ON THE DIFFERENTIAL FORMS ON ALGEBRAIC VARIETIES

YÛSAKU KAWAHARA

Introduction. In the book "Foundations of algebraic geometry"[1] A. Weil proposed the following problem; *does every differential form of the first kind on a complete variety* $\mathbf{U}$ *determine on every subvariety* $\mathbf{V}$ *of* $\mathbf{U}$ *a differential form of the first kind?* This problem was solved affirmatively by S. Koizumi when $\mathbf{U}$ is a complete variety without multiple point.[2] In this note we answer this problem in affirmative in the case where $\mathbf{V}$ is a simple subvariety of a complete variety $\mathbf{U}$ (in §1). When the characteristic is 0 we may extend our result to a more general case but this does not hold for the case characteristic $p \neq 0$ (in §2).

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§1. Let $K = k(x_1, \ldots, x_N) = k(x)$ be a field, generated over a field k by a set of quantities (x), the class $\mathfrak{P}$ of equivalent $(n-1)$-dimensional valuations for K/k is called a prime divisor in the sense of Zariski,[3] n being the dimension of K over k, and its normalized valuation with rational integers as the value group is denoted by $\nu_{\mathfrak{P}}$. Let $F(x, dx)$ be a differential form belonging to the extension $k(x)$ of k. We say that $F(x, dx)$ is *finite at* $\mathfrak{P}$ if $F(x, dx)$ is of the form

$$F(x, dx) = \sum z_{\alpha\beta\ldots} dy_\alpha dy_\beta \ldots ,$$

where $\nu_{\mathfrak{P}}(z_{\alpha\beta\ldots}) \geqq 0,\ \nu_{\mathfrak{P}}(y_\alpha) \geqq 0,\ \nu_{\mathfrak{P}}(y_\beta) \geqq 0, \ldots$.

THEOREM 1. *Let* $\mathbf{U}^n$ *be a complete variety and* k *a field of definition of* $\mathbf{U}^n$ *which is perfect. Let* $\mathbf{P}$ *be a generic point of* $\mathbf{U}^n$ *over* k. *Then, for every differential form* ω *on* $\mathbf{U}$ *defined over* k, $\omega(\mathbf{P})$ *is of the first kind if and only if it is finite at every prime divisor of* $k(\mathbf{P})$.

Proof. Sufficiency. Let (y) be a set of quantities such that $k(\mathbf{P}) = k(y)$ and let P' be a simple point of the locus V^n of (y) over k. If P^* is a generic

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[1] We refer this book by F in this note.

[2] S. Koizumi, *On the differential forms of the first kind on algebraic varieties.* I. Journal of the Mathematical Society of Japan, vol. 1 (1949). II. vol. 2 (1951).

[3] See O. Zariski, *The reduction of the singularities of an algebraic surface.* Annals of Math. vol. 40 (1939).

point of any $(n-1)$-dimensional simple subvariety of V^n over the algebraic closure $\bar{k}$ of k, then $\omega(\mathbf{P})$ is finite at P^* by our hypothesis. Therefore by Prop. 5 in Koizumi's paper[4] $\omega(\mathbf{P})$ is finite at P', which shows that $\omega(\mathbf{P})$ is of the first kind.

Necessity. There exists a set of quantities (y) such that $k(\mathbf{P}) = k(y)$ and that, on the locus V of (y) over k, the center of $\mathfrak{P}$ is an $(n-1)$-dimensional simple subvariety W. V is obtained by a birational transformation such that the center of $\mathfrak{P}$ is an $(n-1)$-dimensional subvariety and by the normalization over k of the resulting variety. Let P' be a generic point of W over k and let $(t_1, \ldots, t_n)$ be a set of uniformizing parameters in $k(y)$ for V at P'. Since $\omega(\mathbf{P})$ is of the first kind,

$$\omega(\mathbf{P}) = \sum w_{ij\ldots} dt_i dt_j \ldots ,$$

where $w_{ij\ldots}$ are in the specialization ring of P' in $k(y) = k(\mathbf{P})$. As $t_1 \ldots, t_n$ are in the specialization ring of P' in $k(y)$ and this specialization ring is identical with the valuation ring of $\mathfrak{P}$, the theorem is proved.

