$$

So we see that (i) has no effect on the definition of covariant differential. Next for (ii) let $a_{\alpha\beta}$ be numbers such that $\omega_\alpha=\sum_\beta a_{\alpha\beta}\bar{\omega}_\beta$ and then

$$\begin{aligned}d\bar{\omega}_i=d\omega_i&=\sum_{(pk)}c_{pki}[\omega_p\omega_k]=\sum_{(jk)}c_{jki}[\omega_j\omega_k]+\sum_{\alpha k}c_{\alpha ki}[\omega_\alpha\omega_k]\\&=\sum_{(jk)}c_{jki}[\omega_j\omega_k]+\sum_{\alpha k}c_{\alpha kj}[\sum_\beta a_{\alpha\beta}\bar{\omega}_\beta,\ \omega_k].\end{aligned}$$

Hence

$$\bar{c}_{\beta ki}=\sum_\alpha c_{\alpha ki}a_{\alpha\beta}$$

$$dv_i-\sum_{\alpha k}\bar{c}_{\alpha ki}\bar{\omega}_\alpha v_k=dv_i-\sum_{\beta\alpha k}c_{\beta ki}a_{\beta\alpha}\bar{\omega}_\alpha v_k=dv_i-\sum_{\beta k}c_{\beta ki}\omega_\beta v_k.$$

[1] The discussi o n and the results of this section are independent of the following sectons.

(ii) has no effect on Dv_i.

As for (iii)

$$d\bar{\omega}_i = d\omega_i = \sum_{(jk)} c_{jki}[\omega_j\omega_k] + \sum_{\alpha k} c_{\alpha ki}[\omega_\alpha\omega_k]$$
$$= \sum_{(jk)} c_{jki}[\omega_j\omega_k] + \sum_{\alpha k} c_{\alpha ki}[\bar{\omega}_\alpha - \sum_j A_{\alpha j}\omega_j,\ \omega_k].$$

Hence
$$\bar{c}_{\alpha ki} = c_{\alpha ki},$$
$$dv_i - \sum_{\alpha j}\bar{c}_{\alpha ji}\bar{\omega}_\alpha v_j = dv_i - \sum_j(\sum_\alpha c_{\alpha ji}\omega_\alpha + \sum_{\alpha k} c_{\alpha ji}A_{\alpha k}\omega_k)v_j$$

and so two sorts of covariant differentials (if they exist) would coincide when and only when $\sum_\alpha c_{\alpha ji}A_{\alpha k} = 0$. But then by virtue of (2.15) we would have $A_{\alpha k} = 0$. Hence if there exist two sorts of covariant differentials which are vectors, they do not coincide. We seek for the condition under which for ω_i, ω_α and $\bar{\omega}_i$, $\bar{\omega}_\alpha$ related by (iii) two differentials are vectors each. Here

$$d\bar{\omega}_\alpha = d\omega_\alpha + \sum_i A_{\alpha i}d\omega_i$$
$$= \sum_{(jk)} c_{jk\alpha}[\omega_j\omega_k] + \sum_{\beta k} c_{\beta k\alpha}[\omega_\beta\omega_k] + \sum_{(\beta\gamma)} c_{\beta\gamma\alpha}[\omega_\beta\omega_\gamma]$$
$$+ \sum_i A_{\alpha i}(\sum_{\beta j} c_{\beta ji}[\omega_\beta\omega_j] + \sum_{(kj)} c_{kji}[\omega_k\omega_j]$$
$$= \sum_{(jk)} c_{jk\alpha}[\omega_j\omega_k] + \sum_{\beta k} c_{\beta k\alpha}[\bar{\omega}_\beta - \sum_i A_{\beta i}\omega_i,\ \omega_k]$$
$$+ \sum_{(\beta\gamma)} c_{\beta\gamma\alpha}[\bar{\omega}_\beta - \sum_j A_{\beta j}\omega_j,\ \bar{\omega}_\gamma - \sum_k A_{\gamma k}\omega_k]$$
$$+ \sum_i A_{\alpha i}(\sum_{\beta k} c_{\beta ki}[\bar{\omega}_\beta - \sum A_{\beta l}\omega_l,\ \omega_k] + \sum_{(kj)} c_{kji}[\omega_k\omega_j]).$$

Hence

(2.17)
$$\bar{c}_{\beta k\alpha} = c_{\beta k\alpha} + \sum_\gamma c_{\gamma\beta\alpha}A_{\gamma k} + \sum_i A_{\alpha i}c_{\beta ki}.$$

For ω_i, ω_α and $\bar{\omega}_i$, $\bar{\omega}_\alpha$ satisfying $c_{\alpha i\beta} = 0$ and $\bar{c}_{\alpha i\beta} = 0$ we have

(2.18)
$$\sum_\gamma c_{\gamma\beta\alpha}A_{\gamma k} + \sum_i c_{\beta ki}A_{\alpha i} = 0.$$

Now we take infinitesimal operators such that $\sum_\alpha A_{\alpha k}X_\alpha = X^{(k)}$, then

(2.19)
$$(X_\beta,\ X^{(k)}) = (X_\beta,\ \sum_\gamma A_{\gamma k}X_\gamma) = \sum_{\alpha\gamma} A_{\gamma k}c_{\beta\gamma\alpha}X_\alpha$$
$$= \sum_{i\alpha} c_{\beta ki}A_{\alpha i}X_\alpha = \sum_i c_{\beta ki}X^{(i)},$$

so $X^{(k)} = \sum_\alpha A_{\alpha k}X_\alpha$ generate an invariant subgroup of the group $\mathfrak{H}$ generated by X_α's. $X^{(i)}$ are not necessarily linearly independent. By choosing transformations (i) and (ii) suitably we can assume without loss of generality that

(2.20)
$$(A_{\alpha i}) = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

where E is a unit matrix of degree ν. In (2.20) 0's in the second row and in the second colomn may vanish. But the discussion is the same. By (2.20) we get

$$X^{(1)}=X_{n+1},\ \ldots,\ X^{(\nu)}=X_{n+\nu},\ X^{(\nu+1)}=\ldots=X^{(n)}=0.$$

For $k\leqq\nu$ we get by (2.19)

$$(X_\beta,\ X^{(k)})=\sum_{i=1}^{\nu}c_{\beta ki}X^{(i)}=\sum_{i=1}^{\nu}c_{\beta ki}X_{n+i}.$$

On the other hand

$$(X_\beta,\ X^{(k)})=(X_\beta,\ X_{n+k})=\sum_{\gamma=n+1}^{r}c_{\beta,n+k,\gamma}X_\gamma.$$

Hence we get the relations

$$(2.21)\qquad \begin{aligned} &c_{\beta,\ n+k,\ n+i}=c_{\beta ki} && (k\leqq\nu,\ i\leqq\nu)\\ &c_{\beta,\ n+k,\ n+i}=0 && (k\leqq\nu,\ i>\nu).\end{aligned}$$

For $k>\nu$

$$0=(X_\beta,\ X^{(k)})=\sum_{i=1}^{\nu}c_{\beta ki}X^{(i)}=\sum_{i=1}^{\nu}c_{\beta ki}X_{n+i}.$$

Hence

$$(2.22)\qquad c_{\beta ki}=0\qquad (k>\nu,\ i\leqq\nu).$$

Denoting by C_β the matrix with $c_{\beta pq}$ as a coefficient of q-th row and p-th colomn we get

$$(2.23)\qquad C_\beta=\begin{pmatrix} B_\beta & 0 & & \\ & & 0 & \\ & & & \\ & & B_\beta & \\ 0 & & & \\ & & 0 & \end{pmatrix}\ \begin{matrix}\uparrow\\ n\\ \downarrow\end{matrix}$$
$$\leftarrow\ n\ \rightarrow$$

where B_β is a square matrix of degree ν. For an element of a linear adjoint group $\tau=(\tau_{pq})$ corresponding to $S_t\in\mathfrak{H}$ we have by (1.5) $d\tau=(\sum_\alpha C_\alpha\omega_\alpha)\tau$. Hence τ is a matrix of the form (2.23). Conversely if τ has this form we have $\bar{c}_{\beta k\alpha}=c_{\beta k\alpha}$ by putting $X^{(i)}=\sum_\alpha A_{\alpha i}X_\alpha$ with $A_{\alpha i}$ given by (2.20). In fact we have then $(X_\beta X^{(k)})=\sum_i c_{\beta ki}X^{(i)}$, hence (2.18), and putting $\bar{\omega}_\alpha=\omega_\alpha+\sum_i A_{\alpha i}\omega_i$ we get $\bar{c}_{\beta k\alpha}=c_{\beta k\alpha}$. In summary

THEOREM 2.2 *Let ω_i and ω_α be principal and secondary relative components* *of a homogeneous space* $\mathfrak{G}/\mathfrak{H}$ *and* $Dv_i = dv_i - \sum_{\alpha k} c_{\alpha ki}\omega_\alpha v_k$ *be covariant differential of a vector which is also a vector. In order that there exists another system of principal and secondary relative components* $\bar{\omega}_i$, $\bar{\omega}_\alpha$ *of our space such that* $\overline{Dv_i} = d\bar{v}_i - \sum_{\alpha k} \bar{c}_{\alpha ki}\bar{\omega}_\alpha \bar{v}_k$ *for any vector* $\bar{v}_i$ *is a component of a vector, it is necessary and sufficient that the following two conditions are satisfied in our space:*

(*a*) *the linear group of isotropy has an invariant linear subspace of dimension* $n-\nu$. *Let the matrix operating on the complementary linear subspace of this invariant subspace be* τ_1 *with a certain choice of base.*

(*b*) *let the part of a linear adjoint group corresponding to an element* S_t *of* $\mathfrak{H}$ *which operates on the secondary relative components be* τ_2, *then for each element of* $\mathfrak{H}$ τ_2 *keeps a certain linear subspace of dimension* ν *invariant and with a suitable choise of base the matrix operating on this part coincides with* τ_1.
As a consequence of (*a*) (*b*) $\mathfrak{H}$ *has a* ν*-parametric invariant subgroup.*

The conditions stated here are rather complicated, but we can get sufficient conditions in simple forms.

