RIEMANN SPACE WITH TWO-PARAMETRIC HOMOGENEOUS HOLONOMY GROUP

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The rotational part of the holonomy group of a Riemann space is called its homogeneous holonomy group. A Riemann space, whose homogeneous holonomy group is one-parametric, was investigated by Liber and an alternative treatment of the same problem was given by S. Sasaki [1]. I will treat here a Riemann space with two-parametric homogeneous holonomy group and prove the following theorem by the method analogous to that of [1]. I thank Prof. T. Ootuki for his kind advice.

THEOREM. *If the homogeneous holonomy group of an n-dimensional Riemann space is two-parametric, the space is a direct sum of two non-flat two-dimensional Riemann spaces and a n − 4-dimensional flat space; namely the line-element of our space is given by*

$$ds^2 = d\sigma_1^2 + d\sigma_2^2 + \sum_{i=5}^{n}(dx^i)^2$$

where

$$d\sigma_1^2 = \sum_{i,\,j=1,\,2} g_{ij}(x^1,\,x^2)dx^i dx^j, \quad d\sigma_2^2 = \sum_{i,\,j=3,\,4} g_{ij}(x^3,\,x^4)dx^i dx^j.$$

0. Let the line element of a general Riemann space be given by

$$ds^2 = \sum_{i=1}^{n}(\omega_i)^2.$$

If we take a rectangular frame in the tangent euclidean space at any point of the Riemann space, a Riemannian connection is given by

(1) $$d\mathbf{A} = \omega_i \mathbf{e}_i, \quad d\mathbf{e}_i = \omega_{ij}\mathbf{e}_j \quad (\omega_{ij} = -\omega_{ji}).$$

If we denote the outer derivative of ω by $d\omega$, we have

(2) $$d\omega_i = [\omega_j \omega_{ji}].$$

Taking the outer derivative of both sides, we get

(3) $$[\omega_j \Pi_{ji}] = 0$$

where Π_{ij} is given by

(4) $\Pi_{ij} = d\omega_{ij} - [\omega_{ik}\omega_{kj}]$.

In general we define a product of two matrices $\Omega = (\omega_{ip})$ and $\Pi = (\pi_{pu})$ with Pfaffians or differential forms as coefficients by a matrix

$$[\Omega\Pi] = ([\omega_{ip}\pi_{pu}])$$

and an outer derivative of Ω by a matrix

$$d\Omega = (d\omega_{ip}) .$$

Then putting

$$\omega = (\omega_1, \ldots, \omega_n), \quad \Omega = (\omega_{ij}), \quad \Pi = (\Pi_{ij}) ,$$

(2), (4), (3) can be written as

(5) $d\omega = [\omega\Omega], \quad \Pi = d\Omega - [\Omega\Omega],$

(6) $[\omega\Pi] = 0 .$

If we rotate a rectangular frame, a new frame is given by $\bar{e}_i = p_{ij}e_j$. Our Riemannian connection is for this new frame determined by

$$\bar{\omega} = \omega P', \quad \bar{\Omega} = P\Omega P' + dP \cdot P' ,$$

where P is the orthogonal matrix (p_{ij}). If P is a constant matrix, we have

(7) $\bar{\Omega} = P\Omega P' .$

Next we state lemmas which are used repeatedly in our treatment.

LEMMA 1. *Let ω_α $(\alpha = 1, \ldots, k)$ be linearly independent Pfaffians with n variables $x^1, \ldots, x^n$. If there exist Pfaffians $\omega_{\alpha\beta}$ such that*

$$d\omega_\alpha = [\omega_\beta, \omega_{\beta\alpha}], \quad \omega_{\alpha\beta} = -\omega_{\beta\alpha} \quad (\alpha, \beta = 1, \ldots, k),$$

then by suitable choice of variables $y^i = y(x^1, \ldots, x^n)$ $(i = 1, \ldots, n)$, $\sum_{\alpha=1}^{k}(\omega_\alpha)^2$ can be written as a quadratic differential form in k variables $y^1, \ldots, y^k$, namely

$$\sum_{\alpha=1}^{k}(\omega_\alpha)^2 = g_{\alpha\beta}(y^1, \ldots, y^k)dy^\alpha dy^\beta .$$

This is seen in E. Cartan [2] p. 296.