Remark. This theorem holds without the assumption that k is a perfect field if each $\mathfrak{P}$ can be uniformized under a birational transformation of $\mathbf{U}$ over k, a fortiori, if $\mathbf{U}$ has no singular point.

The set of elements $(t_1, \ldots, t_n)$ in the proof (necessity) of th. 1 is called *a set of uniformizing parameters at* $\mathfrak{P}$. A differential form is finite at $\mathfrak{P}$ if and only if it is expressed in one and only one way as a polynomial in $dt_1, \ldots, dt_n$ with coefficients in the valuation ring of $\mathfrak{P}$.

LEMMA 1. *Let* $\mathbf{U}^n$ *be a variety defined over* k *and let* $\mathbf{V}^m$ *be a simple subvariety of* $\mathbf{U}^n$ *which is algebraic over* k. *Then there exists a series of algebraic varieties*

$$\mathbf{U}^n = \mathbf{U}_0^n,\ \mathbf{U}_1^{n-1},\ \mathbf{U}_2^{n-2}, \ldots, \mathbf{U}_{n-m}^m = \mathbf{V}^m$$

such that each $\mathbf{U}_i$ *is algebraic over* k *and that* $\mathbf{U}_{i+1}$ *is a simple subvariety of* $\mathbf{U}_i$ $(i = 0, \ldots, n-m-1)$.

Proof. Since it is enough to prove this for affine varieties, we may assume that U^n is contained in affine N-space S^N. Let $P = (y)$ be a generic point of V^m over $\bar{k}$. As P is a simple point of U^n, U^n is defined by a set of equations $F_\mu(X) = 0$, where $F_\mu(X)$ are polynomials in $k[X_1, \ldots, X_N]$ and the rank of the Jacobian matrix $\|\partial F_\mu/\partial y_i\|$ is $N-n$. Further as P is a generic point of V^m, V^m is defined by a set of equations $G_\nu(X) = 0$, where $G_\nu(X)$ are polynomials in $\bar{k}[X_1, \ldots, X_N]$ and the rank of the matrix $\|\partial G_\nu/\partial y_i\|$ is $N-m$. Since we may assume $n > m$, there must exist a ν such that the rank of the matrix $\left\|\begin{matrix}\partial F_\mu/\partial y_i \\ \partial G_\nu/\partial y_i\end{matrix}\right\|$ is $N-n+1$; we may assume without loss of generality that

[4] Loc. cit. [2].

$\nu = 1$. Further we may assume that $G_1(X)$ is irreducible. Let W^{N-1} be the variety defined by $G_1(X) = 0$ in S^N. There exists a component U_1 of the intersection of W^{N-1} and U^n which contains V^m (F. IV$_4$ th. 8). The dimension of U_1 is $n-1$ (F. VI th. 1 Cor. 2) and by the construction it is obvious that V^m is a simple subvariety of U^{n-1}. Thus our assertion follows by induction on n.

LEMMA 2. *Let k be a perfect field and let $P = (x)$ be a set of quantities such that $k(P)$ is a regular n-dimensional extension of k. Let v be an $(n-2)$-dimensional valuation of $k(P)$ of rank 2.*[5] *Then there exists a variety U^n defined over k with a generic point Q such that $k(P) = k(Q)$ and that the center of the valuation v on U is a simple subvariety V^{n-2} of U.*