THEOREM 2.3 *If our space has the following properties, covariant differential of a vector, if exist, is uniquely determined;*

(*a*) *a linear group of isotropy is irreducible*

(*b*) $\mathfrak{H}$ *has not an n-parametric invariant subgroup.*

THEOREM 2.4 *If our space has the following properties, covariant differential of a vector, if exist, is uniquely determined*

(*a*) $\mathfrak{H}$ *has not an essential invariant subgroup*

(*b*) *a linear group of isotropy has not a* $2n-r$*-dimensional invariant linear subspace.*

In the homogeneous spaces which usually appear these conditions are satisfied. But even in symmetric spaces these are not necessarily satisfied. An example will be given in **3.3**.

3. Space with absolute parallelism

3.1 We call hereafter a *space with parallelism* the one in which a covariant differential is also a vector, namely $c_{\alpha i\beta} = 0$ is satisfied. In such a space it is natural to define a parallelism of a vector by

$$Dv_i = dv_i - \sum_{\alpha k} c_{\alpha ki}\omega_\alpha v_k = 0. \tag{3.1}$$

It can be easily seen that by a suitable choise of functions $c_i(t)$ $(i = 1, \ldots, n)$ the solutions of the differential equations

$$\omega_i(x, u, dx) = c_i(t)dt, \qquad \omega_\alpha(x, u, dx, du) = 0$$

give any curve in our space. With such a choice of frames attached to each point of our curve components of vectors which are parallel along the curve are constant.

3.2 When we translate a vector from one point p to another point q along a curve in our space, the resulting vector depends on the path and is not uniquely determined. We call the space, in which for any points p and q and any choice of the initial vector the resulting vector does not depend on the path from p to q, a *space with absolute parallelism.* We seek for necessary and sufficient condition for such a space. This condition is obtained by taking an exterior differential of (3.1), namely

$$-\sum_{\alpha j} c_{\alpha ji}([dv_j,\ \omega_\alpha]+d\omega_\alpha v_j)=0$$

and putting (3.1) into this. Hence

$$\sum_k\Big(\sum_{\alpha\beta j} c_{\alpha ji}c_{\beta kj}[\omega_\beta\omega_\alpha]+\sum_{\alpha(pq)} c_{\alpha ki}c_{pq\alpha}[\omega_p\omega_q]\Big)v_k=0.$$

As the values of v_k are arbitrary we get

$$\sum_{\alpha\beta j} c_{\alpha ji}c_{\beta kj}[\omega_\beta\omega_\alpha]+\sum_{\alpha(pq)} c_{\alpha ki}c_{pq\alpha}[\omega_p\omega_q]=0.$$

Putting the coefficients of $[\omega_j\omega_l]$ to zero we get $\sum_\alpha c_{\alpha ki}c_{jl\alpha}=0$. Hence by (2.15)

$$c_{jl\alpha}=0. \tag{3.2}$$

The coefficients of $[\omega_j\omega_\beta]$ are zero on account of $c_{j\beta\alpha}=0$. As to those of $[\omega_j\omega_\beta]$

$$\sum_j c_{\alpha ji}c_{\beta kj}-\sum_j c_{\beta ji}c_{\alpha kj}+\sum_\gamma c_{\gamma ki}c_{\beta\alpha\gamma}=-\sum_p(c_{\beta kp}c_{p\alpha i}+c_{k\alpha p}c_{p\beta i}+c_{\alpha\beta p}c_{pki})=0.$$

So (3.2) is the required condition. This is equivalent to

$$(X_iX_j)=\sum_k c_{ijk}X_k. \tag{3.3}$$

Moreover we have by virtue of (2.16)

$$(X_\alpha X_i)=\sum_j c_{\alpha ij}X_j. \tag{3.4}$$

Hence we get the following theorem.

THEOREM 3.1 *The necessary and sufficient condition under which a homogeneous space $\mathfrak{G}/\mathfrak{H}$ has the property of absolute parallelism of a vector is that $\mathfrak{G}$ is generated by $\mathfrak{H}$ and an n-parametric invariant subgroup. This invariant subgroup operates on our space simply transitively.*

The latter half of our theorem can be easily verified.

3.3 Now we seek for a symmetric space with absolute parallelism. In a symmetric space we have $c_{ijk}=0$. Hence by (3.3) $(X_iX_j)=0$ and so the group generated by X_i $(i=1, 2, \ldots, n)$ is commutative. By a suitable choice of variables we get $X_i = \frac{\partial}{\partial x_i}$. Putting $X_\alpha = \sum_j X_{\alpha j}(x)\frac{\partial}{\partial x_j}$ we get by (3.4) $X_{\alpha j} = -\sum_i c_{\alpha ij}x_i + a_{\alpha j}$, hence

$$X_\alpha = -\sum_{ij} c_{\alpha ij}x_i\frac{\partial}{\partial x_j} + \sum_j a_{\alpha j}\frac{\partial}{\partial x_j}. \tag{3.5}$$

When we take

$$X_i = \frac{\partial}{\partial x_i}, \quad X_\alpha = -\sum_{ij} c_{\alpha ij}x_i\frac{\partial}{\partial x_j} \tag{3.6}$$

instead of X_i and X_α given by (3.5) we obtain the same structure equation

$$(X_iX_j) = \sum_k c_{ijk}X_k, \quad (X_\alpha X_i) = \sum_j c_{\alpha ij}X_j, \quad (X_\alpha X_\beta) = \sum_\gamma c_{\alpha\beta\gamma}X_\gamma$$

and the homogeneous spaces determined by these two sorts of X_i and X_α are the same except for a transformation of variables. The space determined by (3.6) is an affine space with points as its elements, though a rotation about a point is not necessarily a full linear group. Thus

THEOREM 3.2 *A symmetric space with an absolute parallelism is an affine space with points as its elements, whose fundamental group* $\mathfrak{G}$ *contains the group of all translations, rotation about a point being not necessarily a full linear group.*

Another proof for this theorem can be given in the following way. We have in our case

$$d\omega_i = \sum_{\alpha j} c_{\alpha ji}[\omega_\alpha\omega_j], \quad d\omega_\alpha = \sum_{(\beta\gamma)} c_{\beta\gamma\alpha}[\omega_\beta\omega_\gamma].$$

Putting $\pi_{ji} = -\sum_\alpha c_{\alpha ji}\omega_\alpha$ we get

$$\begin{aligned} d\pi_{ji} &= -\sum_\alpha c_{\alpha ji}d\omega_\alpha = -\sum_{\alpha(\beta\gamma)} c_{\alpha ji}c_{\beta\gamma\alpha}[\omega_\beta\omega_\gamma] \\ &= \sum_{(\beta\gamma)}(\sum_k c_{\gamma jk}c_{k\beta i} + \sum_k c_{j\beta k}c_{k\gamma i})[\omega_\beta\omega_\gamma] = \sum_k[-\sum_\gamma c_{\gamma jk}\omega_\gamma, -\sum_\beta c_{\beta ki}\omega_\beta]. \end{aligned}$$

Hence
$$d\pi_i = \sum_j[\pi_j\pi_{ji}], \quad d\pi_{ji} = \sum_k[\pi_{jk}\pi_{ki}].$$

This shows that our space can be imbedded into an affine space with points as its elements preserving the group theoretical structure.

Example. Transformations

$$x_1' = x_1 + \alpha, \quad x_2' = \lambda x_1 + x_2 + \beta \tag{3.7}$$

give an example of a space with absolute parallelism. Putting $S_a^{-1}S_{a+da}x = x + dx$ we get

$$dx_1 = d\alpha, \quad dx_2 = d\beta - \lambda d\alpha + x_1 d\lambda.$$

Hence
$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = x_1 \frac{\partial}{\partial x_2}$$

$$(X_1X_2) = 0, \quad (X_1X_3) = X_2, \quad (X_2X_3) = 0$$

(3.8)
$$\omega_1 = d\alpha, \quad \omega_2 = d\beta - \lambda d\alpha, \quad \omega_3 = d\lambda$$

$$d\omega_1 = 0, \quad d\omega_2 = [\omega_1\omega_3], \quad d\omega_3 = 0.$$

A translation of a vector $v = (v_1, v_2)$ is given by $dv_1 = 0$, $dv_2 = -d\lambda v_1$, hence for the frames such that $\lambda = 0$ we get $dv_1 = 0$, $dv_2 = 0$. If we take $\bar{\omega}_3 = \omega_3 + c\omega_1$ instead of ω_3 where c is an arbitrary constant, we get

$$d\omega_1 = 0, \quad d\omega_2 = [\omega_1\bar{\omega}_3], \quad d\bar{\omega}_3 = 0.$$

Hence a translation of a vector is given by $dv_1 = 0$, $dv_2 = -d(\lambda + c\alpha)v_1$ and for the frames such that $\lambda = 0$ we get $dv_1 = 0$, $dv_2 = -cd\alpha v_1$ while for the frames such that $\lambda = -c\alpha$ we have $dv_1 = 0$, $dv_2 = 0$. Thus two sorts of parallelism do not coincide, though in each case absolute parallelism of vectors holds.

4. Geodesics

4.1 It is quite natural to define a *geodesic* in our space with parallelism by a curve $x_i = x_i(\sigma)$ determined by a solution of differential equation

(4.1)
$$\frac{d}{d\sigma}\left(\frac{\omega_i}{d\sigma}\right) - \sum_{\alpha j} c_{\alpha i j} \frac{\omega_\alpha}{d\sigma}\frac{\omega_j}{d\sigma} = 0$$

with a suitable choice of parameter σ. This indicates that a vector with a direction ω_i is always parallel along the curve. In appearance (4.1) seems to be differential equation with r variables x_i and u_α but in fact it contains only n variables x_i, because the left side of (4.1) is invariant under the frame transformation about each point.