LEMMA 2. *If the rotational part $\Omega = (\omega_{ij})$ of our connection* (1) *is a direct sum of two matrices, our Riemann space is a direct sum of two Riemann spaces. In other words if $\omega_{\alpha\lambda} = 0$ $(\alpha = 1, \ldots, k; \lambda = k+1, \ldots, n)$ then by a suitable choice of coordinates*

$$ds^2 = g_{\alpha\beta}(x^1, \ldots, x^k)dx^\alpha dx^\beta + g_{\lambda\mu}(x^{k+1}, \ldots, x^n)dx^\lambda dx^\mu.$$
$$(\alpha, \beta = 1, 2, \ldots, k; \lambda, \mu = k+1, \ldots, n)$$

Proof. By virtue of $\omega_{\alpha\lambda} = 0$ and (2) we have

$$dw_\alpha = [\omega_\beta \omega_{\beta\alpha}] + [\omega_\lambda \omega_{\lambda\alpha}] = [\omega_\beta \omega_{\beta\alpha}]$$
$$dw_\lambda = [\omega_\alpha \omega_{\alpha\lambda}] + [\omega_\mu \omega_{\mu\lambda}] = [\omega_\mu \omega_{\mu\lambda}].$$

Using lemma 1 we get lemma 3.

LEMMA 3. *If a Riemann space S is a direct sum of two Riemann spaces S_1 and S_2, the homogeneous holonomy group g of S is the direct product of the homogeneous holonomy group g_1 of S_1 and g_2 of S_2. Especially if g_1 and g_2 are k_1- and k_2-parametric, then g is $k_1 + k_2$-parametric.*

1. Now we consider a Riemann space whose homogeneous holonomy group g is two-parametric. For such a space we can take rectangular frame at each point such that $\Omega = (\omega_{ij})$ is an infinitesimal rotation belonging to our homogeneous holonomy group g. This is evident by the fundamental theorem of E. Cartan about the holonomy group. As g is two-parametric, we have

(8) $$\Omega = (\omega_{ij}) = C_1 \pi_1 + C_2 \pi_2,$$

where π_1 and π_2 are Pfaffians whose ratio is not a constant and C_1 and C_2 are constant skew symmetric matrices such that

$$[C_1 C_2] = C_1 C_2 - C_2 C_1 = \lambda_1 C_1 + \lambda_2 C_2.$$

Owing to the skewness of C_1 and C_2 two constants λ_1 and λ_2 are zero. In fact let λ_1 be not zero, then putting $\lambda_1 C_1 + \lambda_2 C_2 = \overline{C}_1$ and $C_2/\lambda_1 = \overline{C}_2$ we have

$$[\overline{C}_1 \overline{C}_2] = \overline{C}_1.$$

For an orthogonal matrix P $P\overline{C}_1 P'$ and $P\overline{C}_2 P'$ are skew symmetric matrices and moreover

(9) $$[P\overline{C}_1 P', P\overline{C}_2 P'] = P\overline{C}_1 P'.$$

If we select P suitably, we have

(10) $$P\overline{C}_1 P' = \begin{pmatrix} A_1 & & & & & \\ & \cdot & & & & \\ & & \cdot & \cdot A_l & & \\ & & & 0 & \cdot & \\ & & & & \cdot & \\ & & & & & \cdot 0 \end{pmatrix}$$

where

$$A_i = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix}, \quad \alpha_i \neq 0. \quad (i = 1, 2, \ldots, l).$$

Hence calculating the elements of the first row and the second colomn we get by (10) $\alpha_1 = 0$, which contradicts the assumption $\alpha_1 \neq 0$. Thus we get $\lambda_1 = 0$, and similarly $\lambda_2 = 0$. So we have

(11) $$[C_1, C_2] = 0.$$

Now we state one lemma.

LEMMA 4. *Let C_1 and C_2 be skew symmetric matrices such that $[C_1, C_2] = 0$ and we assume there exists a constant m for which the matrix $mC_1 + C_2$ is singular. Then the Riemann space with $\Omega = (\omega_{ij}) = C_1\pi_1 + C_2\pi_2$ decomposes into the direct sum of Riemann spaces.*

Proof. Put $\overline{C}_2 = mC_1 + C_2$. For suitably chosen orthogonal matrix P we have

$$P\overline{C}_2P' = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

where A is non-singular. Let the corresponding formula for PC_1P' be

$$PC_1P' = \begin{pmatrix} K & L \\ M & N \end{pmatrix},$$

namely K and A have the same number of rows and colomns. Then by the relation $[PC_1P', P\overline{C}_2P'] = 0$ we have $AL = 0$ and $MA = 0$. As A is non-singular we get $L = 0$ and $M = 0$. Hence

$$\overline{\Omega} = P\Omega P' = PC_1P'\pi_1 + PC_2P'\pi_2$$
$$= \begin{pmatrix} K\pi_1 + (A - mK)\pi_2 & 0 \\ 0 & N\pi_1 - mN\pi_2 \end{pmatrix}$$

and by lemma 2 and (7) we get the result.