Proof. Let $\mathfrak{O}$ be the valuation ring of v and let $\mathfrak{m}$ be the prime ideal of all the non-units in $\mathfrak{O}$. By our hypothesis, the residue class field $\mathfrak{O}/\mathfrak{m}$ is $(n-2)$-dimensional over k. Let $(u_1, \ldots, u_{n-2})$ be a system of elements in $\mathfrak{O}$ such that they are algebraically independent mod $\mathfrak{m}$ over k. Put $k(u_1, \ldots, u_{n-2}) = K$. Then $k(P)$ is 2-dimensional over K. We can also select $(u_1, \ldots, u_{n-2})$ in such a way that $k(P)$ is separably generated over K. As $v(z) = 0$ for each element $z \neq 0$ in K, we can consider v as a valuation of dimension 0 and rank 2 of $k(P)/K$. By Zariski's local uniformization theorem (cf. O. Zariski, Reduction of algebraic three-dimensional varieties §§ 10-12, § 16),[6] there exists such a set of quantities $(y_1, \ldots, y_m)$ that $k(P) = K(y)$ and that the quotient ring $\mathfrak{O}_{\bar{\mathfrak{p}}}$ of $\bar{\mathfrak{p}} = K[y] \cap \mathfrak{m}$ in $K[y]$ is a regular local ring. Put $Q = (u_1, \ldots, u_{n-2}, y_1, \ldots, y_m)$ and let U be its locus over k. The quotient ring $\mathfrak{O}_{\mathfrak{p}}$ of $\mathfrak{p} = k[u_1, \ldots, u_{n-2}, y_1, \ldots, y_m] \cap \mathfrak{m}$ in $k[u_1, \ldots, u_{n-2}, y_1, \ldots, y_m]$ is identical with $\mathfrak{O}_{\bar{\mathfrak{p}}}$ and hence it is also regular local ring. As k is perfect, $\mathfrak{p}$ defines in U absolutely simple subvariety in the sense of Zariski. Hence there exists a simple point Q' of U whose specialization ring in $k(Q)$ is identical with $\mathfrak{O}_{\mathfrak{p}}$.

THEOREM 2. *Let U^n be a complete variety and $\mathbf{V}$ its simple subvariety. If a differential form ω on $\mathbf{U}$ is of the first kind, then it induces on $\mathbf{V}$ a differential form ω' of the first kind.*

Proof. It is known that a differential form which is finite on $\mathbf{V}$ induces uniquely a differential form ω' on $\mathbf{V}$.[7] We prove that this ω' is of the first kind. We may assume that $\mathbf{U}$, $\mathbf{V}$ and ω have a common field of definition k which is perfect. Let $\mathbf{P}$ be a generic point of $\mathbf{U}$ over k and let $\mathbf{Q}$ be a generic point of $\mathbf{V}$ over k. By lemma 1 we may assume without loss of generality that the dimension of $\mathbf{V}$ is $n-1$. Let $\mathfrak{P}'$ be a prime divisor of $k(\mathbf{Q})$ ($\nu_{\mathfrak{P}'}$ being a $(n-2)$-

5) Loc. cit. 3).

6) O. Zariski, *Reduction of singularities of algebraic three-dimensional varieties*, Annals of Math. vol. 45 (1944).

7) Loc. ict 2) S. Koizumi I. Prop. 6.

dimensional valuation over k). We shall prove that $\omega'(\mathbf{Q})$ is finite at $\mathfrak{P}'$. As $\mathbf{Q}$ is a simple point of $\mathbf{U}$ of dimension $n-1$ over k, it determines a prime divisor $\mathfrak{P}$ in $k(\mathbf{P})$; namely the valuation ring of $\mathfrak{P}$ is identical with the specialization ring of $\mathbf{Q}$ in $k(\mathbf{P})$. We may construct, by virtue of $\mathfrak{P}$ and the prime divisor $\mathfrak{P}'$ of $k(\mathbf{Q})$, a valuation v of dimention $n-2$, and rank 2 of $k(P)$. It follows from lemma 2 that there exists a variety U'^n and a point Q' of U' such that Q' is simple on U' and the specialization ring of Q' is contained in the valuation ring of the valuation v of $k(\mathbf{P})$. Let $(t_1, \ldots, t_n)$ be a system of uniformizing parameters of Q' in $k(\mathbf{P})$. Since ω is of the first kind $\omega(\mathbf{P})$ is of the form