The solution of differential equation

(4.2)
$$\omega_i(x, u, dx) = c_i d\sigma, \quad \omega_\alpha(x, u, dx, du) = 0$$

where c_i's are arbitrary constants gives the solution of (4.1). In fact (4.2) satisfies (4.1) and each solution $x_i = x_i(\sigma)$, $u_\alpha = u_\alpha(\sigma)$ of (4.2) gives rise to that of (4.1) $x_i = x_i(\sigma)$, initial point and initial direction being any assigned ones with suitable choice of the values of c_i. Thus all the geodesics are obtained by solving (4.2). It can be easily seen that each solution of (4.2) gives rise to a curve generated by one-parametric motion of our group. Hence

THEOREM 4.1 *All the geodesics of our space are generated by one-parametric motion of our group.*

Not all the geodesics of our space are necessarily congruent under the motion of our group. If we transform frames at every point of our space by the same element S_t belonging to $\mathfrak{H}$ we get by (2.14)

$$\bar{\omega}_i = \sum_j \tau_{ij}\omega_j, \quad \bar{\omega}_\alpha = \sum_\beta \tau_{\alpha\beta}\omega_\beta.$$

Hence the curve obtained by solving $\omega_i = \sum_j \tau_{ij} c_j d\sigma$, $\omega_\alpha = 0$ is congruent to that obtained from (4.2). So if a linear group of isotropy operates transitively on the direction of vector space all the geodesics are congruent, and the converse is also true. Hence

THEOREM 4.2 *In order that all geodesics are congruent it is necessary and sufficient that the linear group of isotropy operates transitively on the direction of a vector space.*

We call the parameter σ appearing in (4.1) a canonical parameter. A canonical parameter is not uniquely determined but it is easily verified that between two canonical ones σ and ρ there exists the relation $\sigma = a\rho + b$ (a, b const.).

If two sorts of parallelism are defined in our space the geodesics are defined in two different ways. In the example given at the end of **3.3** one sort is a straight line while the other sort is a parabola. This can be verified as follows. If we put in (3.7) $x_1 = x_2 = 0$ we get $x_1' = \alpha$, $x_2' = \beta$. Hence we can put by virtue of (3.8)

$$\omega_1 = dx_1, \quad \omega_2 = dx_2 - \lambda dx_1, \quad \omega_3 = d\lambda$$

and the geodesic corresponding to ω_1, ω_2, ω_3 can be obtained by solving

$$\omega_1 = dx_1 = c_1 d\sigma, \quad \omega_2 = dx_2 - \lambda dx_1 = c_2 d\sigma, \quad \omega_3 = d\lambda = 0$$

and we get a straight line. On the other hand the geodesic corresponding to $\bar{\omega}_3 = \omega_3 + c\omega_1$ can be obtained by solving

$$\omega_1 = dx_1 = c_1 d\sigma, \quad \omega_2 = dx_2 - \lambda dx_1 = c_2 d\sigma, \quad \bar{\omega}_3 = d(\lambda + cx_1) = 0$$

and we get in general a parabola.

5. Invariants of a homogeneous space

5.1 Here we give attention to invariants of a homogeneous space. From a differential $dx = (dx_1, \ldots, dx_n)$ we make exterior forms

$$(5.1) \qquad [dx_{i_1}\, dx_{i_2} \ldots dx_{i_k}] \quad (i_1 < i_2 < \ldots < i_k)$$

where $i_1, i_2, \ldots, i_k$ is any combination taken from $1, 2, \ldots, n$. We denote by X_l exterior form (5.1) which we arrange lexicographically, index l running from 1 to $N = \binom{n}{k}$, and by X a vector with X_l as its components. This we write in short $X = [dx, \ldots, dx]$. Similarly from relative components ω_i we make exterior forms

$$\Omega_l = [\omega_{i_1} \omega_{i_2} \ldots \omega_{i_k}] \quad (i_1 < i_2 < \ldots < i_k) \tag{5.2}$$

and denote by Ω an N-dimensional vector with Ω_l as its components, and we write in short $\Omega = [\omega, \ldots, \omega]$. For a matrix $A = (a_{ij})$ we denote by $[AA \ldots A]$, where the number of A is k, the matrix whose coefficients are the determinants

$$A\begin{pmatrix} i_1 & i_2 & \ldots & i_k \\ j_1 & j_2 & \ldots & j_k \end{pmatrix} \quad (i_1 < i_2 < \ldots < i_k, \quad j_1 < j_2 < \ldots < j_k)$$

which are the minors of A where $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ are any combinations taken from $1, 2, \ldots, n$. Let the coefficients of $[AA \ldots A]$ be arranged in the lexicographical order of $(i_1, \ldots, i_k)$ with respect to row and in the same order of $(j_1, \ldots, j_k)$ with respect to colomn. With these preparation of notations we state a theorem.

THEOREM 5.1 *Let $x = (x_1, \ldots, x_n)$ be coordinates of a point of a homogeneous space. A homogeneous polynomial of $X = (X_1, \ldots, X_N)$ with functions of $x = (x_1, \ldots, x_n)$ as its coefficients, which is invariant under the transformation of our group, can be represented as a homogeneous polynomial of $\Omega = (\Omega_1, \ldots, \Omega_N)$ with constant coefficients.*

Proof. Let the paramaters of our space be x and u, and the transformed ones be $\bar{x}$ and $\bar{u}$. Then we have

$$\omega_i(x, u, dx) = \sum_j a_{ij}(x, u) dx_j = \omega_i(\bar{x}, \bar{u}, d\bar{x}) = \sum_j a_{ij}(\bar{x}, \bar{u}) d\bar{x}_j$$

which is the property of relative components. We denote by $\mathfrak{A}(x, u)$ the matrix $[A(x, u), \ldots, A(x, u)]$ where the number of $A(x, u) = (a_{ij}(x, u))$ is the degree k of our polynomial in question. Then $\Omega = [\omega, \ldots, \omega]$ where the number of ω is also k can be written in the following form

$$\Omega = [\omega, \ldots, \omega] = [A(x, u), \ldots, A(x, u)]X = \mathfrak{A}(x, u)X.$$

Here Ω and X are the matrices with one colomn. As ω_i's are invariant under the transformation of our group we get

$$\Omega = \mathfrak{A}(x, u)X = \mathfrak{A}(\bar{x}, \bar{u})\bar{X}. \tag{5.3}$$

$$X = \mathfrak{A}(x, u)^{-1}\Omega, \quad \bar{X} = \mathfrak{A}(\bar{x}, \bar{u})^{-1}\Omega. \tag{5.4}$$

Non-singularity of $\mathfrak{A}(x, u)$ can be verified by the fact that the inverse of $\mathfrak{A}(x, u)$ can be given by $[A(x, u)^{-1}, \ldots, A(x, u)^{-1}]$. Now let the polynomial of X_l which is invariant be

$$\sum f_{i_1 \ldots i_N}(x) X_1^{i_1} \ldots X_N^{i_N} = \sum f_{i_1 \ldots i_N}(\bar{x}) \bar{X}_1^{i_1} \ldots \bar{X}_N^{i_N}. \tag{5.5}$$

Putting (5.4) into (5.5) we get

(5.6) $$\sum F_{i_1\dots i_N}(x, u)\Omega_1^{i_1}\dots\Omega_N^{i_N}=\sum F_{i_1\dots i_N}(\bar{x}, \bar{u})\Omega_1^{i_1}\dots\Omega_N^{i_N}.$$

As our group operates simply transitively on x and u, $\bar{x}$ and $\bar{u}$ can be put into 0. Hence (5.6) is equal to

(5.7) $$\sum F_{i_1\dots i_N}(0,0)\Omega_1^{i_1}\dots\Omega_N^{i_N}$$

and the proof is completed.

The converse of our theorem is not true. The form (5.7) is of course invariant under the transformation of our group but it depends on the parameter u in general. It is an invariant of our homogeneous space when and only when it is invariant under the transformation of a linear group of isotropy.

THEOREM 5.2 *If a homogeneous space admits an invariant metric* $\sum_i \omega_i^2$ *then the followings are also invariants of our space*

$$\sum_{(ij)}[\omega_i\omega_j]^2,\quad \sum_{(ijk)}[\omega_i\omega_j\omega_k]^2,\ \dots,\ [\omega_1\omega_2\dots\omega_n]^2.$$

Proof. Let $\bar{\omega}=P\omega$ be a transformation of a linear group of isotropy then by our assumption P is an orthogonal matrix. Then $[P,\dots,P]$ which is the transformation matrix from $[\omega_{i_1}\dots\omega_{i_k}]$ to $[\bar{\omega}_{i_1}\dots\bar{\omega}_{i_k}]$ is also orthogonal. It can be verified by making the product of $[P,\dots,P]$ with its transpose $[P,\dots,P]'$. Hence the invariance of the form in question is obtained.

THEOREM 5.3 *Let* $A=(a_{ij})$ *be non-singular symmetric matrix and* $(A\omega, \omega)=\sum_{ij}a_{ij}\omega_i\omega_j$ *be an invariant of a homogeneous space. Then a quadratic form* $([A,\dots,A]\Omega, \Omega)$ *in* Ω *is also invariant, where* $\Omega=(\Omega_1,\dots,\Omega_N)$ *is obtained from* $\Omega_l=[\omega_{i_1}\dots\omega_{i_k}]$.

Proof. By taking a suitable matrix $T=(t_{ij})$ we get $(A\omega, \omega)=(\bar{\omega}, \bar{\omega})$ for $\bar{\omega}=T\omega$, hence $A=T'T$. By the previous theorem $\sum[\bar{\omega}_{i_1}\dots\bar{\omega}_{i_k}]^2$ is invariant and this can be transformed into

$$(\bar{\Omega}, \bar{\Omega})=([T\dots T]\Omega, [T\dots T]\Omega).$$

By calculation we get on account of $A=T'T$

$$[T\dots T]'[T\dots T]=[A\dots A]$$

and the theorem is proved.