Now we treat the case when C_1 is non-singular. Then for a suitable orthogonal matrix P we have

$$PC_1P' = \begin{pmatrix} \alpha_1Q & & \\ & \ddots & \\ & & \alpha_lQ \end{pmatrix}, \quad \text{where} \quad Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and moreover all $\alpha_i > 0$. Put

$$PC_2P' = \begin{pmatrix} U_{11} \ldots U_{l1} \\ \ldots \ldots \\ U_{1l} \ldots U_{ll} \end{pmatrix},$$

where U_{ij}'s $(i, j = 1, 2, \ldots, l)$ are square matrices of degree 2. Then by virtue of $[PC_1P', PC_2P'] = 0$ we have

$$\alpha_iQU_{ij} - U_{ij}\alpha_jQ = 0 \quad (i \neq j),$$

where double indices do not mean summation.

Putting $U_{ij} = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$ we get

$$\alpha_ju - \alpha_iz = 0, \quad \alpha_iu - \alpha_jz = 0, \quad \alpha_jv + \alpha_iw = 0, \quad \alpha_iv + \alpha_jw = 0.$$

As $\alpha_i > 0$, $\alpha_i \neq \alpha_j$ means $\alpha_i^2 \neq \alpha_j^2$ and then $u = v = w = z = 0$, namely $U_{ij} = 0$. For

$\alpha_i = \alpha_j$ $(i \neq j)$ we get $u = z$ and $w = -v$. Thus by suitable change of the order of axes of the frame the rotational part of Riemannian connection (ω_{ij}) decomposes into a direct sum of the matrices of the form

$$\alpha \begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix} \pi_1 + \begin{pmatrix} U_{11} & \ldots & U_{1l} \\ & \ldots & \\ U_{l1} & \ldots & U_{ll} \end{pmatrix} \pi_2$$

where

$$U_{ij} = \begin{pmatrix} u_{ij} & v_{ij} \\ -v_{ij} & u_{ij} \end{pmatrix} \quad (i, j = 1, 2, \ldots, l).$$

Now we take π_1 for $\alpha\pi_1$ and put

$$C_1 = \begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix}, \qquad C_2 = \begin{pmatrix} U_{11} & \ldots & U_{1l} \\ & \ldots & \\ U_{l1} & \ldots & U_{ll} \end{pmatrix}.$$

If $U_{12}, \ldots, U_{1l}$ are all zero matrices, our Riemann space decomposes by virtue of $\Omega = C_1\pi_1 + C_2\pi_2$ and lemma 2. Hence if we treat an indecomposable space, we can assume without loss of generality U_{12} is not zero matrix. We put $U_{12} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$ and take $C_2/\sqrt{u^2 + v^2}$ for C_2. Then we can assume $u^2 + v^2 = 1$. Now we transform C_1 and C_2 by an orthogonal matrix

$$P = \begin{pmatrix} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & u & v & & & \\ & & -v & u & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Then $PC_1P' = C_1$, and taking $PC_2P' - qC_1$ for C_2 where $U_{11} = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}$ we get the reduced formulas

$$(12) \qquad C_1 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & w & z \\ 0 & 0 & 0 & 1 & -z & w \\ -1 & 0 & 0 & p & k & l \\ 0 & -1 & -p & 0 & -l & k \\ & & & & & \\ & & & & & \end{pmatrix}.$$

Moreover we can assume

$$(13) \qquad \det. |\, mC_1 + C_2\,| \neq 0,$$

because in the alternative case our space decomposes by lemma 4.

2. Now we investigate the space for which

$$\Omega = C_1 \pi_1 + C_2 \pi_2 ,$$

where C_1 and C_2 are given by (12) satisfying the relation (13). We get by (5)

$$\Pi = d\Omega - [\Omega\Omega] = C_1 d\pi_1 + C_2 d\pi_2 - [C_1 C_2][\pi_1 \pi_2] ,$$

which by (11) reduces to

(14) $$\Pi = C_1 d\pi_1 + C_2 d\pi_2 .$$

Hence we have by (6)

(15) $$[\omega C_1, \, d\pi_1] + [\omega C_2, \, d\pi_2] = 0 .$$

Here we state one more lemma, which is evident.