$$\omega(\mathbf{P}) = \sum w_{ij\ldots} dt_i dt_j \ldots ,$$

where $w_{ij\ldots}$, t_i, t_j, etc. are contained in the specialization ring of Q'; therefore $v(w_{ij\ldots}) \geqq 0$, $v(t_i) \geqq 0, \ldots$ and $\nu_{\mathfrak{P}}(w_{ij\ldots}) \geqq 0$, $\nu_{\mathfrak{P}}(t_i) \geqq 0$; namely $w_{ij\ldots}$, $t_i, \ldots$ are contained in the specialization ring of $\mathbf{Q}$ in $k(\mathbf{P})$. Therefore the specializations of $w_{ij\ldots}$, t_i, t_j over $\mathbf{P} \rightarrow \mathbf{Q}$ with respect to k are contained in the valuation ring of $\mathfrak{P}'$ in $k(\mathbf{Q})$. This proves that $\omega'(\mathbf{Q})$ is finite at $\mathfrak{P}'$.

2. The case of characteristic 0.

Let $\mathbf{U}^n$ be a complete variety defined over k with a generic point $\mathbf{P}$ over k and let $\mathbf{V}$ be its subvariety defined over k with a generic point $\mathbf{Q}$ over k. If a differential form ω has the following expression

$$\omega(\mathbf{P}) = \sum z_{\alpha\beta\ldots} dy_\alpha dy_\beta \ldots ,$$

where $z_{\alpha\beta\ldots}$, y_α, $y_\beta, \ldots$ are contained in the specialization ring of $\mathbf{Q}$ in $k(\mathbf{P})$,[8] then we can induce ω on $\mathbf{V}$ even if $\mathbf{Q}$ is not a simple point of $\mathbf{U}$.

In this section we assume that the characteristic is 0 and prove that if ω is a differential form of the first kind on $\mathbf{U}$ it induces uniquely on $\mathbf{V}$ a differential form ω' of the first kind.

THEOREN 3. *If a differential form $\omega(\mathbf{P}) = \sum z_{\alpha\beta\ldots} dy_\alpha dy_\beta \ldots$ is finite at $\mathbf{Q}$, then $\omega'(Q) = \sum z'_{\alpha\beta\ldots} dy'_\alpha d'_\beta \ldots$ is uniquely determined by $\omega(\mathbf{P})$, where $z'_{\alpha\beta\ldots}$, y'_α, y'_β are the specializations of $z_{\alpha\beta\ldots}$, y_α, y_β, over $\mathbf{P} \rightarrow \mathbf{Q}$ with respect to k.*

Proof. We prove that if $\omega(\mathbf{P}) = \sum z_{\alpha\beta\ldots} dy_\alpha dy_\beta \ldots = \sum \bar{z}_{\gamma\delta\ldots} d\bar{y}_\gamma d\bar{y}_\delta \ldots$, where $\bar{z}_{\gamma\delta\ldots}$, $\bar{y}_\gamma$, $\bar{y}_\delta, \ldots$ are also contained in the specialization ring of $\mathbf{Q}$ in $k(\mathbf{P})$, then $\sum z'_{\alpha\beta\ldots} dy'_\alpha dy'_\beta \ldots = \omega'(\mathbf{Q})$ and $\sum \bar{z}'_{\gamma\delta\ldots} d\bar{y}'_\gamma d\bar{y}'_\delta \ldots = \bar{\omega}'(\mathbf{Q})$ are identical. If the dimension of $\mathbf{V} < n-1$, then there exists a variety $\mathbf{W}^{n-1}$ which is algebraic over k such that $\mathbf{U} \supset \mathbf{W} \supset \mathbf{V}$. Let $\mathbf{P}'$ be a generic point of $\mathbf{W}$ over k. If z is contained in the specialization ring of $\mathbf{Q}$ in $k(\mathbf{P})$, it is also contained in the specialization ring of $\mathbf{P}'$ in $k(\mathbf{P})$. Further if z^* is the specialization of z