An application will be given at the end of **7.1**.

5.2 Let $v=(v_1,\dots,v_n)$ be a vector on the tangent space at a point in a homogeneous space and P be a homogeneous polynomial in $(V_1,\dots,V_N)$ which are components of a multivector V constructed from $v=(v_1,\dots,v_n)$. The necessary and sufficient condition in order that the polynomial P is invariant under the transformation $\tau = (\tau_{ij})$ of a linear group of isotropy is that, when we put $v_i + dv_i$, where dv_i is given by

$$dv_i = \sum_{\alpha j} c_{\alpha ji}\omega_\alpha(t, dt)v_j, \tag{5.9}$$

in place of v_i in P, the part linear in dv_i vanishes. Here ω_α's are secondary relative components of transformations of $\mathfrak{H}$ and so they are independent Pfaffians.

In a space with parallelism translation of a vector is determined by

$$dv_i = \sum_{\alpha j} c_{\alpha ji}\omega_\alpha(a, da)v_j$$

where $\omega_\alpha(a, da)$ is relative component of a motion along a curve. As $\omega_\alpha(t, dt)$'s in (5.9) are independent a form P which is invariant under the linear group of isotropy is also invariant under the translation of vector along any curve. Thus

THEOREM 5.4 *Vector invariants with respect to linear group of isotropy are also invariant under the translation of vectors.*

For example if $\sum_i v_i^2$ is an invariant of our space then the length of a vector and the volume of a parallelpiped with n vectors as its edges are invariant under the translation of vectors.

5.3 Let $f(\omega) = f(\omega_1, \ldots, \omega_n)$ be an invariant of our space $\mathfrak{G}/\mathfrak{H}$ which is a linear homogeneous function of $\omega = (\omega_1, \ldots, \omega_n)$. Then $\Delta f(\omega)$ vanishes for an infinitesimal variation $\Delta\omega_i = \sum_{\alpha k} c_{\alpha ki}e_\alpha\omega_k$ of ω, namely

$$\sum_{\alpha ki} \frac{\partial f}{\partial \omega_i} c_{\alpha ki}e_\alpha\omega_k = 0. \tag{5.10}$$

We will prove that any curve in $\mathfrak{G}/\mathfrak{H}$ obtained by solving differential equation

$$\omega_i(x, u, dx) = c_i d\sigma, \quad \omega_\alpha(x, u, dx, du) = 0 \quad (c_i \text{ const.}) \tag{5.11}$$

is an extremal for the integral $L = \int f(\omega)$ under a certain condition.

Let a curve obtained by solving (5.11) be c and any two points on it be P_0, P_1, for which the parameters are σ_0 and σ_1 respectively. We consider a one-parametric family of curves joining P_0 and P_1 and let the parameter of the curves be ε, $\varepsilon = 0$ being that of c. Then the coordinates of points on these curves are functions of σ and ε and at P_0 and P_1 σ's are always σ_0 and σ_1 respectively. Now we denote the variation induced by the change of σ by d and that induced by the change of ε by δ. Writing down the relation

$$d\omega_i = \sum_{(jk)} c_{jki}[\omega_j\omega_k] + \sum_{\alpha k} c_{\alpha ki}[\omega_\alpha\omega_k]$$

more fully we get

$$d\omega_i(\delta) - \delta\omega_i(d) = \sum_{jk} c_{jki}\omega_j(d)\omega_k(\delta) + \sum_{\alpha k} c_{\alpha ki}(\omega_\alpha(d)\omega_k(\delta) - \omega_\alpha(\delta)\omega_k(d)).$$

(5.11) means $(\omega_i(d))_{\varepsilon=0} = c_i d\sigma$, $(\omega_\alpha(d))_{\varepsilon=0} = 0$, hence we have

$$(5.12)\qquad (\delta\omega_i(d))_{\varepsilon=0} = -\sum_{jk} c_{jki}c_j d\sigma(\omega_k(\delta))_{\varepsilon=0} + \sum_{\alpha k} c_{\alpha ki}(\omega_\alpha(\delta))_{\varepsilon=0} c_k d\sigma + (d\omega_i(\delta))_{\varepsilon=0}.$$

Now we have

$$(\delta L)_{\varepsilon=0} = \left(\delta\int_{\sigma_0}^{\sigma_1} f(\omega)\right)_{\varepsilon=0} = \int_{\sigma_0}^{\sigma_1}(\delta f(\omega))_{\varepsilon=0} = \int_{\sigma_0}^{\sigma_1}\sum_i\left(\frac{\partial f}{\partial\omega_i}\delta\omega_i(d)\right)_{\varepsilon=0}.$$

As $f(\omega)$ is linear homogeneous $\dfrac{\partial f}{\partial\omega_i}$ is a homogeneous function of degree zero, hence $\left(\dfrac{\partial f}{\partial\omega_i}\right)_{\varepsilon=0} = \dfrac{\partial f}{\partial\omega_i}(c) = \dfrac{\partial f}{\partial\omega_i}(c_1, \ldots, c_n)$. By virtue of (5.12) we get

$$(\delta L)_{\varepsilon=0} = -\sum_{jki} c_{jki}c_j\frac{\partial f}{\partial\omega_i}(c)\int_{\sigma_0}^{\sigma_1}(\omega_k(\delta))_{\varepsilon=0}d\sigma + \sum_{\alpha ki}\frac{\partial f}{\partial\omega_i}(c)c_{\alpha ki}c_k\int_{\sigma_0}^{\sigma_1}(\omega_\alpha(\delta))_{\varepsilon=0}d\sigma + \sum_i\frac{\partial f}{\partial\omega_i}(c)\int_{\sigma_0}^{\sigma_1}(d\omega_i(\delta))_{\varepsilon=0}.$$

The second term on the second side vanishes on account of the relation (5.10) and the third term vanishes as

$$\int_{\sigma_0}^{\sigma_1}(d\omega_i(\delta))_{\varepsilon=0} = [(\omega_i(\delta))_{\varepsilon=0}]_{\sigma_0}^{\sigma_1} = 0.$$

Hence we have

$$(\delta L)_{\varepsilon=0} = -\sum_{jki} c_{jki}c_j\frac{\partial f}{\partial\omega_i}(c)\int_{\sigma_0}^{\sigma_1}(\omega_k(\delta))_{\varepsilon=0}d\sigma$$

and this vanishes when

$$(5.13)\qquad \sum_{ji} c_{jki}c_j\frac{\partial f}{\partial\omega_i}(c) = 0.$$

We call *geodesic* a curve in $\mathfrak{G}/\mathfrak{H}$ obtained by solving (5.11) which in the space with parallelism is the geodesic already defined in **4.1.** Then we get

THEOREM 5.5 *Let $f(\omega)$ be an invariant of $\mathfrak{G}/\mathfrak{H}$ which is linear and homogeneous in $\omega = (\omega_1, \ldots, \omega_n)$ and satisfies the relation $\sum_{ji} c_{jki}\omega_j\frac{\partial f}{\partial\omega_i} = 0$. Then all the geodesics are extremals with respect to the invariant integral $L = \int f(\omega)$. These extremals are generated by one-parametric subgroup.*

In general there may be several invariant integrals. Then geodesics are extremals of these integrals at the same time. Examples will be given in **7.1.** If our space $\mathfrak{G}/\mathfrak{H}$ has an invariant Riemann metric $\sum_i \omega_i^2$ we can take $f(\omega) = \sqrt{\sum_i \omega_i^2}$. Then (5.13) reduces to $\sum_{ji} c_{jki} c_j c_i = 0$, hence $c_{jki} = -c_{ikj}$ as c_i's are arbitrary constants. Thus we get

THEOREM 5.6 *If c_{ijk} is a trivector and $\sum_i \omega_i^2$ is an invariant Riemann metric all the geodesics with respect to this metric are generated by one-parametric subgroups of $\mathfrak{G}$.*

6. Homogeneous Riemann space

6.1 When a linear group of isotropy is orthogonal our space has an invariant Riemann metric $\sum_i \omega_i^2$. Owing to the relation $d\tau_{ij} = \sum_{\alpha k} c_{\alpha ki} \omega_\alpha \tau_{kj}$ a linear group of isotropy is orthogonal when and only when

$$c_{\alpha ki} = -c_{\alpha ik}. \tag{6.1}$$

If the orthogonal matrix $\tau = (\tau_{ij})$ is reducible our space has invariant Riemann metrics whose number is equal to the number of irreducible parts of the linear group of isotropy, hence any linear combination of these metrics with constant coefficients is also invariant.