LEMMA 5. *Let $\omega_1, \ldots, \omega_n$ be linearly independent Pfaffians and let*

$$\Omega_1 = \frac{1}{2} a_{ij}[\omega_i \omega_j], \quad \Omega_2 = \frac{1}{2} b_{ij}[\omega_i \omega_j] \quad (a_{ij} = -a_{ji}, \ b_{ij} = -b_{ji})$$

and moreover

$$[\omega_1 \Omega_1] + [\omega_2 \Omega_2] = 0 ,$$

then we have

$$a_{ij} = 0, \quad b_{ij} = 0 \quad (i \neq 1, 2; \ j \neq 1, 2)$$

and

$$a_{2i} = b_{1i} \quad (i = 3, 4, \ldots, n) .$$

Now by (12) we have

$$\omega C_1 = (-\omega_2, \ \omega_1, \ -\omega_4 \ \omega_3, \ldots),$$
$$\omega C_2 = (-\rho, \ \tau, \ -\xi, \ \eta, \ldots),$$

where

(16)
$$\rho = \omega_3 + w\omega_5 + z\omega_6 + \ldots$$
$$\tau = -\omega_4 + z\omega_5 - w\omega_6 + \ldots$$
$$\xi = -\omega_1 + p\omega_4 + k\omega_5 + l\omega_6 + \ldots$$
$$\eta = \omega_2 + p\omega_3 + l\omega_5 - k\omega_6 + \ldots .$$

So by virtue of (15)

(17) $$[\omega_2 d\pi_1] + [\rho d\pi_2] = 0 , \quad [\omega_1 d\pi_1] + [\tau d\pi_2] = 0$$
(18) $$[\omega_4 d\pi_1] + [\xi d\pi_2] = 0 , \quad [\omega_3 d\pi_1] + [\eta d\pi_2] = 0.$$

By (16) we can take $\omega_1, \omega_2, \rho, \tau, \omega_5, \omega_6, \ldots$ as base. Then by virtue of (17) and lemma 5 $d\pi_1$ and $d\pi_2$ can not have terms containing $\omega_5, \omega_6, \ldots$, and these are of the form

(19) $$A[\omega_1 \omega_2] + B[\omega_1 \rho] + C[\omega_2 \tau] + D[\rho \tau] .$$

On the other hand by (18) $d\pi_1$ and $d\pi_2$ are represented by $\omega_3, \omega_4, \xi, \eta$ only, and

so by ω_3, ω_4, $\xi' = \xi - p\omega_4$, $\eta' = \eta - p\omega_3$. Now by (16)

$$\rho = \omega_3 + w\omega_5 + z\omega_6 + \ldots$$
$$\tau = -\omega_4 + z\omega_5 - w\omega_6 + \ldots$$
$$(20) \qquad \omega_1 = -\xi' + k\omega_5 + l\omega_6 + \ldots$$
$$\omega_2 = \eta' - l\omega_5 + k\omega_6 + \ldots$$

We put these into (19), and equating the coefficients of $[\omega_3\omega_5]$, $[\omega_3\omega_6]$, $[\omega_4\omega_5]$, $[\omega_4\omega_6]$, $[\xi'\omega_5]$, $[\xi'\omega_6]$, $[\eta'\omega_5]$, and $[\eta'\omega_6]$ to zero we get

$$(21) \qquad Bk - Dz = 0, \qquad Bl + Dw = 0$$
$$(22) \qquad -Cl + Dw = 0, \qquad Ck + Dz = 0$$
$$(23) \qquad -Al + Bw = 0, \qquad Ak + Bz = 0$$
$$(24) \qquad Ak - Cz = 0, \qquad Al + Cw = 0.$$

From each row of these we get

$$(25) \qquad B(k^2 + l^2) = D(kz - lw), \quad 0 = D(lz + kw)$$
$$(26) \qquad 0 = -D(lz + kw), \quad C(k^2 + l^2) = -D(kz - lw)$$
$$(27) \qquad B(w^2 + z^2) = -A(kz - lw), \quad 0 = A(lz + kw)$$
$$(28) \qquad 0 = A(lz + kw), \quad C(w^2 + z^2) = A(kz - lw).$$