8) Even if ω is of the first kind, this is not always true.

over $\mathbf{P}\to\mathbf{P}'$ with respect to k, then the specialization of z^* over $\mathbf{P}'\to\mathbf{Q}$ with respect to $\bar{k}$ is identical with the specialization z' of z over $\mathbf{P}\to\mathbf{Q}$ with respect to k. Therefore we can assume without loss of generality that the dimension of $\mathbf{V}$ is $n-1$. Let $\mathbf{U}^*$ be the normalization of $\mathbf{U}$ over k; let $\mathbf{P}^*$ be the corresponding generic point of $\mathbf{P}$, and let $\mathbf{Q}^*$ be a corresponding point of $\mathbf{Q}$ under the natural birational transformation between $\mathbf{U}$ and $\mathbf{U}^*$. Then $\mathbf{Q}^*$ is a simple point of $\mathbf{U}^*$ and $k(\mathbf{Q}^*)$ is an algebraic extension over $k(\mathbf{Q})$. Let ω^* be a differential form on $\mathbf{U}^*$ defined by $\omega^*(\mathbf{P}^*)=\omega(\mathbf{P})$; then since $\mathbf{Q}^*$ is simple $\omega^{*\prime}(\mathbf{Q}^*)$ $=\sum z'_{\alpha\beta}\ldots dy'_\alpha dy'_\beta\ldots$ and $\bar{\omega}^{*\prime}(\mathbf{Q}^*)=\sum \bar{z}'_{\gamma\delta}\ldots d\bar{y}'_\gamma d\bar{y}'_\delta\ldots$ are identical. If $(t_1,\ldots,t_{n-1})$ is a set of elements of $k(\mathbf{Q})$ such that $k(\mathbf{Q})/k(t_1,\ldots,t_{n-1})$ is (separably) algebraic, then $\omega'(\mathbf{Q})-\bar{\omega}'(\mathbf{Q})$ *is expressed in one and only one way as a polynomial of* dt_i $(i=1,\ldots,n-1)$:

$$\omega'(\mathbf{Q})-\bar{\omega}'(\mathbf{Q})=\sum w_{ij\ldots}dt_i dt_j\ldots .$$

Then we have $\omega^{*\prime}(\mathbf{Q}^*)-\bar{\omega}^{*\prime}(\mathbf{Q}^*)=\sum w_{ij\ldots}dt_i dt_j\ldots$. As $k(\mathbf{Q}^*)/k(\mathbf{Q})$ is (separably) algebraic, $k(\mathbf{Q}^*)/k(t_1,\ldots,t_{n-1})$ is also (separably) algebraic, and hence $w_{ij\ldots}$, ect. must be equal to 0, because $\omega^{*\prime}(\mathbf{Q}^*)=\bar{\omega}^{*\prime}(\mathbf{Q}^*)$. Therefore $\omega'(\mathbf{Q})$ $=\bar{\omega}'(\mathbf{Q})$.

THEOREM 4. *Assumptions being as in the above theorem, let* ω *be of the first kind. Then* ω' *is also of the first kind.*

Proof. We use the same notations as in the proof of the preceeding theorem. We may also assume without loss of generality that $\mathbf{V}$ is of dimension $n-1$. As $\mathbf{Q}^*$ is simple on $\mathbf{U}^*$, $\omega^{*\prime}$ is of the first kind on the locus of $\mathbf{Q}^*$ over k in $\mathbf{U}^*$. Therefore the proof may by reduced to the following lemma.