6.2 We calculate the parameters of the Riemannian connection of our space with a Riemann metric

$$ds^2 = \sum_i \omega_i^2. \tag{6.2}$$

Let the parameters of the Riemannian connection be ω_{ij}, then

$$d\omega_i = \sum_j [\omega_j \omega_{ji}] \quad (\omega_{ij} = -\omega_{ji}). \tag{6.3}$$

On the other hand by the structure equation of our group

$$\begin{aligned} d\omega_i &= \sum_{(pj)} c_{pji} [\omega_p \omega_j] = \sum_{(kj)} c_{kji} [\omega_k \omega_j] + \sum_{\alpha j} c_{\alpha ji} [\omega_\alpha \omega_j] \\ &= \sum_j [\omega_j, \ -\frac{1}{2} \sum_k c_{kji} \omega_k - \sum_\alpha c_{\alpha ji} \omega_\alpha]. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4) we get

$$\sum_j [\omega_j \omega_{ji}] = \sum_j [\omega_j, \ -\frac{1}{2} \sum_k c_{kji} \omega_k - \sum_\alpha c_{\alpha ji} \omega_\alpha].$$

Hence

$$\omega_{ji} = -\frac{1}{2} \sum_k c_{kji} \omega_k - \sum_\alpha c_{\alpha ji} \omega_\alpha + \sum_k A_{jik} \omega_k$$

where $$A_{jik} = A_{kij}.$$
By the relations $\omega_{ij} + \omega_{ji} = 0$ and (6.1) we get

$$0 = -\frac{1}{2}\sum_k (c_{kji} + c_{kij})\omega_k + \sum_k (A_{jik} + A_{ijk})\omega_k,$$

hence $$A_{jik} + A_{ijk} = \frac{1}{2}(c_{kji} + c_{kij}),$$

similarly $$A_{ikj} + A_{kij} = \frac{1}{2}(c_{jik} + c_{jki})$$

$$A_{kji} + A_{jki} = \frac{1}{2}(c_{ikj} + c_{ijk}).$$

Adding the first two of these equations and subtracting the third and dividing by 2 we get

$$A_{jik} = \frac{1}{2}(c_{jik} + c_{kij}).$$

Thus we obtain

(6.5) $$\omega_{ji} = -\sum_\alpha c_{\alpha ji}\omega_\alpha + \frac{1}{2}\sum_k (c_{jik} + c_{kij} + c_{jki})\omega_k.$$

6.3 We investigate the relation between the parallelism hitherto studied and the parallelism determined by the Riemannian connection (6.5). We assume that a homogeneous space with parallelism has a Riemann metric (6.2) and call the covariant differential of a vector v_i a covariant differential in Klein connection and denote by $D_K v_i$, namely

(6.6) $$D_K v_i = dv_i - \sum_{\alpha j} c_{\alpha ji}\omega_\alpha v_j.$$

On the other hand denoting a covariant differential of v_i in Riemann connection by $D_R v_i$ we have

(6.7) $$\begin{aligned} D_R v_i &= dv_i + \sum_j \omega_{ji} v_j \\ &= dv_i - \sum_{\alpha j} c_{\alpha ji}\omega_\alpha v_j + \frac{1}{2}\sum_{kj}(c_{jik} + c_{kij} + c_{jki})\omega_k v_j. \end{aligned}$$

In order that $D_K v_i$ and $D_R v_i$ coincide it is necessary and sufficient that the following relations hold

$$c_{jik} + c_{kij} + c_{jki} = 0.$$

Adding this and

$$c_{kji} + c_{ijk} + c_{kij} = 0$$

we get $$c_{kij}=0.$$

As $c_{\alpha i\beta}=0$ holds in our space we get a symmetric space on account of **1. 7.** We call a frame corresponding to ω_i and ω_{ij} determined by (6.2) and (6.5) an adapted frame. Thus

THEOREM 6.1 *In order that in a homogeneous space with parallelism the covariant differential in Klein connectin and the covariant differential in Riemann connection corresponding to an adapted frame coincide, it is necessary and sufficient that the space is symmetric.*

The equation of the geodesic in Klein connection is by (4.1)

$$\frac{d}{d\sigma}\left(\frac{\omega_i}{d\sigma}\right)-\sum_{\alpha j}c_{\alpha ji}\frac{\omega_\alpha}{d\sigma}\frac{\omega_j}{d\sigma}=0, \tag{6.8}$$

and the equation of the geodesic in Riemann connection is

$$\frac{d}{ds}\left(\frac{\omega_i}{ds}\right)+\sum_j\frac{\omega_{ji}}{ds}\frac{\omega_j}{ds}=\frac{d}{ds}\left(\frac{\omega_i}{ds}\right)-\sum_{\alpha j}c_{\alpha ji}\frac{\omega_\alpha}{ds}\frac{\omega_j}{ds}+\sum_{jk}B_{jik}\frac{\omega_k}{ds}\frac{\omega_j}{ds}=0 \tag{6.9}$$

where

$$B_{jik}=\frac{1}{2}(c_{jik}+c_{kij}+c_{jki}). \tag{6.10}$$

From (6.8) we get

$$\frac{d}{d\sigma}\left(\sum_i\left(\frac{\omega_i}{d\sigma}\right)^2\right)=2\sum_i\frac{\omega_i}{d\sigma}\frac{d}{d\sigma}\left(\frac{\omega_i}{d\sigma}\right)=2\sum_{\alpha ji}\frac{\omega_i}{d\sigma}c_{\alpha ji}\frac{\omega_\alpha}{d\sigma}\frac{\omega_j}{d\sigma}=0$$

hence

$$\sum_i\left(\frac{\omega_i}{d\sigma}\right)^2=\text{const.}$$

Similarly we get from (6.9) $\sum_i\left(\frac{\omega_i}{ds}\right)^2=\text{const.}$

So if two sorts of geodesics obtained by solving (6.8) and (6.9) coincide the ratio of ds and $d\sigma$ is constant, and we get by comparing (6.8) and (6.9)

$$\sum_{jk}B_{jik}\frac{\omega_k}{ds}\frac{\omega_j}{ds}=0.$$

Hence $B_{jik}+B_{kij}=0$. We get by (6.10)

$$c_{ikj}=-c_{ijk}. \tag{6.11}$$

Thus the following theorem is obtained (cf. theorem 5.5).

THEOREM 6.2 *In order that the geodesic in Klein connection and that in*

Riemann connection coincide, it is necessary and sufficient that c_{ijk} is a trivector.

Example. Let $\mathfrak{G}$ be a group generated by the differential operators X_1, X_2, X_3, X_4 and $\mathfrak{H}$ be a subgroup generated by X_4, where

$$(X_2X_3) = X_1, \quad (X_3X_1) = X_2, \quad (X_1X_2) = cX_3$$
$$(X_4X_1) = -X_2, \quad (X_4X_2) = X_1, \quad (X_4X_3) = 0.$$

Then the structure equations of a space $\mathfrak{G}/\mathfrak{H}$ are

$$d\omega_1 = [\omega_2\omega_3] + [\omega_4\omega_2], \quad d\omega_2 = [\omega_3\omega_1] - [\omega_4\omega_1], \quad d\omega_3 = c[\omega_1\omega_2], \quad d\omega_4 = 0.$$

Putting $\omega_3 - \omega_4 = \omega$ we get

$$d\omega_1 = [\omega_2\omega], \quad d\omega_2 = [\omega\omega_1], \quad d\omega = c[\omega_1\omega_2].$$

Now we take the case $c > 0$. Then $\sqrt{c}\,\omega_1$, $\sqrt{c}\,\omega_2$, ω satisfy the structure equations of a rotation group of dimension 3. Hence $c(\omega_1^2 + \omega_2^2)$ is a line element of a 2-dimensional sphere, and by a suitable choice of variables we have

$$c(\omega_1^2 + \omega_2^2) = d\theta^2 + \sin^2\theta d\varphi^2$$
$$\sqrt{c}\,\omega_1 = \cos\alpha\, d\theta - \sin\alpha \sin\theta\, d\varphi, \quad \sqrt{c}\,\omega_2 = \sin\alpha\, d\theta + \cos\alpha \sin\theta\, d\varphi.$$

Hence $$\omega = -\cos\theta\, d\varphi + d\alpha, \quad \omega_4 = d\beta.$$

By putting $\alpha + \beta = \psi$ we get

$$\omega_3 = -\cos\theta\, d\varphi + d\psi.$$

Thus the line element of our Riemann space is

$$ds^2 = \frac{1}{c}(d\theta^2 + \sin^2\theta\, d\varphi^2) + (\cos\theta\, d\varphi - d\psi)^2.$$

In this space the condition (6.11) is not satisfied if $c \neq 1$, so two sorts of geodesics do not coincide. This space was given by E. Cartan in his book [5] as an example of the space in which a linear group of isotropy is a rotation group about an axis.

6.4 The discussion in **6.2** is independent of a Klein connection. If (6.1) is satisfied, we have an invariant Riemann metric (6.2). If (6.11) is satisfied the equation of the geodesic is

$$\frac{d}{ds}\left(\frac{\omega_i}{ds}\right) - \sum_{\alpha j} c_{\alpha ji} \frac{\omega_\alpha}{ds} \cdot \frac{\omega_j}{ds} = 0$$

and the solution of this equatin is obtained by solving

$$\omega_i = c_i\, ds, \quad \omega_\alpha = 0$$

and so the geodesic in Riemann connection is a curve generated by one-parametric subgroup of $\mathfrak{G}$. Hence we get again (cf. theorem 5.6)

THEOREM 6.3 *If c_{ijk} is a trivector the geodesic of a Riemann space with the metric $\sum_i \omega_i^2$ is generated by one-parametric subgroup.*

An analogous theorem was obtained by E. Cartan in his book [5] in the case of a group space itself by the use of canonical coordinates. The proof here given is applicable to the case of E. Cartan. In our case the condition that c_{ijk} is a trivector is not necessary. In fact even though c_{ijk} is a trivector $\bar{c}_{ijk}$ is not always a trivector after the transformation (iii) $\bar{\omega}_\alpha = \omega_\alpha + \sum_i A_{\alpha i}\omega_i$ considered in **2.3.**

7. Examples

7.1 Two examples will be given here about the discussion hitherto done. Let the orthonormal system of vectors be $e_1, e_2, \ldots, e_n$ and their infinitesimal change be $de_i = \sum_j \omega_{ij} e_j$. Let e be a matrix with one colomn whose coefficients are $e_1, e_2, \ldots, e_n$ and Ω be (ω_{ij}), then we can write the above relation in the form $de = \Omega e$. If we take a new system of orthonormal vectors $\bar{e} = Pe$, where P is an orthogonal matrix, and put $d\bar{e} = \bar{\Omega}\bar{e}$, we get

$$\bar{\Omega} = P\Omega P' + dP\cdot P'. \tag{7.1}$$

This is the special case of (2.1).