If $k^2 + l^2 \neq 0$ and $w = z = 0$, we get by (25) and (26) $B = C = 0$, hence by (23) $A = 0$. If $w^2 + z^2 \neq 0$ and $k = l = 0$, we get by (27) (28) and (22) $B = C = D = 0$. If $k^2 + l^2 \neq 0$ and $w^2 + z^2 \neq 0$, we get

$$\frac{A}{w^2 + z^2} = \frac{B}{-(kz - lw)} = \frac{C}{kz - lw} = \frac{D}{-(k^2 + l^2)}.$$

In any way $A : B : C : D$ is determined when at least one of k, l, w, z is not zero. We treat this case in the first. Then (19) is determined except for a multiplier, and as $d\pi_1$ and $d\pi_2$ are of this form, we have two cases, namely either $d\pi_2 = 0$ or $d\pi_1 = md\pi_2$ $(d\pi_2 \neq 0)$. If $d\pi_2 = 0$, we get by (17) and (18) $d\pi_1 = 0$, and our space is flat and so this case may be omitted. If $d\pi_1 = md\pi_2$, we get by (17) and (18)

$$(29) \qquad [\rho + m\omega_2, \, d\pi_2] = 0, \quad [\tau + m\omega_1, \, d\pi_2] = 0$$
$$(30) \qquad [\xi + m\omega_4, \, d\pi_2] = 0, \quad [\eta + m\omega_3, \, d\pi_2] = 0.$$

Now we have by (16)

$$\rho + m\omega_2 = m\omega_2 + \omega_3 + w\omega_5 + z\omega_6 + \ldots$$
$$\tau + m\omega_1 = m\omega_1 - \omega_4 + z\omega_5 - w\omega_6 + \ldots$$

which shows that $\rho + m\omega_2$ and $\tau + m\omega_1$ are independent. Hence by (29)

$$(31) \qquad d\pi_2 = h[\rho + m\omega_2, \, \tau + m\omega_1] = h(\ldots - [\omega_3\omega_4] + m[\omega_3\omega_1] + \ldots).$$

Here terms containing ω_1, ω_3, ω_4 only are written. Putting this and

$$\xi + m\omega_4 = -\omega_1 + (p + m)\omega_4 + k\omega_5 + l\omega_6 + \ldots$$

into (30) we get from the consideration of the term $[\omega_1\omega_3\omega_4]$ the **relation** $1 - m(p + m) = 0$. Hence $m = (-p \pm \sqrt{p^2+4})/2$, namely m is a constant. So by (14)

$$\Pi = C_1 d\pi_1 + C_2 d\pi_2 = (mC_1 + C_2) d\pi_2 .$$

By virtue of $[\omega\Pi] = 0$

(32) $$[\omega(mC_1 + C_2), \ d\pi_2] = 0.$$

By (31) and (32) each element of the matrix $\omega(mC_1 + C_2)$ must be a **linear** combination of $\rho + m\omega_2$ and $\tau + m\omega_1$, hence the rank of the matrix $mC_1 + C_2$ is two. So by the assumption (13) C_1 and C_2 are square matrices of **degree 2** and our Riemann space is of dimension 2. Next we treat the case $k = l = w = z = 0$. Then we can assume that in C_2 all the elements of the first four rows except first four colomns are zero, for in the contrary case we can treat as in the above case already treated, namely the case when one of k, l, w, z is not zero. Thus our space decomposes into a 4-dimensional space and another space, and it is sufficient to consider this 4-dimensional part. For this part C_1 and C_2 have components

$$C_1^{(0)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad C_2^{(0)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & p \\ 0 & -1 & -p & 0 \end{pmatrix}.$$

Hence

$$\text{det.} \ |\, mC_1^{(0)} + C_2^{(0)} \,| = (m(p + m) - 1)^2.$$

This is zero for a real number $m = (-p \pm \sqrt{p^2+4})/2$. Hence det. $|\, mC_1 + C_2 \,| = 0$. This contradicts the assumption (13).

Thus we get the result that the Riemann space, for which

$$\varOmega = C_1\pi_1 + C_2\pi_2 ,$$

where C_1 and C_2 are given by (12) satisfying the relation (13), is two-dimensional.

3. Thus we obtain the result that the Riemann space with two-parametric homogeneons holonomy group decomposes into a certain number of two-dimensional non-flat Riemann spaces and a euclidean space. By lemma 3 the number of these two-dimensional spaces is two, and our proof concludes.

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