LEMMA 3. *Suppose that* $k(\mathbf{Q}^*)$ *is an algebraic extension over* $k(\mathbf{Q})$ *and* $\omega^*(\mathbf{Q}^*)=\omega(\mathbf{Q})$. *If* $\omega^*(\mathbf{Q}^*)$ *is of the first kind, then* $\omega(\mathbf{Q})$ *is also of the first kind.*

Proof. If we suppose that this is not true, there must exist a prime divisor $\mathfrak{P}$ of $k(\mathbf{Q})$ such that $\omega(\mathbf{Q})$ is not finite at $\mathfrak{P}$. Let $t_1,\ldots,t_{n-1}$ be a set of uniformizing parameters at $\mathfrak{P}$ in $k(\mathbf{Q})$. Let $\mathfrak{P}^*$ be a prime divisor of $k(\mathbf{Q}^*)$ which is an extension of $\mathfrak{P}$ and let $(t_1^*,\ldots,t_{n-1}^*)$ be a set of uniformizing parameters at $\mathfrak{P}^*$ in $k(\mathbf{Q}^*)$. Suppose $\mathfrak{P}^{*e}\|\mathfrak{P}$. As $\omega(\mathbf{Q})$ is not finite at $\mathfrak{P}$, we can assume that

$$\omega^*(\mathbf{Q}^*)=\omega(\mathbf{Q})=a dt_1\ldots dt_s+\ldots ,$$

where a is an element in $k(\mathbf{Q})$ and $\nu_{\mathfrak{P}}(a)<0$. Since ω^* is of the first kind, $\omega^*(\mathbf{Q}^*)$ is finite at $\mathfrak{P}^*$ and $\theta(\mathbf{Q}^*)=dt_{s+1}\ldots dt_{n-1}$ is finite at $\mathfrak{P}^*$; therefore $\theta_1(\mathbf{Q}^*)$ $=\omega^*(\mathbf{Q}^*)\cdot\theta(\mathbf{Q}^*)=a dt_1\ldots dt_s dt_{s+1}\ldots dt_{n-1}$ is also finite at $\mathfrak{P}^*$. But as $dt_1\ldots$ $dt_{n-1}=b dt_1^*\ldots dt_{n-1}^*$, where b is an element of $k(\mathbf{Q}^*)$ and $\nu_{\mathfrak{P}^*}(b)=e-1$, $\theta_1(\mathbf{Q}^*)$ $=ab dt_1^*\ldots dt_{n-1}^*$, where $\nu_{\mathfrak{P}^*}(ab)\leqq -e+(e-1)<0$. This contradicts to the fact that $\theta_1(\mathbf{Q}^*)$ is finite at $\mathfrak{P}^*$.

An example

In the case of characteristic $p \neq 0$, theorem 4 does not hold in general. Let k be an algebraically closed field of characteristic p and let V^1 be the variety defined over k by $F(X_1, X_2) = X_2^q + X_2 - X_1^m$, where $q = p^r$, $r > 0$, $m > 1$, $q + 1 = mn$. Let (x_1, x_2) be a generic point of V over k. Then dx_1 is a differential of the first kind in $k(x_1, x_2)$. This is the example of F. K. Schmidt.[9] Let t be a quantity such that t and $k(x_1, x_2)$ are independent over k. Put $x_1 = x$, $tx_2 = y$, $P = (1, x, y, t)$. Then $k(x_1, x_2, t) = k(P)$. Let U^2 be the locus of P over k and consider a projective variety $\mathbf{U}^2$ which has a representative $U_0^2 = U^2$ and let $\mathbf{P}$ be a gneric point of $\mathbf{U}$ with the representative $P_0 = P$; let ω be the differential form defined on $\mathbf{U}$ by $\omega(\mathbf{P}) = dx$. Then ω is the differential form of the first kind. However if $\mathbf{Q}$ is a point of $\mathbf{U}$ which has the representative $Q_0 = (1, x, 0, 0)$ and if $\mathbf{W}$ is the locus of $\mathbf{Q}$ over k, then the induced differential form ω' by ω on $\mathbf{W}$ cannot be of the first kind.

Mathematical Institute,
Nagoga University

9) F. K. Schmidt, *Zur arithmetischen Theorie der algebraischen Funktionen* II, § 5. Math. Zeitschrift, Bd. 35 (1939).