The transformation P, which keeps invariant a k-dimensional plane passing through the origin and containing $e_1, e_2, \ldots, e_k$, is of the form $P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ where P_1 is an orthogonal matrix of degree k. Corresponding to this we put

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_0 \\ -\Omega_0' & \Omega_2 \end{pmatrix}.$$

Then we get by (7.1)

$$\bar{\Omega}_0 = P_1\Omega_0 P_2' \tag{7.2}$$

$$\bar{\Omega}_1 = P_1\Omega_1 P_1' + dP_1\cdot P_1', \quad \bar{\Omega}_2 = P_2\Omega_2 P_2' + dP_2\cdot P_2'. \tag{7.3}$$

Hence the principal relative components of a homogeneous space with k-dimensional planes through the origin as its elements, namely a Grassmann manifold, are Ω_0. If we denote in general the product of matrices $\Omega = (\omega_{ip})$, $\Pi = (\pi_{pu})$ with Pfaffians or differential forms as its coefficients by $[\Omega\Pi] = ([\omega_{ip}\pi_{pu}])$ and the exterior differential of Ω by $d\Omega = (d\omega_{ip})$ we can write the structure equation in the form $d\Omega = [\Omega\Omega]$ which in our case reduces to

$$d\Omega_0 = [\Omega_1\Omega_0] + [\Omega_0\Omega_2], \quad d\Omega_1 = [\Omega_1\Omega_1] - [\Omega_0\Omega_0'], \quad d\Omega_2 = [\Omega_2\Omega_2] - [\Omega_0'\Omega_0].$$

Hence by (1.13) (1.14) a Grassmann manifold is symmetric, as is well known.

A vector in this space has components V written in matric form and after a frame transformation V is transformed into $\bar{V}=P_1VP_2'$. Hence

$$d\bar{V}-\bar{\Omega}_1\bar{V}+\bar{V}\bar{\Omega}_2=P_1(dV-\Omega_1V+V\Omega_2)P_2'$$

and the covariant differential of V is

$$DV=dV-\Omega_1V+V\Omega_2. \tag{7.4}$$

The equation of the geodesic is

$$\frac{d}{d\sigma}\left(\frac{\Omega_0}{d\sigma}\right)-\frac{\Omega_1}{d\sigma}\frac{\Omega_0}{d\sigma}+\frac{\Omega_0}{d\sigma}\frac{\Omega_2}{d\sigma}=0, \tag{7.5}$$

and the solution of this equation is obtained by solving

$$\Omega_0=Cd\sigma,\quad \Omega_1=0,\quad \Omega_2=0 \tag{7.6}$$

where C is a matrix with constant coefficients. As C has invariants under the transfomation $\bar{C}=P_1CP_2'$ of linear group of isotropy, not all the geodesics are congruent in a Grassmann manifold (cf. theorem 4.2).

By the transformation $\bar{V}=P_1VP_2'$ $|VV'-\lambda E|$ is invariant and if we put

$$V=(v_{i\alpha})\quad (i=1,\ldots,k;\ \alpha=k+1,\ldots,n)$$

we have vector invariants (cf. W. Blaschke [1] p. 12)

$$\sum_{i,\alpha}v_{i\alpha}^2,\quad \sum_{\substack{(ij)\\(\alpha\beta)}}\begin{vmatrix}v_{i\alpha} & v_{i\beta}\\ v_{j\alpha} & v_{j\beta}\end{vmatrix}^2,\ \ldots\ .$$

By the discussion of 5.2 these are invariant by the parallel translation of vector. As our space is symmetric the parallelism of a vector in the sense of Klein connection here stated and that in the sense of a Riemannian connection associated with the metric $\sum_{i\alpha}\omega_{i\alpha}^2$ are the same. (7.6) gives an extremal of $\int\sqrt{\sum\omega_{i\alpha}^2}$ as well as of $\int\sqrt[4]{\sum\begin{vmatrix}\omega_{i\alpha} & \omega_{i\beta}\\ \omega_{j\alpha} & \omega_{j\beta}\end{vmatrix}^2}$ etc. by the theorem 5.5.

As for the case $n=4$, $k=2$ we have two sorts of Riemann metrics

$$\omega_{13}^2+\omega_{14}^2+\omega_{23}^2+\omega_{24}^2,\quad \begin{vmatrix}\omega_{13} & \omega_{14}\\ \omega_{23} & \omega_{24}\end{vmatrix}=\omega_{13}\omega_{24}-\omega_{23}\omega_{14}.$$

Hence $\omega_{13}^2+\omega_{14}^2+\omega_{23}^2+\omega_{24}^2+2c(\omega_{13}\omega_{24}-\omega_{23}\omega_{14})$ with arbitrary constant c is invariant and we get by the theorem 5.3 two invariants which are polynomials of degree 2 in c. Coefficients of each degree in c are invariant. Thus we get the following invariants.

$$[\omega_{13}\omega_{14}]+[\omega_{23}\omega_{24}]=d\omega_{13},\quad [\omega_{13}\omega_{23}]+[\omega_{14}\omega_{24}]=d\omega_{21}$$

$$[\omega_{13}\omega_{24}]^2+[\omega_{14}\omega_{23}]^2-2[\omega_{13}\omega_{14}][\omega_{23}\omega_{24}]-2[\omega_{13}\omega_{24}][\omega_{23}\omega_{14}]$$

$$[\omega_{13}\omega_{14}\omega_{24}]^2+[\omega_{23}\omega_{24}\omega_{14}]^2+[\omega_{13}\omega_{14}\omega_{23}]^2+[\omega_{23}\omega_{24}\omega_{13}]^2$$

$$[\omega_{13}\omega_{24}\omega_{14}][\omega_{13}\omega_{24}\omega_{23}]-[\omega_{14}\omega_{23}\omega_{13}][\omega_{14}\omega_{23}\omega_{24}]$$

$[\omega_{13}\omega_{14}\omega_{23}\omega_{24}]$.

7.2 Let $x=(x_0, x_1, \ldots, x_n)$ and $x'=(x'_0, x'_1, \ldots, x'_n)$ be homogeneous coordinates of points in an n-dimensional projective space and a projective transformation be $x'=Px$ where P is a matix with the determinant $|P|=\pm 1$. Let the frame be a set of $n+1$ analytic points $A=(A_0, A_1, \ldots, A_n)$. Then $dA=\Omega A$ where $\Omega=(\omega_{ij})$ and $\sum_{i=0}^{n}\omega_{ii}=0$. We consider a set of quadratic surfaces obtained from $\sum_{i=0}^{n}x_i^2=0$ by projective transformations. The transformations which leave invariant $\sum_{i=0}^{n}x_i^2=0$ are orthogonal. Principal relative components of a homogeneous space with quadratic sufaces as its elements are $\Omega+\Omega'=(\omega_{ij}+\omega_{ji})$. Putting $\Pi=\frac{1}{2}(\Omega+\Omega')$, $\Sigma=\frac{1}{2}(\Omega-\Omega')$ we get by virtue of (7.1)

(7.6) $$\bar{\Pi}=P\Pi P', \quad \bar{\Sigma}=P\Sigma P'+dP\cdot P'.$$

A symmetric matrix V which is transformed in such a way that $\bar{V}=PVP'$ is a vector and its covariant differential is given by

(7.7) $$DV=dV-\Sigma V+V\Sigma.$$

The equation of a geodesic is

(7.8) $$\frac{d}{d\sigma}\left(\frac{\Pi}{d\sigma}\right)-\frac{\Sigma}{d\sigma}\cdot\frac{\Pi}{d\sigma}+\frac{\Pi}{d\sigma}\cdot\frac{\Sigma}{d\sigma}=0$$

and its solution can be obtained by solving

(7.9) $$\Pi=Cd\sigma, \quad \Sigma=0$$

where C is a constant symmetric matrix. By a suitable orthogonal matrix C is transformed into a diagonal form. Then by solving $dA=Cd\sigma A$ we get $A_i=a_ie^{c_i\sigma}A_i^{(0)}$. The equations of quadratic surfaces corresponding to the equation $\sum_{i=0}^{n}x_i^2=0$ with respect to $A=(A_0, A_1, \ldots, A_n)$ are $\sum_{i=0}^{n}a_i^{-2}e^{-2c_i\sigma}x_i^2=0$ with respect to a fixed frame. This is the geodesic. The structure equations are

$$d\Pi=[\Sigma\Pi]+[\Pi\Sigma], \quad d\Sigma=[\Pi\Pi]+[\Sigma\Sigma]$$

and our space is symmetric. The invariants of a vector are given by the coefficients of a polynomial in λ $|V-\lambda E|$.

8. Space with Klein connection

8.1 Now we treat a vector in a space with Klein connection. Two-dimensional space with euclidean connection can be determined by giving two independent Pfaffians $\pi_1(a, da)$, $\pi_2(a, da)$ and $\pi_{12}(a, da)$ in two variables $a=(a_1,$

a_2), and the development into a euclidean plane can be given by integrating

$$(8.1)\qquad d\mathbf{A} = \pi_1\mathbf{e}_1 + \pi_2\mathbf{e}_2, \quad d\mathbf{e}_1 = \pi_{12}\mathbf{e}_2, \quad d\mathbf{e}_1 = -\pi_{12}\mathbf{e}_1.$$

Let $\mathbf{A} = (x_1, x_2)$, $\mathbf{e}_1 = (\cos\theta, \sin\theta)$, $\mathbf{e}_2 = (-\sin\theta, \cos\theta)$. Then (8.1) can be written in the form

$$(8.2)\qquad dx_1\cos\theta + dx_2\sin\theta = \pi_1(a, da), \quad -dx_1\sin\theta + dx_2\cos\theta = \pi_2(a, da),$$
$$d\theta = \pi_{12}(a, da).$$

It is easy to formulate a space with Klein connection from this point of view.

Let the relative components of a homogeneous space $\mathfrak{G}/\mathfrak{H}$ be $\omega_i = \omega_i(x, u, dx)$ and $\omega_\alpha = \omega_\alpha(x, u, dx, du)$. Then

$$(8.3)\qquad d\omega_i = \sum_{(pj)} c_{pji}[\omega_p\omega_j], \quad d\omega_\alpha = \sum_{(pq)} c_{pq\alpha}[\omega_p\omega_q].$$

Let us consider an n-dimensional differentiable manifold with $a = (a_1, a_2, \ldots, a_n)$ as coordinates of a point, and attach to each point of the manifold n independent Pfaffians $\pi_i(a, da)$ and $r-n$ Pfaffians $\pi_\alpha(a, da)$. We integrate the differential equation

$$(8.4)\qquad \omega_i(x, u, dx) = \pi_i(a, da), \quad \omega_\alpha(x, u, dx, du) = \pi_\alpha(a, da)$$

along a curve $a = a(\sigma)$ in our manifold with the initial condition $x_i = x_i^{(0)}$, $u_\alpha = u_\alpha^{(0)}$ at $\sigma = 0$. The curve thus obtained in the homogeneous space is called a *development of a curve* $a = a(\sigma)$.

By the frame transformation from S_cR to S_cS_tR where S_t is an element of $\mathfrak{H}$ relative components are transformed into

$$(8.5)\qquad \bar{\omega}_i = \sum_j \tau_{ij}\omega_j, \quad \bar{\omega}_\alpha = \sum_p \tau_{\alpha p}\omega_p + \omega_\alpha(t, dt).$$

So if we take in the place of π_i and π_α

$$(8.6)\qquad \bar{\pi}_i = \sum_j \tau_{ij}\pi_j, \quad \bar{\pi}_\alpha = \sum_p \tau_{\alpha p}\pi_p + \omega_\alpha(t, dt)$$

and solve

$$(8.7)\qquad \omega_i(x, u, dx) = \bar{\pi}_i(a, t, da), \quad \omega_\alpha(x, u, dx, du) = \bar{\pi}_\alpha(a, t, da, dt)$$

where $a = a(\sigma)$, $t = t(\sigma)$, we get a curve congruent to the solution of (8.4) in $\mathfrak{G}/\mathfrak{H}$. We call a differentiable manifold of dimension n with Pfaffians $\pi_i(a, da)$, $\pi_\alpha(a, da)$ *a space with Klein connection* and $\mathfrak{G}/\mathfrak{H}$ *a fundamental space.* We speak of $\pi_i(a, da)$, $\pi_\alpha(a, da)$ in relation with a frame R and if we take another frame $\bar{R} = S_tR$ ($S_t \in \mathfrak{H}$) we attach to a point a $\bar{\pi}_i$ and $\bar{\pi}_\alpha$ determined by (8.6) in the place of π_i and π_α. Let us call π_i, π_α *parameters of Klein connection* and (8.6) the transformation of these parameters. $\bar{\pi}_i$ and $\bar{\pi}_\alpha$ are called *generalized parameters.*

8.2 We attach to each point of a space with Klein connection a vector space and let the transformation of a linear adjoint group of isotropy be a transformation in our vector space. In general we define a tensor by a set of numbers attached to a frame S_tR which are transformed by a group of linear transformation homomorphic to $\mathfrak{H}$ in accordance with a frame transformation S_t. Especially $v=(v_1, \ldots, v_n)$, whose transformation is given by $\bar{v}_i = \sum_j \tau_{ij} v_j$, is a vector in our sense. Then putting

$$\Omega_i = d\pi_i - \sum_{(pj)} c_{pji}[\pi_p \pi_j], \quad \Omega_\alpha = d\pi_\alpha - \sum_{(pq)} c_{pq\alpha}[\pi_p \pi_q] \tag{8.8}$$

we get the following theorem.

THEOREM 8.1 *Ω_p is a tensor and Ω_i is a vector. If the fundamental homogeneous space is a space with parallelism Ω_α is also a tensor.*

Proof. The assertion that Ω_p is a tensor is lemma 2 in 2.1. Ω_i is a vector on account of the relation $\bar{\Omega}_i = \sum_j \tau_{ij}\Omega_j$ because of $\tau_{i\alpha}=0$. In a space with parallelism $\tau_{\alpha j}=0$ holds and so $\bar{\Omega}_\alpha = \sum_\beta \tau_{\alpha\beta}\Omega_\beta$ which shows that Ω_α is a tensor.

We call Ω_i a *torsion vector* of a space with a Klein connection and if the fundamental space is a space with parallelism we call Ω_α a *curvature tensor*.

A space whose torsion vanishes is called *a space without torsion*. In such a space Ω_α is a tensor, which we call a *curvatnre tensor* as in the previous case.

It is often convenient to take

$$\bar{\pi}_i = \sum_j \tau_{ij}\pi_j, \quad \bar{\pi}_\alpha = \sum_p \tau_{\alpha p}\pi_p + \omega_\alpha(t, dt) \tag{8.9}$$

in the place of π_i, π_α. If we consider $a=(a_1, \ldots, a_n)$ and $t=(t_{n+1}, \ldots, t_r)$ as independent variables $\bar{\pi}_i, \bar{\pi}_\alpha$ are independent Pfaffians. If Ω_i and Ω_α vanish the relations

$$d\bar{\pi}_i = \sum_{(pj)} c_{pji}[\bar{\pi}_p \bar{\pi}_j], \quad d\bar{\pi}_\alpha = \sum_{(pq)} c_{pq\alpha}[\bar{\pi}_p \bar{\pi}_q] \tag{8.10}$$

hold for $\bar{\pi}_i, \bar{\pi}_\alpha$ as well as for π_i, π_α. Hence $\bar{\pi}_i$ and $\bar{\pi}_\alpha$ are relative components of our homogeneous space. Thus

THEOREM 8.2 *A space with Klein connection coincides with the fundamental homogeneous space when and only when torsion vector and curvature tensor vanish.*

Next we take exterior differential of the equations

$$d\pi_i = \sum_{(pj)} c_{pji}[\pi_p \pi_j] + \Omega_i, \quad d\pi_\alpha = \sum_{(pq)} c_{pq\alpha}[\pi_p \pi_q] + \Omega_\alpha$$

and substitute in them the right side of these equations in the place of $d\pi_i$,

$d\pi_\alpha$. Then owing to the group property of $\mathfrak{G}$ the terms not containing Ω_i, Ω_α vanish. Hence we get

$$d\Omega_i + \sum_{\alpha j} c_{\alpha ji}[\pi_j \Omega_\alpha] - \sum_{\alpha j} c_{\alpha ji}[\pi_\alpha \Omega_j] - \sum_{jk} c_{jki}[\pi_j \Omega_k] = 0$$
$$d\Omega_\alpha - \sum_{pq} c_{pq\alpha}[\pi_p \Omega_q] = 0.$$

If our space has not torsion we get

$$\sum_{\alpha j} c_{\alpha ji}[\pi_j \Omega_\alpha] = 0 \tag{8.11}$$

$$d\Omega_\alpha - \sum_{\beta\gamma} c_{\beta\gamma\alpha}[\pi_\beta \Omega_\gamma] + \sum_{i\beta} c_{\beta i\alpha}[\pi_i \Omega_\beta] = 0. \tag{8.12}$$

The latter is the generalization of Bianchi's identity.

8.3 We consider a vector field $v = v(a)$ on a space with Klein connection. Let the components of $v(a)$ with respect to a frame R be $(v_1, \ldots, v_n)$. Now we take $\bar{\pi}_i$ and $\bar{\pi}_\alpha$ defined by (8.6) instead of π_i and π_α, and denote them by π_i and π_α anew. So π_i, π_α are Pfaffians in variables $a_1, \ldots, a_n, t_{n+1}, \ldots, t_r$. We define a covariant differential of a vector v by

$$Dv_i = dv_i - \sum_{\alpha j} c_{\alpha ji} \pi_\alpha v_j. \tag{8.13}$$

Then by the same arguement with that of theorem **2.1** we get

THEOREM 8.3 *In order that in a space with Klein connection Dv_i defined by* (8.13) *is a component of a vector it is necessary and sufficient that the fundamental homogeneous space is a space with parallelism, namely* $c_{\alpha i\beta} = 0$.

Proof. In **2.2** structure equations for ω_i, ω_α and $\bar{\omega}_i$, $\bar{\omega}_\alpha$ which in our case correspond to π_i, π_α and $\bar{\pi}_i$, $\bar{\pi}_\alpha$ were not used, and only the relation (2.2) and the property of linear adjoint group were used. Hence the discussion there is applicable to the present case.

Under the assumption $c_{\alpha i\beta} = 0$ a translation of a vector can be defined by

$$Dv_i = dv_i - \sum_{\alpha j} c_{\alpha ji} \pi_\alpha v_j = 0 \tag{8.14}$$

and a *geodesic* by the solution of

$$\frac{d}{d\sigma}\left(\frac{\pi_i}{d\sigma}\right) - \sum_{\alpha j} c_{\alpha ji} \frac{\pi_\alpha}{d\sigma}\frac{\pi_j}{d\sigma} = 0. \tag{8.15}$$

If we take generalized parameters $\pi_i(a, t, da)$, $\pi_\alpha(a, t, da, dt)$ the discussion of **4.1** is applicable to our space. Hence the solution of (8.15) can be given by solving

$$\pi_i(a, t, da) = c_i d\sigma, \quad \pi_\alpha(a, t, da, dt) = 0. \tag{8.16}$$

When we develop the geodesic thus defined in the space with Klein connection into the fundamental homogeneous space we get a geodesic of the space. The vector invariants with respect to the linear group of isotropy are also invariant under the translation of vectors. For example if our space has an invariant Riemannian metric the length of a vector is invariant by a translation, as is well known.

8.4 Now we treat a space with Klein connection which admits an absolute parallelism of a vector. Then the differential equation

$$dv_i - \sum_{\alpha j} c_{\alpha j i} \pi_\alpha v_j = 0$$

is completely integrable, hence taking an exterior differential of this equation and substituting $dv_i = \sum_{\alpha j} c_{\alpha j i} \pi_\alpha v_j$ we get

$$\Omega_\alpha = -\sum_{(jl)} c_{jl\alpha} [\pi_j \pi_l] \tag{8.17}$$

by a calculation analogous to that of **3.2**. As $c_{jl\alpha} = 0$ is a necessary and sufficient condition for the fundamental space to admit an absolute parallelism we obtain the following

THEOREM 8.4 *In order that a space with Klein connection admits an absolute parallelism of a vector it is necessary and sufficient that* (8.17) *and* $c_{\alpha i \beta} = 0$ *hold. In particular if the fundamental homogeneous space admits an absolute parallelism of a vector the space with Klein connection admitting an absolute parallelism has a vanishing curvature and if moreover it has not torsion it is nothing but the fundamental homogeneous space.*

Now we investigate a space without torsion which admits an absolute parallelism of a vector. For such a space we have

$$d\pi_i = \sum_{(pj)} c_{pji} [\pi_p \pi_j], \quad d\pi_\alpha = \sum_{(\beta\gamma)} c_{\beta\gamma\alpha} [\pi_\beta \pi_\gamma].$$

If we take $\bar{\pi}_i = \pi_i(a, t, da)$, $\bar{\pi}_\alpha = \pi_\alpha(a, t, da, dt)$ determined by (8.9) instead of π_i, π_α yet we have

$$d\bar{\pi}_i = \sum_{(pj)} c_{pji} [\bar{\pi}_p \bar{\pi}_j], \quad d\bar{\pi}_\alpha = \sum_{(\beta\gamma)} c_{\beta\gamma\alpha} [\bar{\pi}_\beta \bar{\pi}_\gamma] \tag{8.18}$$

because $\bar{\Omega}_\alpha = -\sum_{(jl)} c_{jl\alpha} [\bar{\pi}_j \bar{\pi}_l]$ holds after the transformation. This relation can be verified by the relations $\Omega_\alpha = \sum_\beta \tau_{\alpha\beta} \Omega_\beta$, $\bar{\pi}_i = \sum_j \tau_{ij} \pi_j$, $\sum_\beta \tau_{\alpha\beta} c_{jl\beta} = \sum_{kh} c_{kh\alpha} \tau_{kj} \tau_{hl}$. As c_{pqs}'s are constants $\bar{\pi}_i$, $\bar{\pi}_\alpha$ are relative components of a certain homogeneous space by virtue of (8.18). Hence we get

THEOREM 8.5 *A space without torsion which admits an absolute parallelism of a vector is a homogeneous space with an absolute parallelism which is different from the fundamental homogeneous space.*

In a space without torsion we have by (8.11) $\sum_{\alpha j} c_{\alpha ji}[\pi_j \Omega_\alpha] = 0$ and so for a space with an absolute parallelism we get by (8.17)

$$\sum_{\alpha jkl} c_{kl\alpha} c_{\alpha ji} [\pi_j \pi_k \pi_l] = 0.$$

Hence on account of the independence of π_i we get

$$\sum_\alpha (c_{kl\alpha} c_{\alpha ji} + c_{lj\alpha} c_{\alpha ki} + c_{jk\alpha} c_{\alpha li}) = 0 \tag{8.19}$$

and so

$$\sum_h (c_{klh} c_{hji} + c_{ljh} c_{hki} + c_{jkh} c_{hli}) = 0. \tag{8.20}$$

Conversely if (8.20) is satisfied we easily see that there exists a homogeneous space with

$$\bar{c}_{ijk} = c_{ijk}, \quad \bar{c}_{ij\alpha} = 0, \quad \bar{c}_{\alpha ji} = c_{\alpha ji}$$
$$\bar{c}_{\alpha i\beta} = c_{\alpha i\beta} = 0, \quad \bar{c}_{\alpha\beta i} = c_{\alpha\beta i} = 0, \quad \bar{c}_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}$$

as structure constants. In fact among the identities between c_{pqs}'s

$$\sum_s (c_{pqs} c_{stu} + c_{qts} c_{spu} + c_{tps} c_{squ}) = 0$$

those containing $c_{ij\alpha}$'s are

$$\sum_p (c_{ijp} c_{pkl} + c_{jkp} c_{pil} + c_{kip} c_{pjl}) = 0$$
$$\sum_h (c_{klh} c_{hj\alpha} + c_{ljh} c_{hk\alpha} + c_{jkh} c_{hl\alpha}) = 0$$
$$\sum_\alpha c_{ij\alpha} c_{\alpha\beta\gamma} + \sum_k c_{j\beta k} c_{ki\gamma} + \sum_k c_{\beta ik} c_{kj\gamma} = 0$$

and these are satisfied even when we put $c_{ij\alpha}$ to zero. Hence $\bar{c}_{pqs}$'s defined above are structure constants of a certain group and determine a certain homogeneous space. This space can be considered as a space with Klein connection whose fundamental space is $\mathfrak{G}/\mathfrak{H}$ and which has not torsion and admits an absolute parallelism.

THEOREM 8.6 *In order that there exists a space with Klein connection which has not torsion and admits an absolute parallelism of a vector it is necessary and sufficient that the fundamental homogeneous space has the property that there exists a group which has c_{ijk} as structure constants.*

In particular there exists a space which has a symmetric space as its fundamental space and admits an absolute parallelism of a vector. This space was treated in **3.2.** For example there exists a space with spherical connection which has not torsion and admits an absolute parallelism of a vector. This space is nothing but a euclidean space.

8. 5 The result in **5. 3** can be extended to the case of the space with Klein connection which has not torsion. In **5. 3** we called a geodesic of $\mathfrak{G}/\mathfrak{H}$ the curve obtained by solving (5.11). Now we call a *geodesic* of our space a curve which can be developed into a geodesic of the homogeneous space $\mathfrak{G}/\mathfrak{H}$. Such a curve can be obtained by solving

$$\pi_i(a,\ t,\ da) = c_i d\sigma, \quad \pi_\alpha(a,\ t,\ da,\ dt) = 0 \quad (c_i \text{ const.})$$

where π_i and π_α are the generalized ones, namely $\bar{\pi}_i$, $\bar{\pi}_\alpha$ given by (8.6).

As our space has not torsion we have

$$d\pi_i = \sum_{(jk)} c_{jki}[\pi_j\pi_k] + \sum_{\alpha k} c_{\alpha ki}[\pi_\alpha\pi_k].$$

Hence the discussion of **5. 3** can be applicable to our case by the same calculation. Thus we get

THEOREM 8.7 *Let $f(\omega) = f(\omega_1, \ldots, \omega_n)$ be an invariant of a homogeneous space $\mathfrak{G}/\mathfrak{H}$ which is linear and homogeneous in principal relative components and satisfies the relation $\sum_{ji} c_{jki}\omega_j \frac{\partial f}{\partial \omega_i} = 0$. Then all the geodesics of a space with Klein connection, whose fundamental space is $\mathfrak{G}/\mathfrak{H}$ and which has not torsino, are extremals of the integral $\int f(\pi) = \int f(\pi_1, \ldots, \pi_n)$.*

If the fundamental space satisfies the relation $c_{ijk} = 0$ our condition is always satisfied. A symmetric space is such a space. If the fundamental space is a euclidean space we get the well known theorem that the curve of a Riemann space which in development gives a straight line is a geodesic in the sense of the Riemann metric, whose proof is usually accomplished by the calculation rather roundabout.

8. 6 A space with Klein connection whose fundamental homogeneous space has the property $c_{\alpha ij} = -c_{\alpha ji}$ admits a Riemann metric $\sum_i \pi_i^2$ which is invariant under the linear group of isotropy. If the space has not torsion we have

$$d\pi_i = \sum_{(jk)} c_{jki}[\pi_j\pi_k] + \sum_{\alpha j} c_{\alpha ji}[\pi_\alpha\pi_j].$$

Hence by the same argument with that of **6. 2** we get the parameter π_{ij} of the Riemannian connection associated with the metric $\sum_i \pi_i^2$ and so

$$\pi_{ji} = -\sum_\alpha c_{\alpha ji}\pi_\alpha + \sum_k B_{jik}\pi_k \tag{8.21}$$

where

$$B_{jik} = \frac{1}{2}(c_{jik} + c_{kij} + c_{jki}). \tag{8.22}$$

By calculation we get a Riemannian curvature tensor

$$(8.23)\qquad \begin{aligned} \Omega_{ij} &= d\pi_{ij} - \sum_k [\pi_{ik}\pi_{kj}] \\ &= -\sum_\alpha c_{\alpha ij}(\Omega_\alpha + \sum_{(hl)} c_{hl\alpha}[\pi_h\pi_l]) \\ &\quad + \sum_{(hl)}(\sum_k c_{hlk}B_{jik} + \sum_k B_{jkh}B_{kil} - \sum_k B_{ikh}B_{kjl})[\pi_h\pi_l]. \end{aligned}$$

Hence a space without torsion which admits an absolute parallelism and has an invariant Riemann metric, is not always a Riemann space without a curvature. For such a space we have $\Omega_\alpha + \sum_{(jl)} c_{jl\alpha}[\pi_j\pi_l] = 0$ and only the first term on the right side of (8.23) vanishes.

Concerning the relation between the parallelisms of a vector and the geodesics in the sense of Klein connection and in the sense of Riemann connection the results are quite the same with those of **6.3.**

Let $\mathfrak{G}/\mathfrak{H}$ be a homogeneous space which admits a Riemann metric $\sum_i \omega_i^2$. For a Riemann space arbitrarily given it is not always possible to introduce a Klein connection without torsion which has $\mathfrak{G}/\mathfrak{H}$ as its fundamental homogeneous space. It is possible when and only when (8.21) holds, where π_{ij} is a parameter of the Riemannian connection determined by a suitable decomposition of the Riemann metric $ds^2 = \sum_i \pi_i^2$. For a general Riemann space only a euclidean connection and a spherical connection without torsicn are possible.

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